

A two-dimensional numerical model of estuarine circulation using cubic splines

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A two-dimensional, laterally integrated, numerical model has been developed to represent the velocity and salinity distribution along an estuary. The governing equations, which express the conservation of mass, momentum, and salt or heat content, are solved by a finite difference method in combination with spline functions.

The model has been applied to the estuary of the Great Whale River in the James Bay region and results of the simulation are presented and compared with tide gauge data. The sensitivity of the model to various values of the stratification parameters that influence vertical diffusion has been studied and some preliminary results presented.

The use of the cubic spline formulation in computational hydraulics has been found to be promising and warrants further development.

Un modèle numérique bi-dimensionnel résultant de l'intégration suivant la composante latérale (largeur) des équations différentielles de base a été développé pour simuler la répartition de la vitesse et de la salinité le long d'un estuaire. Les équations exprimant la conservation de masse, de quantité de mouvement, de chaleur et de la salinité, ont été résolues par une méthode de différences finies conjointement avec les fonctions "splines."

Le modèle a été appliqué à l'estuaire de la Rivière Grande Baleine dans la région de la Baie James. Les résultats numériques sont comparés avec les données expérimentales enregistrées sur marégraphe. La sensibilité du modèle aux différentes valeurs des paramètres de stratification qui influencent la diffusion dans le sens vertical, a été analysée et quelques résultats sont présentés.

L'utilisation de la formulation des "spline" cubiques s'est avérée prometteuse et justifie des développements subséquents.

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Introduction

In engineering and environmental impact studies of estuaries it is of paramount importance to be able to accurately predict hydraulic and dispersive behaviour under the combined influence of natural and artificially imposed constraints. One of the current areas of interest arising from these investigations is the density-induced circulation, which has a major influence on the salt balance and the dilution of thermal or pollutant discharges. Numerical models of estuaries incorporating simulation of the density-induced circulation are increasingly being developed and applied as a major component in the assessment of the dispersive characteristics of an estuary.

In general, numerical models may have one, two, or three dimensions in space and one in time since the physical processes being modelled are basically unsteady. One-dimensional models applied to, say, the prediction of salinity intrusions in estuaries have been found unsatisfactory since the important effect of the density-induced circulation must be included as a longitudinal dispersion term with considerable uncertainty as to the appropriate formulation. Laterally averaged, two-dimensional (commonly referred to as $X-Z$)

models, on the other hand, correctly treat the problem as a convection rather than a dispersion phenomenon. Three-dimensional models have been developed (Leendertse and Liu 1975; Caponi 1976) but these are extremely expensive to run so that a vast majority of engineers are forced to content themselves with numerical models in one or two dimensions.

Several $X-Z$ models have been reported in the literature, such as those of Boericke and Hogan (1977) and Blumberg (1977) among others. A common feature of these models is the discretization of the dependent variables in space and time by the method of finite differences. Typically, the solution domain is divided into a grid of uniform mesh points and approximate numerical solutions are generated at the nodes of the grid.

When developing laterally averaged $X-Z$ models, certain disadvantages in using the finite difference formulation become evident. Firstly, the number of horizontal layers between grid points has to be adjusted according to the bathymetry of the estuary, while the thickness of the surface layer must be allowed to vary with time in order to correctly include the periodic tidal effects.

An alternative formulation in numerical modelling is the use of finite element techniques. These have the advantage of being able to treat (at least in principle) irregular bathymetry and nonuniform spacing in the coordinate directions but suffer from the disadvantage

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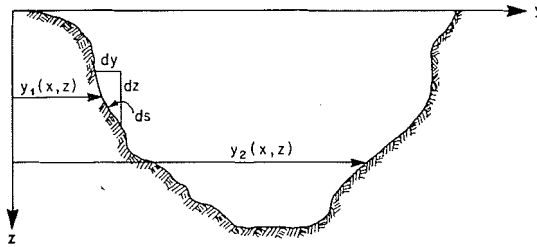


FIG. 1. Definition of cross-section terms.

of requiring a complex formulation as well as being relatively uneconomic in computing time.

The purpose of this paper is to explore a third alternative, namely the use of spline integration techniques to develop computational hydraulic models.

The main advantages of using spline integration techniques are that:

(i) the requirement of uniformly spaced grid points in the solution domain may be relaxed, thus simplifying treatment of irregular regions;

(ii) for implicit procedures, a tridiagonal matrix formulation may still be obtained as in finite differences;

(iii) third-order accuracy for the first derivative is obtained, fourth-order if equal grid point spacing is used (Rubin and Khosla 1976);

(iv) since the first or second derivatives are evaluated directly, boundary conditions containing derivatives may be incorporated into the solution procedure, thus avoiding the difficulty that exists with conventional finite difference schemes.

It appears therefore that the spline formulation possesses the advantages of finite element techniques without the disadvantages of high computer cost and complex problem formulation.

Applications of the method to problems in fluid mechanics have been discussed in the papers by Rubin and Khosla (1976), Rubin and Graves (1975), and Panton

$$[2] \quad \frac{\partial u}{\partial t} + \frac{1}{b} \left[\frac{\partial(bu^2)}{\partial x} + \frac{\partial(buw)}{\partial z} \right] + g \left[\frac{\partial \xi}{\partial x} + \frac{1}{\rho} \int_{-\xi}^{\cdot} \frac{\partial \rho}{\partial x} dz' \right] = \frac{1}{b} \frac{\partial}{\partial z} \left(b \epsilon_z \frac{\partial u}{\partial z} \right) - \frac{1}{b} \left(\frac{f}{8} u |u| \right) \left(\frac{ds}{dz} \Big|_{y_2} + \frac{ds}{dz} \Big|_{y_1} \right)$$

For salt or heat balance,

$$[3] \quad \frac{\partial(bS)}{\partial t} + \frac{\partial(buS)}{\partial x} + \frac{\partial(bwS)}{\partial z} = \frac{Q''' b}{\rho C_p} + \frac{\partial}{\partial x} \left(b K_x \frac{\partial S}{\partial x} \right) + \frac{\partial}{\partial z} \left(b K_z \frac{\partial S}{\partial z} \right)$$

The equation of state is written as:

$$[4] \quad \rho = \frac{p_0}{\lambda + \alpha_0 p_0}$$

where $\lambda = 1779.5 + 11.25T - 0.0745T^2 - (3.80 + 0.01T)S$, $\alpha_0 = 0.698$, $p_0 = 5890 + 38T - 0.375T^2 + 3S$, and T is temperature ($^{\circ}\text{C}$). This formulation is due to Eckart (1958).

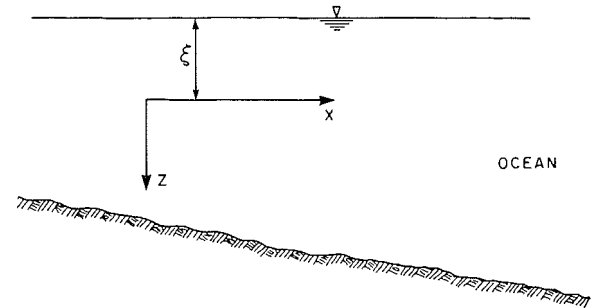


FIG. 2. Definition of coordinates.

and Sallee (1975) among others. There does not appear to have been any attempt made to apply the method to computational hydraulics.

Accordingly, the development of a preliminary $X-Z$ numerical model of estuarine circulation incorporating cubic splines in the solution procedure is outlined here. The model was applied to the estuary of the Great Whale River in the James Bay development region. The results of the numerical simulation agree reasonably well with tidal measurements taken in the estuary. Unfortunately, no salinity or velocity data exist with which further comparison may be made.

Governing equations

The set of equations describing the dynamics of a partially mixed estuary consists of (in two dimensions) the continuity, momentum balance, and salt and/or heat balance equations, together with an equation of state.

In laterally averaged form, the equations may be written as follows. The continuity equation is:

$$[1] \quad \frac{\partial(bu)}{\partial x} + \frac{\partial(bw)}{\partial z} = 0$$

For the x -momentum,

Equations [1]–[3] have been obtained by integration in the y direction from one shoreline $y = y_1(x, z)$ to the other, $y = y_2(x, z)$ (see Fig. 1). Here b is the local width, u , w are the longitudinal and vertical laterally averaged velocities (Fig. 2), ξ is water surface elevation above a datum, ρ the density, S salinity, Q''' heat source or sink, C_p specific heat, and ϵ_z , K_z , and K_x the turbulent viscosity and exchange coefficients re-

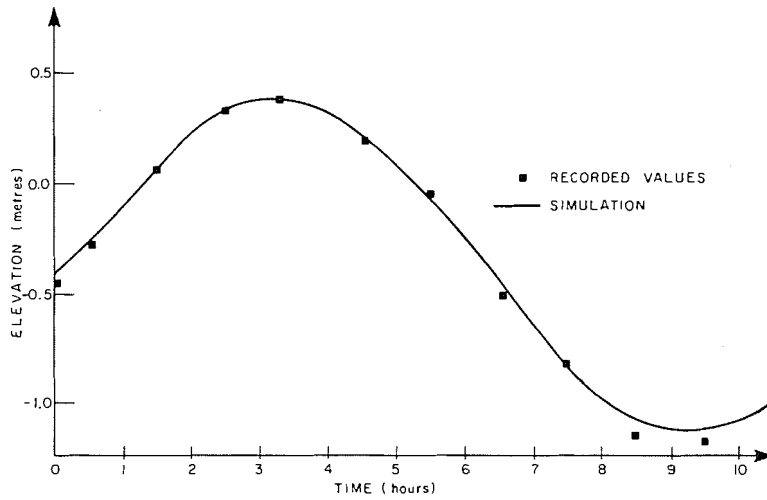


FIG. 3. Simulation compared with tide gauge records.

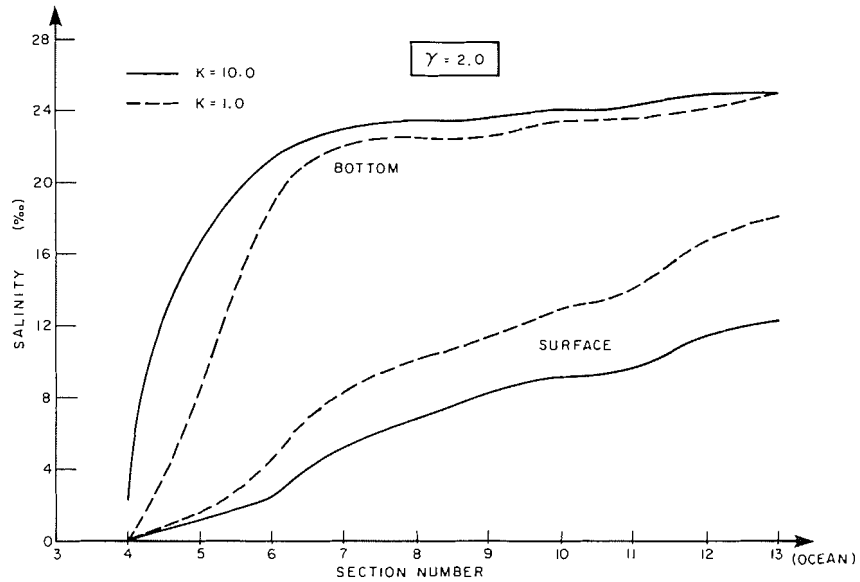


FIG. 4. Influence of K on longitudinal salinity distribution.

spectively. The third term on the left-hand side of the x -momentum equation has been obtained from the pressure gradient term $(1/\rho)(\partial p/\partial x)$ of the corresponding Navier-Stokes equation; $(ds/dz)|_{y_1}$, and $(ds/dz)|_{y_2}$ are the bed slopes as defined in Fig. 1.

Coordinate transformation

Following Boericke and Hogan (1977) and Perrels

and Karelse (1978), the solution domain, which is in general irregular, is transformed into a rectangular region by the nondimensional variable $\eta = (h - z)/(h + \xi)$, where $h(x)$ is the local depth. Here η varies between 0 at the bottom to 1 at the surface. With this transformation, the equations reduce to:

$$[5] \quad b \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} \left[H \int_0^1 bu \, d\eta \right] = 0$$

$$[6] \quad \frac{\partial u}{\partial t} + g \left[\frac{\partial \xi}{\partial x} + \frac{H}{\rho} \int_{\eta}^1 \frac{\partial \rho}{\partial x} \, d\eta \right] + CVT = \frac{1}{H^2 b} \frac{\partial}{\partial \eta} \left(b \epsilon_z \frac{\partial u}{\partial \eta} \right) - \frac{1}{b} \frac{f}{8} u |u| \left(\left. \frac{ds}{dz} \right|_{y_1} + \left. \frac{ds}{dz} \right|_{y_2} \right)$$

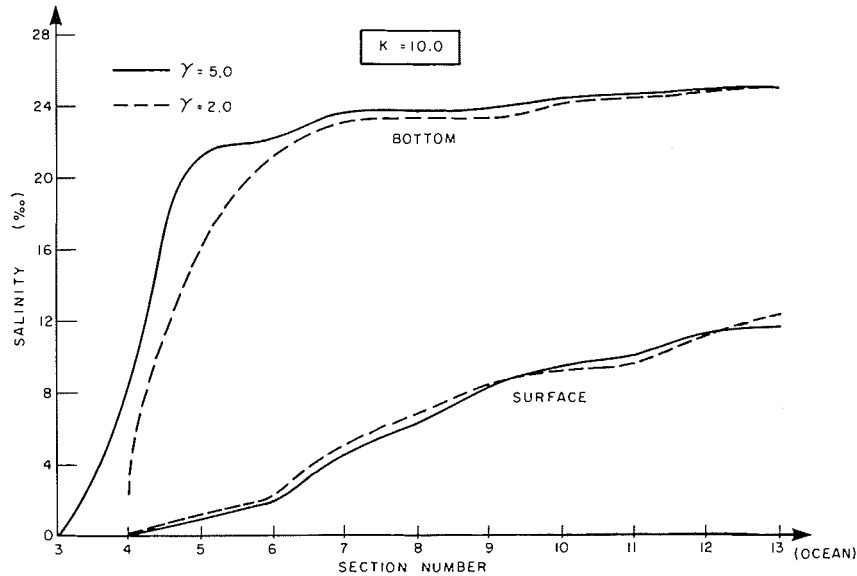


FIG. 5. Influence of γ on longitudinal salinity distribution.

$$[7] \quad \frac{\partial(HS)}{\partial t} + \frac{1}{b} \frac{\partial}{\partial x} (bHuS) + \frac{H}{b} \frac{\partial}{\partial \eta} (bWS) = \frac{Q'''H}{\rho_0 C_p} + \frac{H}{b} \frac{\partial}{\partial x} \left(bK_x \frac{\partial S}{\partial x} \right) + \frac{1}{Hb} \frac{\partial}{\partial \eta} \left(bK_z \frac{\partial S}{\partial \eta} \right)$$

where *CVT* represents the convection terms and *W* is a transformed vertical velocity (Boericke and Hogan 1977). This system of equations resembles closely the equations of the authors mentioned above, except for the inclusion of the convection terms and a correct representation of the density-induced forcing term.

The convective terms in the *x*-momentum equation are generally considered unimportant for estuaries without sudden changes in cross section. Numerical experimentation on the model estuary described here tended to confirm this. Nevertheless, the computer program that was developed included these terms in the numerical procedure.

Shear stress and eddy diffusivities

The shear force on the bottom boundary is calculated from Manning's equation, so that $f/8 = gn^2/d^{1/3}$ where *n* is Manning's coefficient and *d* is the hydraulic depth.

The vertical eddy viscosity is obtained from Prandtl's mixing length hypothesis,

$$\epsilon_z = \frac{l^2}{\phi_m} \left| \frac{\partial u}{\partial z} \right|$$

in which the mixing length is

$$l = \kappa(h - z) \frac{z + \xi}{h + \xi}$$

where $H = \xi + h$ is the total instantaneous depth, *K* is the Von Karman constant, assigned a value of 0.38 in these computations, ϕ_m is an empirically specified function of the Richardson number,

$$Ri = \frac{g}{\rho} \frac{\partial \rho}{\partial z} / (\partial u / \partial z)^2$$

which accounts for the important effect of density stratification on the vertical exchange of heat, mass, and momentum. A negative Richardson number indicates an increase in this exchange due to the potential energy available from buoyancy effects and a positive value results in the ultimate suppression of turbulence.

Many semiempirical formulas for the function ϕ_m have been proposed. At present, the sensitivity of the model to various expressions for ϕ_m is being assessed. The expression used here for stable conditions is $\phi_m = (1 + \gamma Ri)^{1/2}$ for $Ri > 0$ with $K_z = \epsilon_z / (1 + K\gamma Ri)$, where γ is typically of the order of 10. This formulation is a modified adaptation of the expression proposed by Munk and Anderson (1948). For an unstably stratified medium ($Ri < 0$), $\phi_m = 0.5(1.0 - e^{Ri})$ and $K_z = \epsilon_z$, which corresponds approximately to the relation obtained from the data of Charnock (1967).

The longitudinal dispersion coefficient K_x in [7] has been simply represented as $K_x = C_d(x)b\bar{u}$ in which C_d is an empirically adjusted coefficient, and \bar{u} the mean velocity at the cross section.

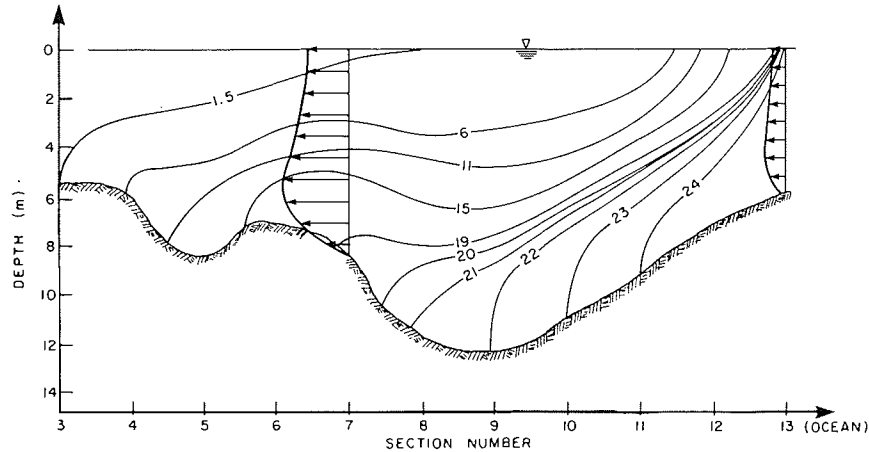


FIG. 6. Isohalines at high tide.

Boundary conditions

The system of equations requires either the tidal elevation or the velocities in each vertical layer to be specified at the upper and lower borders. In the present application the tidal elevations were specified at the upstream and downstream boundaries. The boundary conditions for the salinity were specified as a constant value (zero for this case) upstream, while the downstream salinity was set at a user-specified value (here the ocean value was used) whenever flow entered the model at any horizontal level. For flow leaving the model, the salinity was specified using an interpolated Lagrangian approach. The same approach was used in the computations with the energy equation.

The boundary conditions imposed at the free surface and at the bottom were in the form of first derivatives which are easily incorporated into the spline formulation. For the velocities, the no-slip boundary condition was imposed at the bottom.

Initial conditions

Initial values for the tidal surface elevation, velocity, salinity, and temperature distribution must be specified. The initial values need not be exact because the model approaches the same final periodic steady-state solution regardless of the initial data employed. It is reasonable to expect, however, that a steady-state periodic cycle

will be approached faster if good initial data are employed. For these simulations, a linear vertical velocity distribution was specified and zero salinity was assumed everywhere. The upstream water surface elevation was adjusted by a small bias value to give the correct freshwater inflow. With these initial conditions the present model, when applied to the estuary of the Rivière Grande Baleine (Great Whale River) reached steady-state conditions in about three or four tidal cycles.

Numerical method

The numerical method used to solve the continuity and momentum equations is a time and space staggered scheme coupled with a spline formulation for the evaluation of the first and second derivatives. The vertical direction is treated implicitly to avoid the time-step restrictions that result if an explicit scheme is used.

The tidal elevations are determined at locations that are staggered by half a mesh space and half a time step from the velocity and salinity nodes. At each velocity node the velocity and salinity are calculated at vertical locations evenly spaced from surface to bottom.

Spline formulation

If the symbols $m_{i,j}$, $M_{i,j}$ are used to represent $(\partial u / \partial \eta)_{i,j}$ and $(\partial^2 u / \partial \eta^2)_{i,j}$ respectively, then the longitudinal momentum equation [6] may be written:

$$\begin{aligned}
 [8] \quad \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = & \frac{g}{\Delta x} (\zeta_{i+1/2,j}^{n+1/2} - \zeta_{i-1/2,j}^{n+1/2}) - g \frac{H_i^n}{\rho} \left(\int_{\eta}^1 \frac{\partial \rho}{\partial x} d\eta \right)_{i,j}^n - CB_{i,j} u_{i,j}^{n+1} |u_{i,j}^n| \\
 & + \frac{m^{n+1}}{H_i^2 b_{i,j}} \left\{ \frac{(b\epsilon_z)_{i,j+1}^n - (b\epsilon_z)_{i,j}^n}{\Delta \eta} \right\} + M_{i,j}^{n+1} \left(\frac{\epsilon_z}{H_i^2} \right)_{i,j}^n
 \end{aligned}$$

The convective terms have been omitted (since they are computed separately) and second-order accurate finite

differences have been used to evaluate the space derivatives in the longitudinal direction.

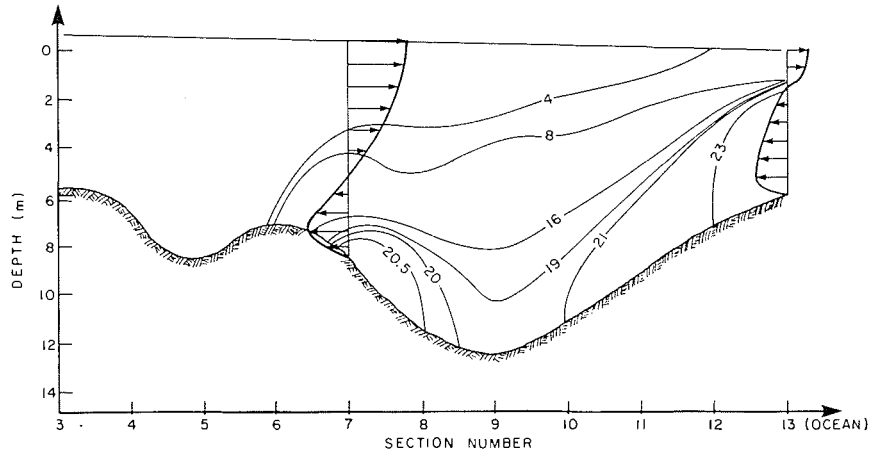


FIG. 7. Isohalines at mean tide.

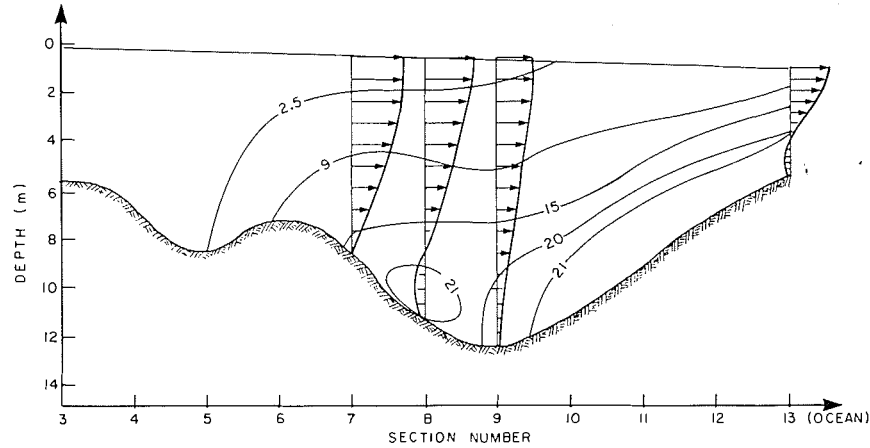


FIG. 8. Isohalines at low tide.

After some rearrangement, [8] may be written in the following form:

$$[8.1] \quad u_{i,j}^{n+1} = F_{i,j} + R_{i,j}m_{i,j}^{n+1} + Q_{i,j}M_{i,j}^{n+1}$$

where $F_{i,j}$, $R_{i,j}$, and $Q_{i,j}$ are known coefficients evaluated at previous time steps.

This equation, when combined with [A.2] in the Appendix, may be written in tridiagonal form as

$$[8.2] \quad a_{i,j}u_{i,j-1}^{n+1} + b_{i,j}u_{i,j}^{n+1} + c_{i,j}u_{i,j+1}^{n+1} = d_{i,j}$$

Equation [8.2] is easily solved for the longitudinal velocities $u_{i,j}^{n+1}$ by use of the Thomas algorithm. The spline relations are now used to compute the first derivatives $m_{i,j}^{n+1}$ which are used in the evaluation of the Richardson number.

The continuity equation [5] was solved by a similar technique for the tidal surface elevations and the vertical velocities.

The salt and heat transport equation was solved using the SADI (spline alternating direction implicit) procedure outlined by Rubin and Graves (1975). In this method the spline implicit procedure was used in both the η and x directions, except for the advective terms which were treated by a second-order upwinding procedure discussed by Roache (1976). This results in the following two-step procedure: the first step is calculation of the salinity at time $n + 1/2$ using an equation of the form,

$$[8.3] \quad A_{i,j}M_{i-1,j}^{n+1/2} + B_{i,j}M_{i,j}^{n+1/2} + C_{i,j}M_{i+1,j}^{n+1/2} = D_{i,j}$$

together with the appropriate spline relation given in the Appendix; the second step advances the salinity to time $n + 1$ using

$$[8.4] \quad a_{i,j}m_{i,j-1}^{n+1} + b_{i,j}m_{i,j}^{n+1} + c_{i,j}m_{i,j+1}^{n+1} = d_{i,j}$$

In [8.3] and [8.4], $m = \partial S / \partial \eta$ and $M = \partial^2 S / \partial x^2$.

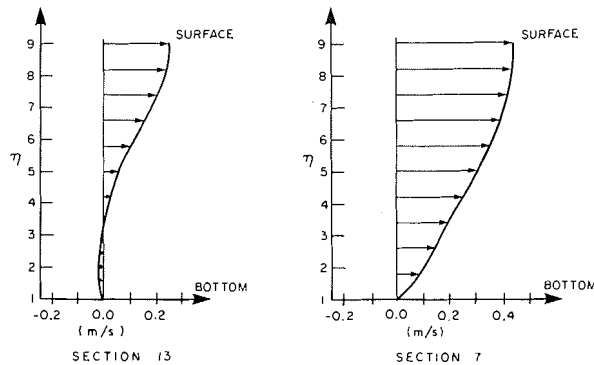


FIG. 9. Computed velocity profiles at two sections (low tide).

Time-step restrictions

Since the longitudinal derivative is treated explicitly in the momentum equation, the allowable time step is governed by the well-known Courant-Friedrichs-Lewy criterion. Hence,

$$\Delta t \leq \frac{\Delta x}{|u| + gH}$$

In the application of the model to the estuary of the Rivière Grande Baleine (Great Whale River), 12 computational reaches were employed with nine vertical stations at each reach. The upstream elevation was adjusted by a small bias value to generate the correct freshwater river discharge of $260 \text{ m}^3/\text{s}$.

Results and discussion

The results of the simulation are presented in Figs. 3-10. Figure 3 is a comparison between the model prediction and tide gauge recordings at a station situated 6 km upstream of the river mouth. For these computations, values for the Manning coefficient suggested from field observations were used.

In order to examine the influence of the stratification parameters γ and K on the salinity distribution, a program of experimentation with different K and γ values was initiated. Figure 4 indicates the influence of K on the longitudinal variation of salinity and Fig. 5 was plotted using two different values for γ and a fixed value for K . As expected, stronger stratification is apparent at the higher value for K of 10. Figures 6-8 are representations of the salinity fields obtained during different stages of the tidal cycle.

Vertical velocity profiles as computed by the model are presented in Figs. 9 and 10. The effect of the longitudinal salinity gradient on the vertical velocity profiles is quite marked, as expected, with a dominant upstream current in the lower layers even at ebb tide. At high tide the flow at the river mouth reverses completely.

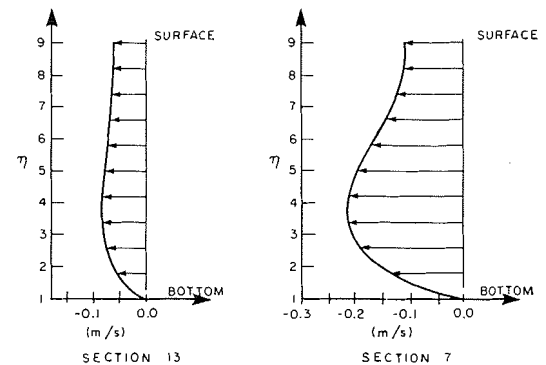


FIG. 10. Computed velocity profiles at two sections (high tide).

Conclusions

The $X-Z$ equations of motion for a partially mixed estuary have been solved using numerical integration techniques based on splines. The results obtained from the numerical model have been encouraging as far as the tidal propagation terms are concerned.

An unfortunate drawback has been the absence of reliable field data or salinity distributions with which to calibrate the model. It is hoped, however, that the specification of the two independent parameters K and γ will permit sufficient flexibility in calibration with field data. It is perhaps not premature to indicate here that, very recently, an improved version of the model was run on an estuary where salinity measurements were available. Preliminary comparisons of salinity distributions have been very encouraging and will be reported in the near future.

Further work is required; this is in progress and is the incorporation of a two-parameter turbulence model into the solution procedure together with a generalized formulation of the spline procedure to treat the shallow-water equations.

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Appendix

In this section, the results of some simple developments are presented.

For example, after an equation of the type [8.1] has been obtained, it may be converted with the aid of the spline formulae into three useful forms containing exclusively either the function values, the first derivatives, or the second derivatives.

Consider the following system of equations:

$$[A.1] \quad u_i^{n+1} = F_i^{n+1} + R_i^{n+1} m_i^{n+1} + Q_i^{n+1} M_i^{n+1}$$

(1) The transformed system containing function values only (and with the time index $n + 1$ omitted) may be written,

$$[A.2] \quad A_i u_{i-1} + B_i u_i + C_i u_{i+1} = D_i; \quad i = 1, \dots, N$$

where

$$A_i = \frac{e_i h_i}{6c_i} - \frac{1}{h_i}$$

$$B_i = \frac{d_i h_i}{6c_i} + \frac{e_{i+1}(h_i + h_{i+1})}{3c_{i+1}} - \frac{R_{i+1}(3h_{i+1}^2 + R_i h_{i+1} - 6Q_i)}{36c_i} + \frac{h_i + h_{i+1}}{h_i h_{i+1}}$$

$$C_i = \frac{d_{i+1}(h_i + h_{i+1})}{3c_{i+1}} - \frac{R_i(2h_{i+1}^2 - 3R_{i+1}h_{i+1}) - 6Q_i(h_{i+1} - R_{i+1})}{36c_{i+1}} - \frac{1}{h_{i+1}}$$

$$D_i = \frac{a_i h_i}{6c_i} + \frac{a_{i+1}(h_i + h_{i+1})}{3c_{i+1}} - \frac{F_{i+1}(2R_i h_{i+1}^2 - 6Q_i h_{i+1}) + F_i R_{i+1} h_{i+1}^2}{36c_{i+1}}$$

$$a_i = \frac{F_i R_{i-1} h_i}{6} + F_{i-1} \left(\frac{R_i h_i}{3} + Q_i \right)$$

$$c_i = \frac{R_i R_{i-1} h_i^2}{36} - \frac{(R_i h_i + 3Q_i)(R_{i-1} h_i - 3Q_{i-1})}{9}$$

$$d_i = R_{i-1} \left(\frac{h_i}{6} - \frac{R_i}{2} - \frac{Q_i}{h_i} \right)$$

$$e_i = R_i \left(\frac{h_i}{3} + \frac{R_{i-1}}{2} \right) + Q_i \left(1 + \frac{R_{i-1}}{h_i} \right)$$

$$h_i = X_i - X_{i-1}$$

(2) The equations containing first derivative values only are

$$A_i m_{i-1} + B_i m_i + C_i m_{i+1} = D_i; \quad i = 1, \dots, N$$

where

$$A_i = \frac{1}{3h_i} - \frac{2Q_i + 4Q_{i-1} - R_{i-1}h_i}{h_i^3 \Delta_i}$$

$$B_i = \frac{2}{3} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) - \frac{2Q_{i+1} + 4Q_i - R_i h_{i+1}}{h_{i+1}^3 \Delta_{i+1}} - \frac{2Q_{i-1} + 4Q_i + R_i h_i}{h_i^3 \Delta_i}$$

$$C_i = \frac{1}{3h_{i+1}} - \frac{2Q_i + 4Q_{i+1} + R_{i+1} h_{i+1}}{h_{i+1}^3 \Delta_{i+1}}$$

$$D_i = \frac{F_{i+1} - F_i}{h_{i+1}^2 \Delta_{i+1}} + \frac{F_i - F_{i-1}}{h_i^2 \Delta_i}$$

$$\Delta_i = 1 + 6 \left(\frac{Q_i + Q_{i-1}}{h_i^2} \right)$$

(3) The system with second derivatives only is

$$[A.3] \quad A_i M_{i-1}^{n+1} + B_i M_{i+1}^{n+1} + C_i M_{i+1}^{n+1} = D_i; \quad i = 1, \dots, N$$

where

$$A_i = \frac{h_i}{6} + \frac{R_i + 2R_{i-1}}{6\Delta_i} - \frac{Q_{i-1}}{h_i \Delta_i}$$

$$B_i = \frac{h_i + h_{i+1}}{3} - \frac{R_{i+1} + 2R_i}{6\Delta_{i+1}} + \frac{2R_i + R_{i-1}}{6\Delta_i} + Q_i \left(\frac{1}{\Delta_{i+1} h_{i+1}} + \frac{1}{\Delta_i h_i} \right)$$

$$C_i = \frac{h_{i+1}}{6} - \frac{2R_{i+1} + R_i}{6\Delta_{i+1}} - \frac{Q_{i+1}}{h_{i+1} \Delta_{i+1}}$$

$$D_i = \frac{F_{i+1} - F_i}{\Delta_{i+1} h_{i+1}} - \frac{F_i - F_{i-1}}{\Delta_i h_i}$$

$$\Delta_i = 1 - \frac{R_i - R_{i-1}}{h_i}$$