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ON THE LEADING TERM OF THE OUTER ASYMPTOTIC EXPANSION OF VAN DER POL'S EQUATION*

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Abstract. It is proved rigorously that the leading term in the outer asymptotic expansion of the time dependent solution describing relaxation oscillations of the Van der Pol equation is correct. This is accomplished by constructing rigorous estimates of the difference between the exact solution and the outer asymptotic solution as constructed by J. D. Cole. These estimates are both rigorous and numerically computable.

1. Introduction. We continue the rigorous verification, begun in [9], of the validity of the leading terms in the asymptotic expansions of periodic solutions of Van der Pol's equation. We write Van der Pol's equation in the form used by Cole [5]:

$$\varepsilon\frac{d^2y}{dt^2}-(1-y^2)\frac{dy}{dt}+y=0. \tag{1a}$$

Specifically, we are interested in proving the validity of asymptotic approximations that describe relaxation oscillations which appear when ε is small. This leads to a singular perturbation problem, as is well known. (For a comprehensive review of the concepts of singular perturbation techniques and notation the reader is referred to Lagerstrom and Casten [7].)

It is well known that if we specify

$$y(0;\varepsilon)=0 \tag{1b}$$

and

$$\frac{dy}{dt}(0;\varepsilon)<0; \tag{1c}$$

then there is a unique nontrivial periodic solution for the system (1a, b, c), whose image in the phase plane is the limit cycle.

The enormous literature on the subject of relaxation oscillations seems to divide itself into three main approaches: the construction of a narrow annulus in the phase plane or Lienard plane (e.g. LaSalle [8], Ponzo and Wax [11]); use of comparison equations (e.g. Cartwright [3], [4], Ponzo and Wax [12]; and asymptotic solutions of the singular perturbation problem (e.g. Dorodnicyn [6], Bavinck and Grasman [1], Carrier [2], Cole [5]). These last two approaches include work of the time domain, as well as in the phase plane and the Lienard plane.

A study of the literature leads one to the conclusion that convincing proofs of the validity of asymptotic expansions do not exist. This is hardly surprising, since even the formal construction of an asymptotic solution carried out for the first few terms requires many involved calculations. Dorodnicyn's work perhaps comes the closest to being accepted as rigorous, but there are steps in his analysis which are missing or are suspect. In any case, as Cartwright [4] says, "his analysis is difficult to follow". Even his formal results, and specifically the method of matching used, have been questioned [1]. Further comments on Dorodnicyn's work appear in our discussion in § 4.

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Recently, MacGillivray [9] has proved the validity of the leading term of the inner asymptotic expansion as constructed by Cole [5] in the (y, t) domain. In particular, it proved there that

(2a) $$y(t^*; \varepsilon) = g_0(t^*) + O(\varepsilon^{1/3})$$

uniformly for

(2b) $$t^* \in [-\varepsilon^{-1/3}/2, \varepsilon^{-2/3}],$$

where t^* is the inner variable, defined by

$$t = \varepsilon t^*,$$

and $g_0(t^*)$ is defined implicitly in [5] by

(2c) $$\frac{1}{3}\log(1-g_0) - \frac{1}{1-g_0} - \frac{1}{3}\log(g_0+2) = t^* - \left(1 + \frac{1}{3}\log 2\right).$$

It is also proved in [9] that

(2d) $$\frac{dy}{dt^*} = \frac{dg_0}{dt^*} + O(\varepsilon^{1/3})$$

uniformly for $t^* \in [0, \varepsilon^{-2/3}]$; these results will be needed in the analysis to follow, when $g_0(t^*)$ will be used to describe $y(t; \varepsilon)$ at and near the zeros of $y(t; \varepsilon)$. Important properties possessed by $g_0(t^*)$ that we shall be needing are

(3a) $$g_0(t^*) \to 1 \quad \text{as } t^* \to -\infty,$$

and

(3b) $$g_0(t^*) \to -2 \quad \text{as } t^* \to +\infty,$$

which, with (2b), implies $y(t; \varepsilon) \sim 1$ when $t = \varepsilon t^*$ is in the set $-\varepsilon^{2/3}/2 \ll \eta(\varepsilon) \ll -\varepsilon$, and $y(t; \varepsilon) \sim -2$ when t is in the set $\varepsilon \ll \eta(\varepsilon) \ll \varepsilon^{1/3}$. Similar statements can be made for the vicinity of the first zero of $y(t; \varepsilon)$ to the left of zero, that is, at $-T_\varepsilon/2$, where T_ε denotes the period of the oscillation. Thus, $y(t; \varepsilon) \sim +2$ when t is the set $-(T_\varepsilon/2) + \varepsilon \ll \eta(\varepsilon) \ll -(T_\varepsilon/2) + \varepsilon^{1/3}$. (For notation conventions the reader is referred to [7, Definitions 3.13, 3.14].)

In this paper we continue our study by proving the validity of the leading term of Cole's outer asymptotic expansion. Since the proof automatically provides a domain of validity, the possibility of verification of an overlap domain (basic to matching) presents itself. It turns out that the domain of validity of the outer term indeed overlaps with that previously found for the inner. This gives one of the very few instances where overlap is actually proved for a nontrivial problem.

The situation is schematically summarized in Fig. 7. In fact, much of the story can be told in pictures, and consequently several sketches have been included in the text. It is hoped the reader will find these clarify the ideas behind the propositions, whose proofs, though simple, tend at times to be tedious.

One final remark. Each order estimate is constructed in a manner which lends itself to numerical computation. That is, the ubiquitous phrase "for all sufficiently small ε" can be made numerically precise.

2. Analysis of period. From Cole [5], the leading term in the formal outer asymptotic expansion is given implicitly by

$$\log u_0 - \frac{u_0^2}{2} = t - \frac{1}{2}, \qquad t \leqq 0 \tag{4}$$

and is expected to be valid on most of the open interval $(-(T_\varepsilon/2), 0)$, where T_ε is the period.

Recall the problem we are concerned with is

$$\varepsilon \frac{d^2y}{dt^2} - (1-y^2) + y = 0, \tag{5a}$$

$$y(0;\varepsilon) = 0, \tag{5b}$$

$$\frac{dy}{dt}(0;\varepsilon) < 0, \tag{5c}$$

$$y(t;\varepsilon) \text{ periodic.} \tag{5d}$$

The existence of a unique solution has long since been established. We define $h(t;\varepsilon)$ for $t \leqq 0$ by

$$y(t;\varepsilon) = u_0(t) + h(t;\varepsilon). \tag{6}$$

Substituting into (5a) yields a differential equation for h:

$$\varepsilon \frac{d^2h}{dt^2} - (1-y^2)\frac{dh}{dt} + h + \varepsilon \frac{d^2u_0}{dt^2} + (2u_0h + h^2)\frac{du_0}{dt} = 0. \tag{7}$$

Since

$$\frac{du_0}{dt} = \frac{u_0}{u_0^2 - 1} < 0 \tag{8}$$

and

$$\frac{d^2u_0}{dt^2} = -\frac{u_0(1+u_0^2)}{(u_0^2-1)^3} < 0, \tag{9}$$

Equation (7) can be rewritten and the original problem (5) replaced by

$$\varepsilon \frac{d^2h}{dt^2} = (1-y^2)\frac{dh}{dt} + \frac{1+u_0y}{u_0^2-1}h + \varepsilon \frac{u_0(1+u_0^2)}{(u_0^2-1)^3}, \tag{10a}$$

$$h(0;\varepsilon) = -1, \tag{10b}$$

$$\lim_{t\uparrow 0}\left(\frac{dh}{dt} + \frac{u_0}{1-u_0^2}\right) > 0, \tag{10c}$$

$$h(t;\varepsilon) + u_0(t) \text{ periodic for } t \leqq 0. \tag{10d}$$

The formal analysis of Cole and others yields the leading term in the expansion of the period,

$$T_\varepsilon = 3 - 2\log 2 + o(1). \tag{11}$$

Our first two propositions will show the right side of (11) is a lower bound for T_ε. This will be followed by a sequence of propositions which show the right side of

(11) is also an upper bound for T_ε, thereby establishing (11) rigorously. Finally, it will be shown in Proposition 7 that $u_0(t)$ is a uniformly valid approximation of $y(t;\varepsilon)$ on a domain which overlaps the domain of validity of the leading term of the inner asymptotic expansion.

PROPOSITION 1. *$h(t;\varepsilon)$ is negative for all $t \leqq 0$.*

Proof. Please refer to Fig. 1. Noting that $u_0(t) \geqq 1$ for all $t \leqq 0$, and that $y(t;\varepsilon)$ has zeros at $-nT_\varepsilon/2$, $n = 0, 1, 2, \cdots$, it follows that

$$h\left(-\frac{nT_\varepsilon}{2};\varepsilon\right) < 0. \tag{12}$$

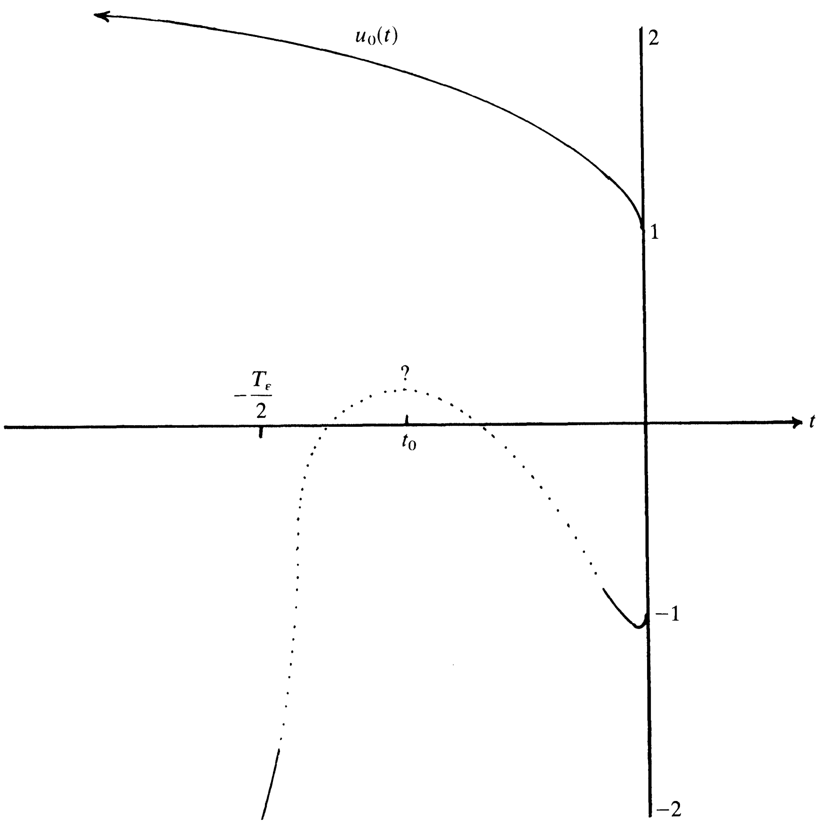

FIG. 1. *Dotted curve shows tentative $h(t)$ becoming nonnegative. Periodicity requires $h(t)$ to be negative at each half period, so $h(t)$ would have a nonnegative maximum. But (10a) gives $d^2h/dt^2 > 0$ there, a contradiction. Consequently, $h(t) < 0$ for all $t \leqq 0$. The sketch of $u_0(t)$ is schematic only.*

Consequently, if $h(t;\varepsilon)$ is ever nonnegative for $t \leqq 0$, it must possess a nonnegative local maximum, at t_0, say. But there

$$\left.\frac{dh}{dt}\right|_{t_0} = 0, \tag{13a}$$

and

$$h(t_0;\varepsilon) \geqq 0. \tag{13b}$$

But now (10a) tells us

$$\varepsilon \frac{d^2h}{dt^2}\bigg|_{t_0} > 0, \tag{13c}$$

a contradiction. This completes the proof of Proposition 1.

PROPOSITION 2.

$$T_\varepsilon \geqq 3 - 2\log 2 + o(1) \quad as\ \varepsilon \downarrow 0. \tag{14}$$

Proof. Please refer to Fig. 2. We assume tentatively there is an $\alpha > 0$ and arbitrarily small values of ε for which

$$T_\varepsilon < 3 - 2\log 2 - \alpha. \tag{15}$$

(For the remainder of the proof, ε will be restricted to the set for which (15) is satisfied.) We focus our discussion on the interval $[-3/2 + \log 2, 0]$. Fig. 2 displays the following:

$$-\frac{T_\varepsilon}{2} > -\frac{3}{2} + \log 2 + \frac{\alpha}{2}, \tag{16a}$$

$$y\left(-\frac{T_\varepsilon}{2}; \varepsilon\right) = 0, \tag{16b}$$

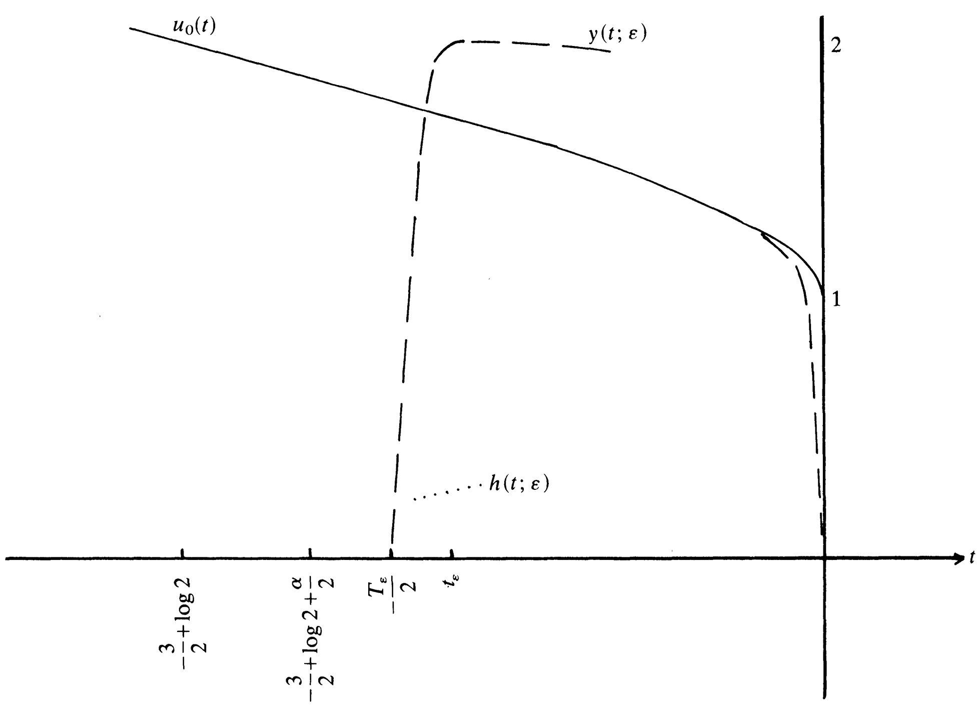

FIG. 2. *(See Proposition* 2.) *At* $t_\varepsilon > -T_\varepsilon/2 > -3/2 + \log 2 + \alpha/2$, $y(t_\varepsilon) \sim 2$, $u_0(t_\varepsilon) < 2$, $h(t_\varepsilon) = y(t_\varepsilon) - u_0(t_\varepsilon) \approx 2 - u_0(t_\varepsilon) > 0$, *contradicting Proposition* 1. *Dotted curve is* $h(t;\varepsilon)$ *in the vicinity of* t_ε.

(16c) $$y(0;\varepsilon)=0,$$

(16d) $$y(t;\varepsilon)>0, \qquad -\frac{T_\varepsilon}{2}<t<0.$$

The analysis in [9] of the leading term of the inner asymptotic expansion proved that there is an interval of t-values within $O(\varepsilon^{1/3})$ to the right of $-T_\varepsilon/2$ where $y(t;\varepsilon)=2+O(\varepsilon^{1/3})$. More precisely, choose t_ε as follows:

(17a) $$t_\varepsilon=-\frac{T_\varepsilon}{2}+e_1(\varepsilon)$$

with

(17b) $$\varepsilon \ll e_1(\varepsilon) \ll \varepsilon^{1/3}.$$

Of course, since $e_1(\varepsilon)>0$, we have

(17c) $$t_\varepsilon>-\frac{3}{2}+\log 2+\frac{\alpha}{2}.$$

Then, using the monotone property of $u_0(t)$, and $u_0(-3/2+\log 2)=2$, we have

(18) $$\begin{aligned} h(t_\varepsilon;\varepsilon)&=y(t_\varepsilon;\varepsilon)-u_0(t_\varepsilon)=[2+O(\varepsilon^{1/3})]-u_0(t_\varepsilon)\\ &>[2+O(\varepsilon^{1/3})]-u_0\left(-\frac{3}{2}+\log 2+\frac{\alpha}{2}\right)\\ &>0 \quad \text{if } \varepsilon \text{ is sufficiently small.}\end{aligned}$$

This contradicts Proposition 1, and the proof of Proposition 2 is complete.

COROLLARY.

(14a) $$T_\varepsilon \geqq 3-2\log 2+O(\varepsilon^{1/3}).$$

Proof. Replace α in the above proof by $K\varepsilon^{1/3}$ with K a *large* constant.

Remark. In what follows, there is considerable simplification achieved if the estimates found are expressed as $o(1)$ instead of $O(\varepsilon^{1/3})$, etc. The more precise estimates can be found using the same lines of reasoning, but at the expense of adding details. We shall therefore pursue the weaker estimates, and leave the calculation of sharper estimates to the interested reader.

PROPOSITION 3. *Assume there is an $\alpha>0$ and arbitrarily small ε-values for which*

(19) $$T_\varepsilon>3-2\log 2+\alpha.$$

Then if ε is sufficiently small, there exists

$$t_1\in\left[-\frac{T_\varepsilon}{2},-\frac{T_\varepsilon}{2}+\frac{3\alpha}{8}\right]\subset\left[-\frac{T_\varepsilon}{2},-\frac{3}{2}+\log 2-\frac{\alpha}{8}\right],$$

such that

(19a) $$h(t_1;\varepsilon)<-\left[u_0\left(-\frac{T_\varepsilon}{2}\right)-2\right]+\frac{\beta}{2},$$

and

(19b) $$\left.\frac{dh}{dt}\right|_{t_1}\leqq 1,$$

where

(19c)
$$\beta=\min\left\{u_0\left(-\frac{3}{2}+\log 2-\frac{\alpha}{4}\right)-2;\frac{1}{4}\right\}.$$

Remark. As in the previous proposition, we always assume ε is taken from the set of ε-values for which $T_\varepsilon>3-2\log 2+\alpha$.

Proof. Please refer to Fig. 3.

Defining t_ε as specified in (17a, b), we obtain

(20)
$$y(t_\varepsilon;\varepsilon)=2+o(1).$$

Now choose ε small enough to ensure

(21)
$$t_\varepsilon\leqq-\frac{T_\varepsilon}{2}+\frac{\alpha}{8}\leqq-\frac{3}{2}+\log 2-\frac{3\alpha}{8}.$$

Then

(22)
$$h(t_\varepsilon;\varepsilon)=y(t_\varepsilon;\varepsilon)-u_0(t_\varepsilon)=[2+o(1)]-u_0(t_\varepsilon).$$

If $dh/dt|_{t_\varepsilon}\leqq 1$, then the observation that in any case

(22a)
$$h(t_\varepsilon;\varepsilon)=[2+o(1)]-\left[u_0\left(-\frac{T_\varepsilon}{2}\right)+o(1)\right]<-\left[u_0\left(-\frac{T_\varepsilon}{2}\right)-2\right]+\frac{\beta}{4}$$

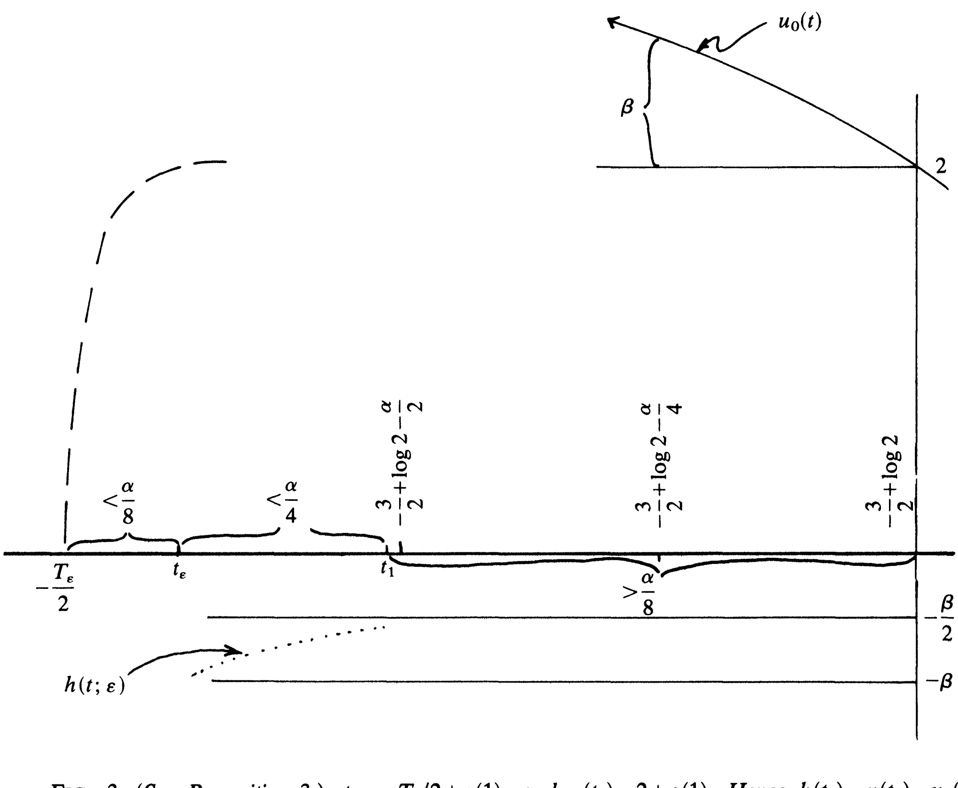

FIG. 3. (*See Proposition* 3.) $t_\varepsilon=-T_\varepsilon/2+o(1)$, *and* $y(t_\varepsilon)=2+o(1)$. *Hence* $h(t_\varepsilon)=y(t_\varepsilon)-u_0(t_\varepsilon)\approx 2-u_0(t_\varepsilon)<2-u_0(-3/2+\log 2-\alpha/4)=-\beta$. *$dh/dt$ decreases to unity at t_1, with $h(t)$ still as negative as $-\beta/2$.*

if ε is sufficiently small certainly enables us to satisfy (19a, b) simply by setting $t_1=t_\varepsilon$.

Now assume

$$\left.\frac{dh}{dt}\right|_{t_\varepsilon}>1.$$

We first compare the last two terms on the right side of (10a) on the interval $[t_\varepsilon, -3/2+\log 2]$. Their ratio $R(t;\varepsilon)$ satisfies

$$|R(t;\varepsilon)|=\frac{u_0(1+u_0^2)}{(u_0^2-1)^2}\frac{\varepsilon}{|h|}. \tag{23}$$

The maximum value of the expression $u_0(1+u_0^2)/(u_0^2-1)^2$ occurs at the right endpoint of the interval, i.e. at $u_0=2$. Hence

$$|R(t;\varepsilon)|\leqq\frac{10}{9}\frac{\varepsilon}{|h(t;\varepsilon)|}\quad\text{for } t\in\left[t_\varepsilon, -\frac{3}{2}+\log 2\right]. \tag{24}$$

Now, (22a), (19), (19c), and (9) imply

$$h(t_\varepsilon)<-\beta+\frac{\beta}{4}=-\frac{3\beta}{4}. \tag{25}$$

When (25) is substituted into (24), we find, at t_ε,

$$|R(t_\varepsilon;\varepsilon)|<\frac{40}{27}\frac{\varepsilon}{\beta}. \tag{26}$$

Thus

$$|R(t_\varepsilon;\varepsilon)|\leqq\tfrac{1}{2} \tag{27}$$

if

$$\varepsilon\leqq\tfrac{9}{80}\beta. \tag{27a}$$

With this restriction on ε, (27) will remain valid to the right of t_ε on the interval $[t_\varepsilon, -3/2+\log 2]$ as long as $h(t;\varepsilon)\leqq-\beta/4$. This in turn implies

$$\varepsilon\frac{d^2h}{dt^2}\leqq(1-y^2)\frac{dh}{dt}+\frac{1}{2}\frac{1+u_0y}{u_0^2-1}h. \tag{28}$$

This is further simplified by choosing ε small enough to make (see (20))

$$1-[y(t_\varepsilon;\varepsilon)]^2<-2. \tag{29}$$

Then, as long as $dh/dt\geqq 0$, and $t\in[t_\varepsilon, t_\varepsilon+\alpha/4]$,

$$\begin{aligned} y(t;\varepsilon)&=y(t;\varepsilon)-y(t_\varepsilon;\varepsilon)+y(t_\varepsilon;\varepsilon)\\ &>u_0(t)-u_0(t_\varepsilon)+\sqrt{3}>-\beta+\sqrt{3}\geqq-\frac{1}{4}+\sqrt{3}\end{aligned} \tag{29a}$$

where we have used our assumption that $\beta\leqq 1/4$.

Consequently,

$$1-[y(t;\varepsilon)]^2<1-[\sqrt{3}-1/4]^2<-1 \tag{29b}$$

as long as $dh/dt\geqq 0$ and $t\in[t_\varepsilon, t_\varepsilon+\alpha/4]$.

Combining (29b) with (28) gives

$$\varepsilon \frac{d^2h}{dt^2} < \frac{dh}{dt} \tag{30}$$

as long as $h(t) \leqq -\beta/4$ and $dh/dt \geqq 0$ on the interval $[t_\varepsilon, t_\varepsilon + \alpha/4]$.

An integration yields immediately

$$\frac{dh}{dt} \leqq \left.\frac{dh}{dt}\right|_{t_\varepsilon} e^{-(t-t_\varepsilon)/\varepsilon}. \tag{31}$$

From the inner expansion analysis [9] we know

$$\left.\frac{dh}{dt}\right|_{t_\varepsilon} = O(\varepsilon^{-2/3}) = o(\varepsilon^{-1}), \tag{32}$$

so ε can be chosen to make

$$\left.\frac{dh}{dt}\right|_{t_\varepsilon} \leqq \frac{\beta}{4\varepsilon},$$

and so

$$\frac{dh}{dt} \leqq \frac{\beta}{4\varepsilon} e^{-(t-t_\varepsilon)/\varepsilon}. \tag{33}$$

Another integration yields

$$h(t;\varepsilon) \leqq h(t_\varepsilon;\varepsilon) + \frac{\beta}{4}(1 - e^{-(t-t_\varepsilon)/\varepsilon}) < h(t_\varepsilon;\varepsilon) + \frac{\beta}{4}$$

$$< -\left[u_0\left(-\frac{T_\varepsilon}{2}\right) - 2\right] + \frac{\beta}{2} \quad \text{(using (22a))} \tag{34a}$$

$$< -\frac{\beta}{2}, \tag{34b}$$

so long as $h(t;\varepsilon) \leqq -\beta/4$ and $dh/dt \geqq 0$ on $[t_\varepsilon, t_\varepsilon + \alpha/4]$.

Noting $-\beta/2 < -\beta/4$, we see that as long as $dh/dt \geqq 1$ on $[t_\varepsilon, t_\varepsilon + \alpha/4]$, $h(t;\varepsilon)$ will remain bounded below by (34a) and (34b). We exploit this observation by defining t_1' such that

$$\frac{1}{4}\frac{\beta}{\varepsilon} e^{-(t_1'-t_\varepsilon)/\varepsilon} = 1. \tag{35}$$

That is,

$$t_1' = t_\varepsilon + \varepsilon \log \frac{\beta}{4\varepsilon} \leqq t_\varepsilon + \frac{\alpha}{4}$$

if ε is sufficiently small. Combining this with (33), we conclude dh/dt drops to unity by the time t_1' is reached. We denote by t_1 the first such time. That is,

$$dh/dt|_{t_1} = 1 \quad \text{and} \quad t_1 \in \left[t_\varepsilon, t_\varepsilon + \frac{\alpha}{4}\right].$$

With (34b) this completes the proof of Proposition 3.

Remark. We note that

$$t_1 < t_\varepsilon + \frac{\alpha}{4} < -\frac{3}{2} + \log 2 - \frac{3\alpha}{8} + \frac{\alpha}{4} = -\frac{3}{2} + \log 2 - \frac{\alpha}{8}.$$

Our next result shows that dh/dt reaches zero before $-3/2+\log 2$, and does so with h still at least as negative as $-\beta/4$.

PROPOSITION 4. *Assume the existence of an α and an ε-set as described in Proposition 3. For ε sufficiently small, there exists $t_2 \in [-T_\varepsilon/2, -3/2+\log 2]$ such that $dh/dt|_{t_2} \leqq 0$ and $h(t_2; \varepsilon) < -\beta/4$.*

Proof. Please refer to Fig. 4. Choose ε sufficiently small to give us t_1 as described in Proposition 3. Recall

$$\left.\frac{dh}{dt}\right|_{t_1} \leqq 1, \tag{36}$$

$$h(t_1; \varepsilon) < -\left[u_0\left(-\frac{T_\varepsilon}{2}\right) - 2\right] + \frac{\beta}{2}, \tag{37a}$$

which in turn implies

$$h(t_1; \varepsilon) < -\frac{[u_0(-T_\varepsilon/2) - 2]}{2} \tag{37b}$$

and

$$h(t_1; \varepsilon) < -\frac{\beta}{2}. \tag{37c}$$

Of course, if $dh/dt|_{t_1} \leqq 0$, we simply set $t_2 = t_1$. Assume then that $dh/dt|_{t_1} > 0$, and consider the interval $[t_1, t_1+\beta/4] \subset [t_\varepsilon, t_\varepsilon + 3\alpha/8] \subset [-T_\varepsilon/2, -3/2+\log 2]$. (This inclusion uses the fact that $\beta < \alpha/4$, which in turn follows from $du_0/dt \geqq -2/3$ for $t \leqq -3/2+\log 2$.)

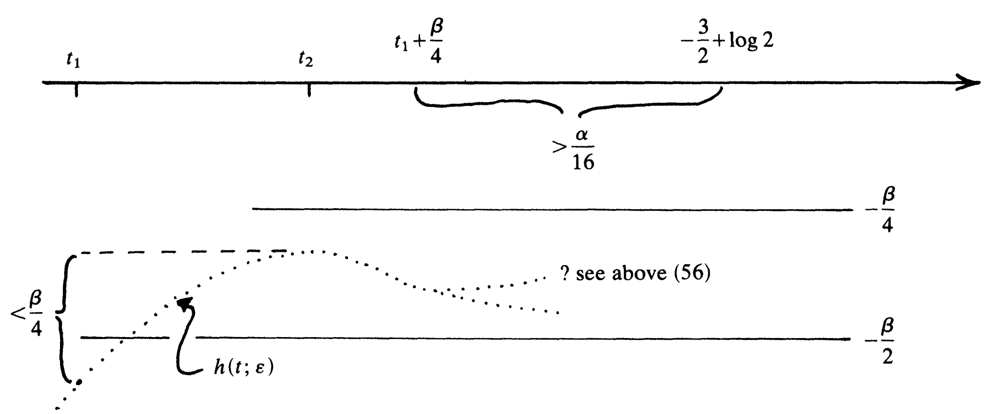

FIG. 4. *$h(t_1) < -\beta/2$ and $dh/dt|_{t_1} \leqq 1$. Curvature of h to the right of t_1 remains sufficiently negative to force graph of $h(t; \varepsilon)$ to turn around (at t_2) within an interval of length $\beta/4$ of t_1, and in so doing remains more negative than $-\beta/4$.*

As long as $dh/dt \geqq 0$ to the right of t_1 on the interval $[t_1, t_1+\beta/4]$, we have

(38a) $$y(t;\varepsilon)=y(t;\varepsilon)-y(t_\varepsilon;\varepsilon)+y(t_\varepsilon;\varepsilon)>u_0(t)-u_0(t_\varepsilon)+\sqrt{3}>-\frac{3\beta}{2}+\sqrt{3}\geqq-\frac{3}{8}+\sqrt{3}.$$

(The $\sqrt{3}$ estimate comes from (29).)

This in turn implies

(38b) $$1-[y(t;\varepsilon)]^2<0$$

as long as $dh/dt \geqq 0$ on the interval $[t_1, t_1+\beta/4]$. Under these same circumstances, we of course have from (28)

(39) $$\frac{d^2h}{dt^2}\leqq\frac{1}{2\varepsilon}\frac{1+u_0y}{u_0^2-1}h,$$

as long as $h\leqq-\beta/4$. From (37c), (39) is certainly valid at t_1, and it remains valid to the right of t_1 on the interval $[t_1, t_1+\beta/4]$ as long as $dh/dt\geqq 0$ and $h(t;\varepsilon)\leqq-\beta/4$.

Our next step is to find a lower bound for

$$\frac{1+u_0y}{u_0^2-1}|h| \quad \text{on} \left[t_1, t_1+\frac{\beta}{4}\right].$$

From (38b) we know $y(t;\varepsilon)>1$, so

$$\frac{1+u_0y}{u_0^2-1}|h|>\frac{1}{u_0-1}|h|=\frac{|h|}{(u_0-2)+1}.$$

From (39), we know dh/dt remains less than unity to the right of t_1, so that

$$\begin{aligned} h(t;\varepsilon)&<h(t_1;\varepsilon)+\frac{\beta}{4}\\ &<-\left[u_0\left(-\frac{T_\varepsilon}{2}\right)-2\right]+\frac{\beta}{2}+\frac{\beta}{4} \qquad \text{(from (37a))}\\ &<-\left[u_0\left(-\frac{T_\varepsilon}{2}\right)-2\right]+\frac{3\beta}{4}\end{aligned}$$

(40) $$<-\frac{1}{4}\left[u_0\left(-\frac{T_\varepsilon}{2}\right)-2\right] \quad \text{on} \left[t_1, t_1+\frac{\beta}{4}\right].$$

Then

(41) $$\frac{1+u_0y}{u_0^2-1}|h|>\frac{[u_0(-T_\varepsilon/2)-2]/4}{[u_0(-T_\varepsilon/2)-2]+1}\geqq\frac{1}{4}\frac{\beta}{\beta+1}$$

(since $u_0(-T_\varepsilon/2)-2\in[\beta,\infty)$).

Noting that (40) implies $h(t;\varepsilon)<-\beta/4$ on $[t_1, t_1+\beta/4]$, we conclude from (39) and (41) that

(42) $$\frac{d^2h}{dt^2}\leqq-\frac{1}{8\varepsilon}\frac{\beta}{\beta+1}$$

on $[t_1, t_1+\beta/4]$ as long as $dh/dt\geqq 0$, and an integration immediately yields

(43) $$\frac{dh}{dt}\leqq 1-\frac{1}{8\varepsilon}\frac{\beta}{\beta+1}(t-t_1).$$

Finally, choose ε small enough to make

$$1-\frac{1}{8\varepsilon}\frac{\beta}{\beta+1}\cdot\frac{\beta}{4}\leqq 0.$$

From (43) it is now obvious there is a first $t_2\in[t_1, t_1+\beta/4]$ such that

$$\left.\frac{dh}{dt}\right|_{t_2}=0, \tag{44}$$

and, reminding ourselves that (40) implies $h(t;\varepsilon)<-\beta/4$ on that same interval, our proof of Proposition 4 is finished.

Our final effort is to show $h(t;\varepsilon)$ remains less than $-\beta/4$ for t-values close to zero, in fact close enough to reach t-values where (from the inner expansion analysis) $y(t;\varepsilon)=1+o(1)$. For such t, $u_0(t)=1+o(1)$, so that $h(t;\varepsilon)=o(1)$, contradicting $h(t;\varepsilon)<-\beta/4$. To do this we need to see how close to $t=0$ the ratio of the last two terms in (10a) remains less than unity in magnitude. (We already know this ratio is bounded by $1/2$ to the left of $-3/2+\log 2$ if h is bounded above by $-\beta/4$, and if ε is sufficiently small; see (26), (27), (27a). The difficulty to the right of $-3/2+\log 2$ is of course due to the factor $(u_0^2-1)^2$ in the denominator of (23), since $u_0(t)\downarrow 1$ as $t\uparrow 0$. We proceed as follows:

$$\begin{aligned}|R(t;\varepsilon)|&=\frac{u_0(1+u_0^2)}{(u_0^2-1)^2}\frac{\varepsilon}{|h|}\leqq\frac{2(1+4)}{(u_0-1)^2(u_0+1)^2}\cdot\frac{\varepsilon}{|h|}\\&\leqq\frac{10}{4}\cdot\frac{1}{(u_0-1)^2}\cdot\frac{\varepsilon}{|h|},\end{aligned}\tag{45}$$

for $t\in[-3/2+\log 2, 0)$ where $u_0(t)\in[2, 1)$.

We refine this by estimating $(u_0-1)^2$.

From (4),

$$t=\frac{1-u_0^2}{2}+\log(1+(u_0-1)). \tag{46}$$

It is easily verified that $\log(1+\xi)$ and $f(\xi)=\xi-\frac{1}{2}\xi^2$ agree at $\xi=0$ and that

$$\log(1+\xi)>\xi-\tfrac{1}{2}\xi^2\quad\text{for } 0<\xi\leqq 1. \tag{47}$$

Substituting (47) into (46) yields

$$t>-\frac{(u_0-1)(u_0+1)}{2}+(u_0-1)-\frac{1}{2}(u_0-1)^2 \tag{48}$$

for $-3/2+\log 2\leqq t<0$. The right-hand member of this last inequality simplifies to $-(u_0-1)^2$, so we have

$$t>-(u_0-1)^2, \tag{49}$$

or

$$(u_0(t)-1)^2>-t$$

or

$$\frac{1}{(u_0(t)-1)^2}<\frac{1}{|t|}. \tag{50}$$

Substitution into (45) yields

$$|R(t;\varepsilon)| < \frac{5}{2}\frac{\varepsilon}{|t|\cdot|h|}. \tag{51}$$

The right member of (51) is bounded by unity if

$$|t| \geqq \frac{5}{2}\frac{\varepsilon}{|h|}. \tag{52}$$

Thus, if

$$h \leqq -\frac{\beta}{4} \tag{53}$$

holds, we have

$$|R(t;\varepsilon)| < 1 \quad \text{if } |t| \geqq \frac{10\varepsilon}{\beta}. \tag{54}$$

Summarizing, we see that as long as $h(t;\varepsilon) \leqq -\beta/4$ on the interval $[t_2, -10\varepsilon/\beta]$, then $|R(t;\varepsilon)| < 1$, and we have the strict inequality

$$\varepsilon\frac{d^2h}{dt^2} < (1-y^2)\frac{dh}{dt}. \tag{55}$$

(At this point the reader may wish to refer to Fig. 5.) Now, because $h(t_2;\varepsilon) < -\beta/4$, and $dh/dt|_{t_2} \leqq 0$, (55) tells us that dh/dt is negative on some interval to the right of t_2. If dh/dt ever again reaches zero, there is a first time, t_0 say, and there $h(t_0;\varepsilon) < -\beta/4$. If $t_0 < -10\varepsilon/\beta$, then (55) yields $d^2h/dt^2|_{t_0} < 0$, a contradiction. Consequently dh/dt

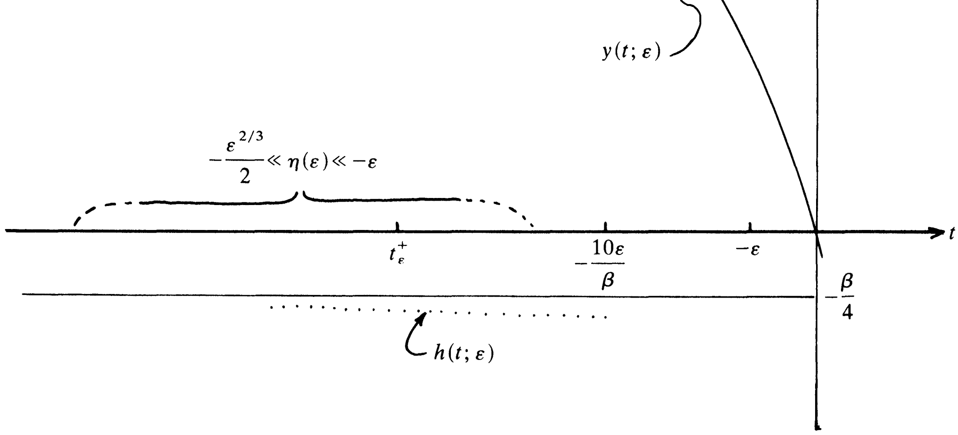

FIG. 5. *For t_ε^+ chosen from the interval $-\varepsilon^{2/3}/2 \ll \eta(\varepsilon) \ll -\varepsilon$, $y(t_\varepsilon^+;\varepsilon)1+O(\varepsilon^{1/3})$, from the inner solution analysis. Since $h(t_\varepsilon^+;\varepsilon) < -\beta/4$ whereas $u_0(t_\varepsilon^+) = 1+o(1)$, we have instead $y(t_\varepsilon^+;\varepsilon) < 1-\beta/4+o(1)$, a contradiction, and the possibility of an α as tentatively assumed in Proposition 3 is rejected.*

remains negative, and $h(t;\varepsilon)$ remains strictly less than $-\beta/4$ on the interval $(t_2, -10\varepsilon/\beta]$:

$$h(t;\varepsilon)<-\beta/4, \qquad t\in\left(t_2, -\frac{10\varepsilon}{\beta}\right]. \tag{56}$$

We now use one more result from the inner expansion analysis: there exists t_ε^+, $-\varepsilon^{2/3}/2<t_\varepsilon^+\ll-\varepsilon$ (for example $t_\varepsilon^+=\varepsilon\log\varepsilon$ will work if ε is small) such that

$$y(t_\varepsilon^+;\varepsilon)=1+o(1). \tag{57}$$

But $t_\varepsilon^+<-10\varepsilon/\beta$ is ε is sufficiently small. Thus (56) implies

$$y(t_\varepsilon^+;\varepsilon)=u_0(t_\varepsilon^+)+h(t_\varepsilon^+;\varepsilon)<[1+o(1)]-\frac{\beta}{4}, \tag{58}$$

contradicting (57). The chain of arguments which leads to this contradiction was started by the assumption (in Proposition 3) that there were arbitrarily small ε-values for which $T_\varepsilon>3-2\log 2+\alpha$. The contradiction implies no such α exists, and we have proved:

PROPOSITION 5.

$$T_\varepsilon\leqq 3-2\log 2+o(1). \tag{59}$$

Combining Propositions 2 and 5, we have proved the well known formula in:

PROPOSITION 6.

$$T_\varepsilon=3-2\log 2+o(1). \tag{60}$$

Remark. The reader may have noticed that the $o(1)$ estimates could, with the help of the inner expansion analysis, be made into $O(\varepsilon^{1/3})$ estimates. It will also be noted that each time we have used the phrase "ε sufficiently small", an algorithm for actually calculating an ε is displayed. Thus our estimates are in fact all computable, albeit with a certain amount of effort.

In the next section we turn our attention to our main goal which is to show that the leading term in Cole's outer asymptotic expansion, $u_0(t)$, is an $O(1)$ approximation to the exact solution $y(t;\varepsilon)$ on some domain of validity. We shall in fact establish an overlap domain on which leading terms of the inner and outer expansions are both uniformly valid. Fortunately, our work in Propositions 3 and 4 makes this final task relatively easy.

3. Validity of the outer approximation.

PROPOSITION 7.

(a)

$$y(t;\varepsilon)=u_0(t)+o(1) \tag{61a}$$

uniformly on the domain

$$-\frac{T_\varepsilon}{2}+\varepsilon\ll\eta(\varepsilon)\ll-\varepsilon. \tag{61b}$$

(b) *The domain of validity of $u_0(t)$ as an $O(1)$ approximation for $y(t;\varepsilon)$ overlaps the domain of validity of $g_0(t/\varepsilon)$ the leading term of Cole's inner asymptotic expansion,* *on the intervals*

(62a) $$-\frac{T_\varepsilon}{2}+\varepsilon \ll \eta(\varepsilon) \ll -\frac{T_\varepsilon}{2}+\varepsilon^{1/3}$$

and

(62b) $$-\frac{\varepsilon^{2/3}}{2} \ll \eta(\varepsilon) \ll -\varepsilon.$$

Proof. Please refer to Fig. 6.

(a) Let $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$ satisfy

$$-\frac{T_\varepsilon}{2}+\varepsilon \ll \eta_1(\varepsilon) \ll -\frac{T_\varepsilon}{2}+\varepsilon^{1/3}, \qquad -\frac{\varepsilon^{2/3}}{2} \ll \eta_2(\varepsilon) \ll -\varepsilon,$$

and consider $u_0(t)$ on the closed interval $[\eta_1(\varepsilon), \eta_1(\varepsilon)]$. Let $-\beta/2$ denote a fixed negative number. As mentioned in our previous discussions, we know from our inner solution analysis that $y(t;\varepsilon)=2+o(1)$ uniformly on the interval $-T_\varepsilon/2+\varepsilon \ll \eta(\varepsilon) \ll -T_\varepsilon/2+\varepsilon^{1/3}$. Thus

(63) $$\begin{aligned} h(t;\varepsilon) = y(t;\varepsilon)-u_0(t) &= 2+o(1)-u_0(-\tfrac{3}{2}+\log 2+o(1)) \\ &= 2+o(1)-u_0(-\tfrac{3}{2}+\log 2)+o(1) \\ &= o(1) \end{aligned}$$

uniformly for the same interval.

Now choose ε small enough so that $h(t;\varepsilon)>-\beta/4$ there. Then, in particular,

(64) $$h(\eta_1(\varepsilon);\varepsilon) > -\frac{\beta}{4}.$$

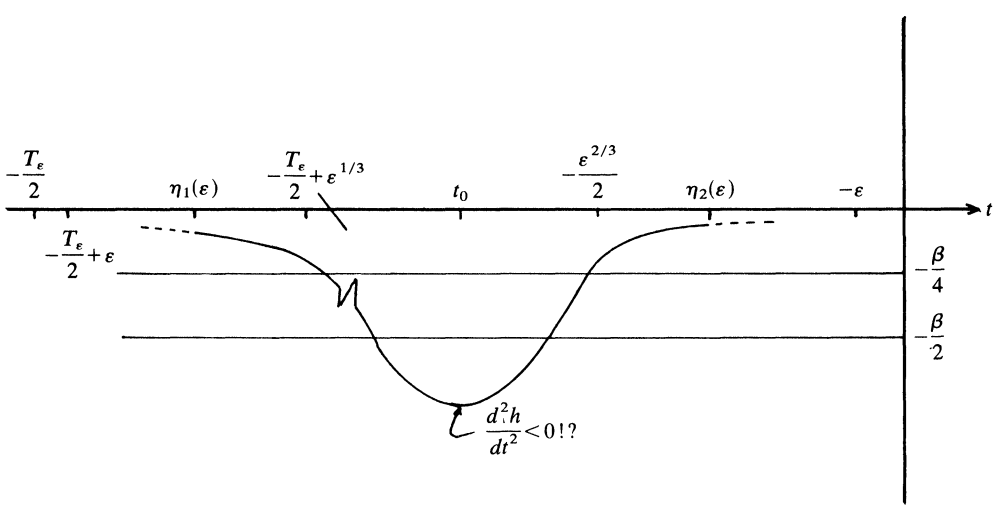

FIG. 6. (*See Proposition* 7, *part* (a).) *Given* $\beta>0$, $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$ *as shown. If graph of* $h(t;\varepsilon)$ *on* $[\eta_1(\varepsilon), \eta_2(\varepsilon)]$ *falls as low as* $-\beta/2$, *h attains its minimum at* $t_0 \in (\eta_1(\varepsilon), \eta_2(\varepsilon))$. *It turns out that if* ε *is sufficiently small,* $d^2h/dt^2|_{t_0}$ *is negative, a contradiction.*

A similar argument leads to

$$h(t;\varepsilon)=o(1) \tag{65}$$

uniformly on $-\varepsilon^{2/3}/2 \ll \eta(\varepsilon) \ll -\varepsilon$, so we can choose ε small enough to make

$$h(\eta_2(\varepsilon);\varepsilon) > -\frac{\beta}{4}. \tag{66}$$

Thus, if $h(t;\varepsilon)$ is as negative as $-\beta/2$ at some point on $[\eta_1(\varepsilon), \eta_2(\varepsilon)]$, the minimum of $h(t;\varepsilon)$ will occur at some $t_0 \in (\eta_1(\varepsilon), \eta_2(\varepsilon))$, and there

$$\left.\frac{dh}{dt}\right|_{t_0}=0 \tag{67a}$$

and

$$h(t_0;\varepsilon) \leqq -\frac{\beta}{2}. \tag{67b}$$

Finally, if ε is small enough to ensure $\eta_2(\varepsilon) \leqq -10\varepsilon/\beta$ the discussion summarized after (54) tells us

$$|R(t_0;\varepsilon)|<1. \tag{68}$$

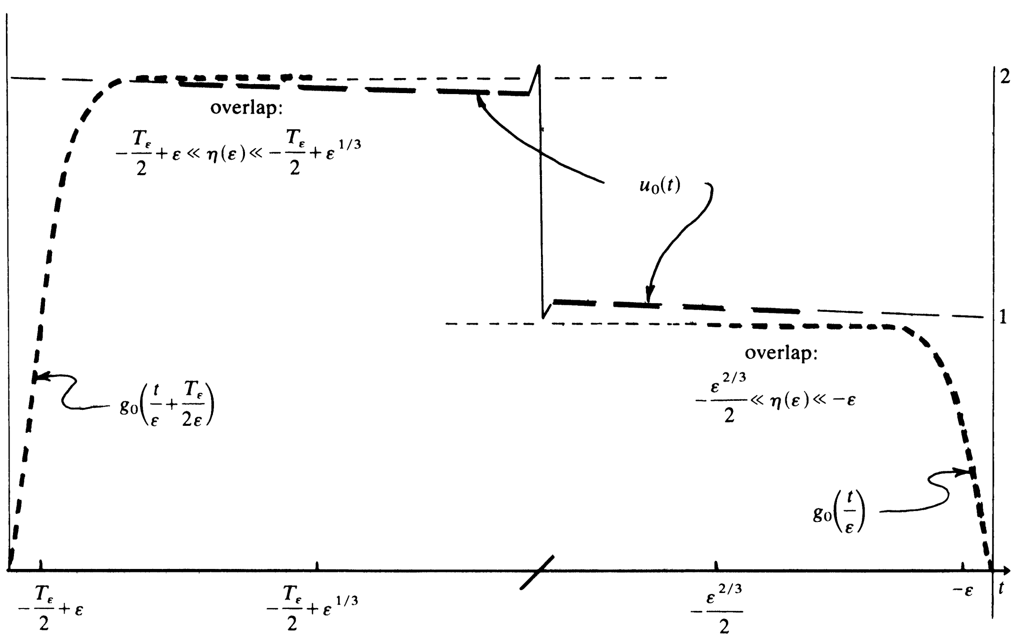

FIG. 7. *Short dashes show schematic graph of inner solution,* g_0; *the heavy dashes show where it has been proved to be a valid approximation for* $y(t;\varepsilon)$. *The long dashes show graph of outer solution* $u_0(t)$, *with the heavy long dashes indicating where it has been proved to be a valid approximation for* $y(t;\varepsilon)$. *The two overlap domains are schematically indicated as the two intervals where the heavy dashes agree. Note the graph of* $y(t;\varepsilon)$ *on the complete half period* $[-T_\varepsilon/2, 0]$ *can now easily be sketched in (not shown).*

Thus, at t_0, (10a) tells us

$$\varepsilon \frac{d^2h}{dt^2}\bigg|_{t_0} < 0, \tag{69}$$

a contradiction.

We conclude that given any β, there exists ε_0 such that, if $\varepsilon \leqq \varepsilon_0$, then $h(t;\varepsilon) > -\beta/2$ in $[\eta_1(\varepsilon), \eta_2(\varepsilon)]$. Since $h(t;\varepsilon) < 0$ (Proposition 1), we have proved part (a) of Proposition 7.

(b) To establish overlap is now easy. (Please refer to Fig. 7.) From the inner expansion analysis

$$y(t;\varepsilon) = g_0\left(\frac{t}{\varepsilon}\right) + o(1) \tag{70a}$$

uniformly for

$$-\frac{\varepsilon^{2/3}}{2} \leqq t \leqq 0 \tag{70b}$$

and

$$y(t;\varepsilon) = g_0\left(\frac{t + T_\varepsilon/2}{\varepsilon}\right) + o(1) \tag{71a}$$

for

$$-\frac{T_\varepsilon}{2} \leqq t \leqq -\frac{T_\varepsilon}{2} + \varepsilon^{1/3}, \tag{71b}$$

using periodicity and symmetry.

The intersection of (70b) with (61b) is $-\varepsilon^{2/3}/2 \leqq \eta(\varepsilon) \ll -\varepsilon$, proving (62b). A similar argument lead to (62a); we leave the computation to the reader. This completes the proof of Proposition 7.

4. Discussion and conclusion. In the preceding analysis we have presented a straightforward and rigorous proof that the leading term in Cole's outer asymptotic expansion is a valid $O(1)$ approximation to $y(t;\varepsilon)$, uniformly on the order interval

$$-\frac{T_\varepsilon}{2} + \varepsilon \ll \eta(\varepsilon) \ll -\varepsilon. \tag{72}$$

A corresponding result for Cole's inner asymptotic expansion was proved in MacGillivray [9], and combining the domains of validity obtained there with (72) gives us a rigorous proof of overlap domains to $O(1)$:

$$-\frac{T_\varepsilon}{2} + \varepsilon \ll \eta(\varepsilon) \ll -\frac{T_\varepsilon}{2} + \varepsilon^{1/3} \tag{73a}$$

in the vicinity of $-T_\varepsilon/2$, and

$$-\frac{\varepsilon^{2/3}}{2} \ll \eta(\varepsilon) \ll -\varepsilon \tag{73b}$$

in the vicinity of zero. (It should be mentioned that existence of overlap domains is basic in the Kaplun–Lagerstrom theory of matched asymptotic expansions, and that rigorous proofs of existence of such domains for nontrivial examples are rare.)

Our analysis incidentally confirms the standard results for the period and amplitude:

(74a) $$T_\varepsilon = 3 - 2\log 2 + o(1)$$

and

(74b) $$A_\varepsilon = 2 + o(1).$$

As mentioned in the introduction, treatment of relaxation oscillations of the Van der Pol equation as a singular perturbation problem using phase plane or Lienard plane variables (instead of in the original (y, t) variables) has been done by Bavinck and Grasman [1] and by Dorodnicyn [6]. The formal analysis of Bavinck and Grasman very carefully produces higher order terms in each of five asymptotic expansions, valid in five corresponding (formal) domains of validity which possess overlapping regions which facilitate (formal) matching.

Dorodnicyn's analysis also involves the construction of asymptotic expansions, but he uses only four domains of validity. His constructions are sometimes thought to be rigorous, though his paper does not provide complete proofs. A consensus opinion of his work seems typified by the quotation from Cartwright [4] mentioned in the introduction. A specific question arises in going from his region I to region II, where there seems to be needed the (unstated not to mention unproved) assumption that his x_1 is sufficiently close to his a_1. This assumption implies an overlap in this case, which in turn is necessary for his version of matching (see (3.5)). Finally, he obtains a result (see (3.7)) which is *consistent* with his original assumption. Thus, success in (formal) matching is achieved. However, as Lagerstrom (see e.g. [7, p. 95]) has emphasized, such success does not imply the matching is valid.

In contrast, the present work features the rigorous construction of domains of validity as an integral part of the analysis. It requires only two domains of validity, but this is probably due to the fact that only the leading terms of the expansions are verified correct. In particular, the analysis in [9] suggests the intermediate expansion required by Cole will also be required in the rigorous discussion of the next higher order term. Although the notion of an intermediate expansion is important in the Kaplun–Lagerstrom theory of matched asymptotic expansions, its rigorous verification is rare, and such verification for a system as well known as the Van der Pol equation would seem to be a worthwhile contribution to singular perturbation theory. Of course, the study of higher order terms always makes the analysis more complicated, even in formal expansions. However, the rigorous study of such higher order terms has been done; see [10], where the existence of switchback in a model equation of Lagerstrom is proved.

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