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Influence of Noise on Duffing-Van der Pol Oscillators

Die stochastischen Bewegungen von Duffing-Van der Pol-Oszillatoren, die durch weißes Rauschen angetrieben werden, werden mit Hilfe verschiedener theoretischer Methoden untersucht: Analytische und numerische Berechnungen der stationären Wahrscheinlichkeitsverteilung, Untersuchung der Hopf-Bifurkation, Computer-Simulationen sowie das Studium der Lyapunov-Exponenten. Man beobachtet einen Bruch der radialen Symmetrie und rauschinduzierte Oszillationen.

The stochastic motions of Duffing-Van der Pol oscillators driven by white noise are investigated by using several theoretical methods: analytical and numerical calculations of the stationary probability distribution, analysis of the Hopf bifurcation, computer simulations and study of Lyapunov exponents. One observes a break of the radial symmetry and noise induced oscillations.

Исследуются стохастические движения осцилляторов типа Дуффинг-Ван-дер-Поля с внешным воздействием белого шума с помощью следующих теоретических методов: Аналитическое и численное вычисление стационарных функций распределения вероятности, исследование бифуркации Хопфа, компютерные симуляции как и показателей Ляпунова. Наблюдаются разрушение радиалной симметрии и осцилляции произведенные шумом.

1. Introduction

Many authors [1, 2] investigated the bifurcations and the transition to chaotic motions in periodically driven Duffing-Van der Pol oscillators

$$\ddot{x} - (\alpha - \gamma x^2 - \delta \dot{x}^2)\,\dot{x} + x + \eta x^3 = \beta \cos(\omega t)\,. \tag{1}$$

The surprising result of these and other investigations on related oscillators driven by periodic excitations [3, 4] was an exceedingly rich wealth of qualitatively different regimes of motion and a quite complicate bifurcation scenario.

These results suggest the investigation of Duffing-Van der Pol oscillators driven by white noise forces. As well known white noise may be considered as a kind of superposition of a continuous set of periodic components with constant spectral density. We also expect that nonlinear oscillators driven by white noise (i.e. by an infinite number of periodic excitations) show many interesting phenomena. This work is a continuation of our earlier work on stochastically driven biochemical and mechanical oscillators [5, 6, 11].

2. Solution of the stationary Fokker-Planck equation

Let us first consider the problem from a more principal point of view. If a process is described by a system of stochastic differential equations with white noise sources

$$\dot{x}_i = f_i(\boldsymbol{x}) + g_{ik}(\boldsymbol{x})\,\xi_k(t)\,, \qquad \langle \xi_k(t)\rangle = 0\,, \qquad \langle \xi_k(t)\,\xi_l(t')\rangle = \delta_{kl}\delta(t - t')\,, \tag{2}$$

the stationary Fokker-Planck equation reads [7]

$$\left.\begin{aligned} -\partial_t P &= \partial_i[K_i P - \partial_j(D_{ij}P)] = \partial_i G_i(\boldsymbol{x}) = 0,\\ K_i(\boldsymbol{x}) &= f_i(\boldsymbol{x}) + \tfrac{1}{2}\, g_{jk}\,\partial_j g_{ik}; \qquad D_{ij} = \tfrac{1}{2}\, g_{ik} g_{jk}\,. \end{aligned}\right\} \tag{3}$$

Here $G_i(\boldsymbol{x})$ represents the stationary probability flow and $P(\boldsymbol{x})$ the stationary probability density.

Let us note that here and in the following one has to sum up over repeated indices (Einstein convention). The problem of finding the stationary probability which we shall write in exponential form

$$P(\boldsymbol{x}) = N \exp\left(-\Phi(\boldsymbol{x})\right) \tag{4}$$

(N — normalization factor) is not trivial for oscillating systems which are characterized by rotating probability flows. Many papers are devoted to the problem of finding the stationary distribution [11—14]. One of the most successful methods is based on phase-averaging [7, 11, 13]. Since this method uses averages over oscillation periods, some of the details concerning possible asymmetries in the phase space are lost. Here we are especially interested in those asymmetries which result from non-constant velocities along the orbit. Therefore we shall use here another solution method which goes back to TOLSTOPJATENKO [8] and one of the authors [9]. In order to explain the procedure we start from the observation that there exist many systems with the same probability which are connected by the transformation

$$\tilde{G}_i = G_i - \partial_j[d_{ij}(\boldsymbol{x})\,P(\boldsymbol{x})]\,, \qquad d_{ij} = -d_{ji}\,. \tag{5}$$

The antisymmetry of d_{ij} guarantees the disappearance of the additional term in the Fokker-Planck equation. The transformation by means of the matrices d_{ij} defines an equivalence class of stochastic systems with different diffusion matrices $\tilde{D}_{ij}$ but with the same stationary probability [10]. In this equivalence class exists a special stochastic system with zero flow

$$\tilde{G}_i(\boldsymbol{x}) = 0\,.$$

This stochastic system is called centre of the equivalence class [11]. In the centre representation $P(\boldsymbol{x})$ follows by simple quadratures. Since the centre representation is defined by

$$G_i(\boldsymbol{x}) - \partial_j\big(d_{ij}P(\boldsymbol{x})\big) = 0 \tag{6}$$

we get by using (4) and (6)

$$\partial_j\Phi = -(D_{ij} + d_{ij})^{-1}\,[K_i - \partial_j(D_{ij} + d_{ij})] = -L_j(\boldsymbol{x})\,. \tag{7}$$

In this way the potential Φ follows in a contour integral representation

$$\Phi(\boldsymbol{x}) = -\int L_j(\boldsymbol{x})\,\mathrm{d}x_j\,. \tag{8}$$

In many cases and especially for oscillation systems we are interested in the transformation $d_{ij}(\boldsymbol{x})$ is more easily found than the potential $\Phi(\boldsymbol{x})$. This suggests the following procedure of solving the Fokker-Planck equations for oscillatory systems:

(i) Determine the transformation matrix $d_{ij}(\boldsymbol{x})$ for the centre representation by using the integrability condition

$$\partial_j L_i = \partial_i L_j\,. \tag{9}$$

(ii) Calculate $\Phi(x)$ by quadratures and in this way also $P(\boldsymbol{x})$.

(iii) Find the probability flow from

$$G_i(\boldsymbol{x}) = \partial_j\big(d_{ij}(\boldsymbol{x})\,P(\boldsymbol{x})\big)\,.$$

A bifurcation in a stochastic system is usually defined as a topological change of the probability function $P(\boldsymbol{x})$ or of the vector field $G_i(\boldsymbol{x})$. In the following we apply the procedure given above to oscillators driven by white noise

$$\dot{x} = y\,, \qquad \dot{y} = K(x, y) + \sqrt{\varepsilon}\,\xi(t) \tag{10}$$

where ε is the constant noise strength. For this case the transformation matrix d_{ij} has the special form [9]

$$d_{ij}(\boldsymbol{x}) = \varepsilon\begin{pmatrix} 0 & -\Psi^{-1} \\ \Psi^{-1} & 0 \end{pmatrix}$$

where $\Psi(\boldsymbol{x})$ is a function satisfying the partial differential equation (9)

$$\varepsilon\frac{\partial^2\Psi}{\partial y^2} = y\frac{\partial\Psi}{\partial x} + \Psi\frac{\partial K}{\partial y} + K\frac{\partial\Psi}{\partial y} + \Psi^2 + 2y\Psi\frac{\partial\Psi}{\partial y} \tag{11}$$

with the boundary condition

$$|\Psi(\boldsymbol{x})| \to \infty \quad \text{if} \quad |\boldsymbol{x}| \to \infty\,.$$

If $\Psi(x, y)$ is found ,the integration of (8) yields

$$P(x, y) = N\Psi\exp\left\{\frac{1}{\varepsilon}\left[\int\left(y\Psi^2 + K\Psi - \varepsilon\frac{\partial\Psi}{\partial y}\right)\mathrm{d}x - \int y\Psi\,\mathrm{d}y\right]\right\}. \tag{12}$$

In the following section the procedure developed so far will be applied to Duffing-Van der Pol oscillators.

3. Hopf bifurcation of Duffing-Van der Pol oscillators

Corresponding to (10) the K-function for Duffing-Van der Pol oscillators is given by

$$K(x, y) = (\alpha - \gamma x^2 - \delta y^2)\,y - x - \eta x^3\,.$$

The nonlinear p.d.e. which results for the Ψ-function is quite difficult and not easy to solve. However near to the Hopf bifurcation the problem may be simplified very much by using polynomial expansions [7, 12]. In the stochastic picture a Hopf bifurcation corresponds to the change from a Gaussian peak of the probability to a craterlike shape. This topological change can locally be described by Taylor expansions. By using the quadratic ansatz

$$\Psi(\boldsymbol{x}) = \Psi_0 + \Psi_i x_i + \tfrac{1}{2}\Psi_{ij}x_i x_j$$

and inserting this into the p.d.e. (11) we get by comparison of the coefficients

$$\Psi_0 = -\sqrt{\varepsilon}/2 - \alpha/2 - O(\alpha^2/\sqrt{\varepsilon})\,, \qquad \Psi_x = \Psi_y = 0\,, \qquad \Psi_{xx} = \Psi_{yy} = 1/4\,, \qquad \Psi_{xy} = -3\Psi_0/4\,.$$

After integration we get finally the probability

$$P(x, y) = N\exp\left\{\frac{\alpha}{2\varepsilon}(x^2 + y^2) + 3\delta xy/4 - 3\gamma xy/4 - \frac{\gamma + 3\delta}{16\varepsilon}(x^2 + y^2)^2 + \frac{\alpha\eta}{4\varepsilon}x^4 + \dots\right\} \tag{13}$$

if

$$\alpha^2 \ll \varepsilon \ll 1\,, \qquad \eta < \frac{\gamma + 3\delta}{4\alpha}.$$

In two limiting cases this procedure gives exact probability distributions:

(i) $\gamma = \delta = 0$.

In this case $\Psi(x) = -\alpha$ follows; if further an Einstein relation i.e.

$$\varepsilon = -\alpha kT$$

for the negative friction holds, we obtain the Maxwellian

$$P(x, y) = N \exp\left\{-\frac{1}{kT}\left(\frac{x^2 + y^2}{2} + \frac{\eta}{4} x^4\right)\right\}.$$

(ii) If $\gamma = \delta$, $\eta = 0$, we obtain the distributions for the modified Van der Pol oscillator [11]

$$P(x, y) = N \exp\left\{\frac{1}{\varepsilon}(\alpha H - \gamma H^2)\right\}, \qquad H = \tfrac{1}{2}(x^2 + y^2).$$

We want to discuss now the stochastic Hopf bifurcation in the representative case $\gamma = 1$, $\delta = \eta = 0$. We distinguish three parameter regions:

(i) The stable focus region: $\alpha < -3\varepsilon/4$
The deterministically damped system corresponds to an one-peak distribution around the stable focus.

(ii) The transition region: $-3\varepsilon/4 < \alpha < 3\varepsilon/4$
In the close vicinity of the bifurcation point $\alpha = 0$ the nonlinearity and the noise dominate. Two maxima arise $x_{1,2} = \pm(2\alpha + 3\varepsilon/2)^{1/2}$, $y_{1,2} = \mp(2\alpha + 3\varepsilon/2)^{1/2}$ and the radial symmetry is broken in this way.

(iii) The region of developed oscillations: $\alpha > 3\varepsilon/4$
The probability distribution is shaped like a crater with two maxima and two saddle points on the crest around the minimum at the unstable focus (Fig. 1).

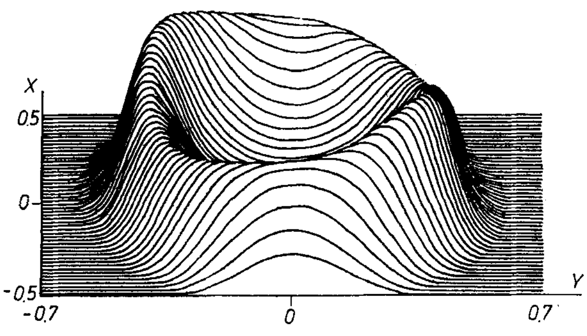

Fig. 1. Probability distribution computed with Taylor expansions ($\alpha = 0.04$; $\gamma = 1$; $\delta = 0$; $\eta = 0$; $\varepsilon = 0.0005$)

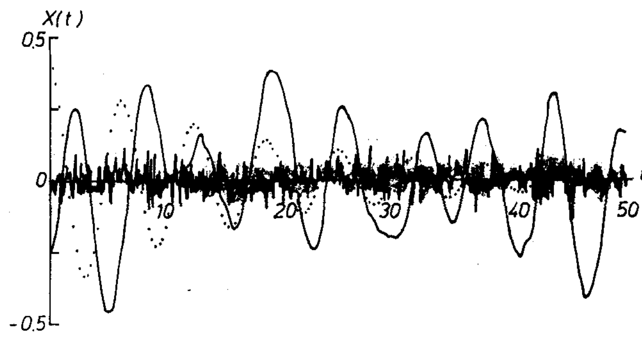

Fig. 2. Noise-induced oscillations, noise and deterministically damped oscillations (dotted line) ($\alpha = -0.1$; $\gamma = 1$; $\delta = 0$; $\eta = 0$; $\varepsilon = 0.008$)

The most important result of the analytical considerations given so far are:

(i) The classical Hopf bifurcation is essentially modified by the fluctuations. The reason is that the stochastic effects dominate near the bifurcation point.

(ii) The method of phase averaging which is usually applied for the study of excited oscillations [7, 11, 13] does not represent the finer details of the distribution. The distribution given here has in general no rotational symmetry [9, 19]. In special cases where this symmetry holds both methods yield identical results.

4. Numerical simulations and local Lyapunov exponents

Computer simulations of the Langevin equations (10) confirm the results of chapter 3. They are a useful tool especially for the region of the fully developed oscillations where the application of the Taylor expansions is limited. We simulate the noise with Gaussian random numbers and compute in this way stochastic realizations, correlation functions, and represent the stationary probability distribution as histograms or stroboscopes. In the parameter region of (deterministically) weakly damped oscillations the stochastic systems shows large oscillations with a rather long correlation time [5]. The relatively regular oscillations (Fig. 2) are due to the amplification of short-correlated noise. They are qualitatively different from chaotic oscillations [6].

We can compute the variance of these oscillations using (13):

$$\text{region (i):} \quad \langle x^2 + y^2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy (x^2 + y^2)\, P(x, y) \approx \frac{2\varepsilon}{|\alpha|} = 2kT\ ,$$

$$\text{region (ii):} \quad \langle x^2 + y^2\rangle = 4(\varepsilon/\gamma\pi)^{1/2}\ .$$

The result of numerical simulations for the transition point $\alpha = 0$ is shown in Fig. 3. It confirms the existence of two maxima of the stationary probability distribution near the Hopf bifurcation.

Now we focus our attention to the fully developed oscillations. The radial symmetry of the probability distribution is only preserved in the case of the modified Van der Pol oscillator. Otherwise the different "speed" in the phase space along the limit cycle and the amplitude fluctuations lead to different height and width of the crater.

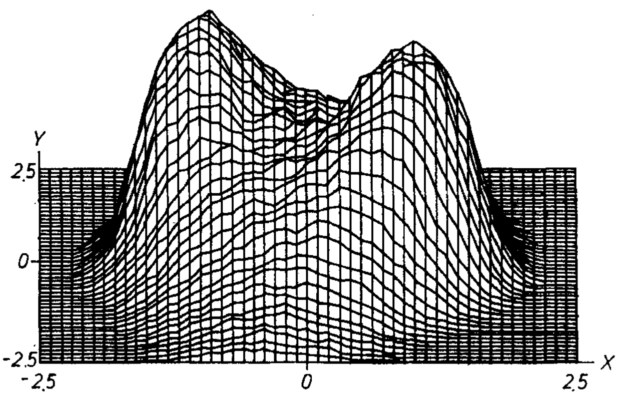

Fig. 3. Histogram of numericl simulations at the bifurcation point ($\alpha = 0$; $\gamma = 1$; $\delta = 0$; $\eta = 0$; $\varepsilon = 0.4$)

In order to estimate the influence of fluctuations we analyse in the following the relaxation of infinitesimal deviations $\boldsymbol{q}$ from the deterministic trajectory $\boldsymbol{x}(t)$

$$\dot{x}_i = f_i(\boldsymbol{x}), \qquad \dot{q}_i = \frac{\partial f_i}{\partial x_j} q_j = \mathcal{J}_{ij}\,(\boldsymbol{x}(t))\, q_j \tag{14), (15}$$

The evolution of the deviation $\boldsymbol{q}$ follows the linearized dynamical equations. The Jacobian $\mathcal{J}$ depends on the trajectory $\boldsymbol{x}(t)$. In this way (14) and (15) are coupled. We get using (15)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,||\boldsymbol{q}|| = \frac{q_i \dot{q}_i}{||\boldsymbol{q}||} = \frac{q_i \mathcal{J}_{ij} q_j}{||\boldsymbol{q}||^2}\,||\boldsymbol{q}||, \qquad ||\boldsymbol{q}(t)|| = ||\boldsymbol{q}(0)||\exp\left\{\int\limits_0^t \frac{q_i \mathcal{J}_{ij} q_j}{||\boldsymbol{q}||^2}\,\mathrm{d}t'\right\}. \tag{16}$$

We call the quantity $\lambda(\boldsymbol{q}, \boldsymbol{x}) = q_i\mathcal{J}_{ij}q_j/||\boldsymbol{q}||^2$ local Lyapunov exponent since its time average leads to the usual Lyapunov exponents. This quantity describes the exponential decrease or increase of fluctuations and depends on the direction of the deviation $\boldsymbol{q}$ and the state $\boldsymbol{x}(t)$. In the case of limit cycle oscillations we consider phase and amplitude fluctuations:

(i) Deviations along the orbit:

$$q_1 = \dot{x}\,, \qquad q_2 = \dot{y}\,.$$

The corresponding (averaged) Lyapunov exponent λ_1 is zero indicating the metastability with respect to phase diffusion.

(ii) Deviations perpendicular to the orbit:

$$q_1 = -\dot{y}\,, \qquad q_2 = \dot{x}\,.$$

The Lyapunov exponent λ_2 (Floquet exponent) is negative. It reflects the stability of the limit cycle averaged over one period. More information the local Lyapunov exponents contain represented in Fig. 4 which describe the local behaviour of phase and amplitude fluctuations. The time dependence of $\lambda_2(t)$ is in agreement with the result of simulations which are represented in Fig. 5.

Periodic excitation of the double-well oscillator may lead to deterministic chaos confirmed by a positive Lyapunov exponent [15]

$$\dot{x} = y\,, \qquad \dot{y} = -0.2y + 10x - 100x^3 + 1.8\cos(0.6t)\,,$$

$$\lambda_1 = 0.15 \pm 0.01\,, \qquad \lambda_2 = 0\,, \qquad \lambda_3 = -0.2 - \lambda_1 = -0.35 \pm 0.01\,.$$

The time dependence of the local Lyapunov exponent shows that contraction and expansion change rapidly (Fig. 6a). We added instead of a periodic excitation short-correlated noise ($\varepsilon = 0.05$). It is seen in Fig. 6c that noise produces quite similar behaviour (compare Fig. 6b and 6c), with orbital instability: $\lambda_1 = 0.30 \pm 0.02$.

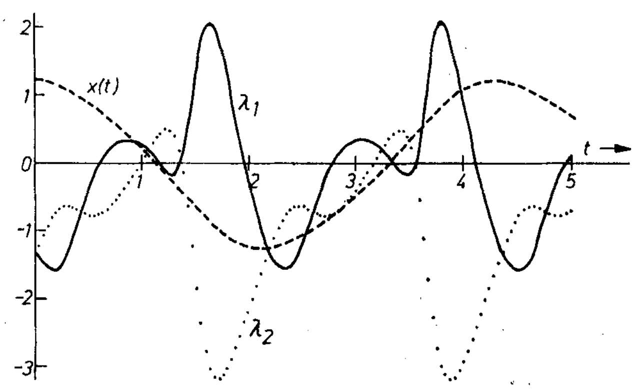

Fig. 4. Local Lyapunov exponents (λ_1 full line; λ_2 dotted line) and the corresponding $x(t)$ (broken line) ($\alpha = 0.4$; $\gamma = 1$; $\delta = 0$; $\eta = 1$)

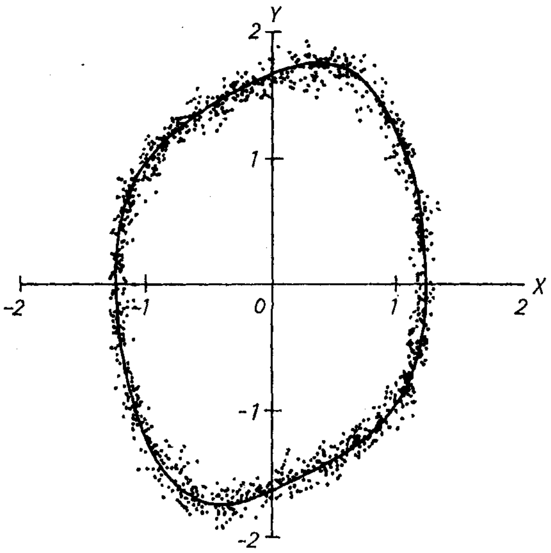

Fig. 5. Stroboscopic representation of fluctuations along the limit cycle (full line). Parameters like Fig. 4, $\varepsilon = 0.00035$

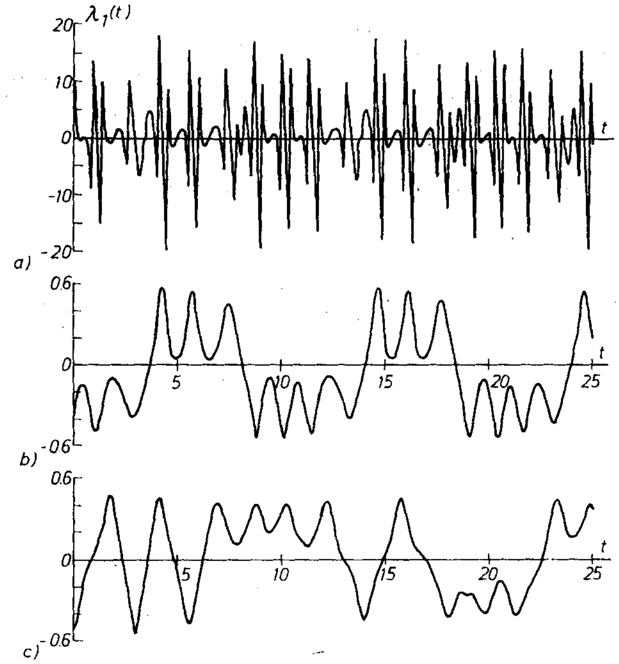

Fig. 6. The local expansion rate $\lambda_1(t)$ in the case of deterministic chaos (6 a, b) and nonperiodic oscillations due to noise (6 c) ($\varepsilon = 0.1$)

5. Summary

Short correlated noise is always present in real mechanical systems. The case of broad band noise excitation of mechanical systems has been studied by STRATONOVICH [7], SCHMIDT [13, 14] and others.

In the present work analytical and numerical results are presented which avoid the commonly used phase averaging. As we have shown the phase asymmetry is of importance in many cases and modifies especially the details of the Hopf bifurcation. Except for special symmetrical cases the probability crater becomes unsymmetrical and shows on the crest two maxima. The mechanical system studied here corresponds to the special case $e = 0$ and $\varepsilon = 0$ of the system studied by SCHMIDT [13]. Therefore the comparison of our unsymmetrical distributions which have been confirmed by numerical simulations with the symmetrical distributions given by SCHMIDT enable us to study the validity of the phase averaging method. A fine measure of the effects along a trajectory are the local Lyapunov exponents.

A special result is that somewhat below the Hopf bifurcation fluctuations are amplified by orders of magnitude. That is consistent with the results of a linearized theory and especially with the fluctuations-dissipation theorem. We remark in addition that noise is of great importance if the deterministic system exhibits different attractors which could be caused by a bistable potential [15] or by periodic excitation [16]. Then fluctuations lead to irregular transitions between the attractors. That is a possible source of the $1/f$ noise [17]. We want to underline the differences between irregularities by noise (Fig. 2) and deterministic chaos. Generalizations of quantities like Lyapunov exponents in the presence of noise are necessary. In a forthcomming paper we want to compare averaged local Lyapunov exponents with other quantities to estimate the stability of trajectories:

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