Brief Communication

Computation of Optimal Controls of a Stochastic Van der Pol Type Oscillator

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I. Introduction

This paper deals with the computation of the optimal feedback control law for a random van der Pol type oscillator. It is shown that, in order to implement the optimal feedback control law, a nonlinear partial differential equation has to be solved. A finite differences algorithm for the solution of this equation is suggested, and its efficiency and applicability are demonstrated with examples.

In the investigation of self-excited, mutually coupled oscillator systems with external excitation, one is often led to the forced van der Pol equation. [For details see Ref. (**1**), pp. 240–242.]

Let a random van der Pol type oscillator be given by

$$\frac{\mathrm{d}^2 Y(t)}{\mathrm{d}t^2} - \zeta_2(t)\,[1-a(Y(t))^2]\frac{\mathrm{d}Y(t)}{\mathrm{d}t} + \zeta_1(t)\,Y(t) = Ku(t), \quad t>0, \tag{1}$$

where a and K are given positive numbers. Let u be the control function and $\{(\zeta_1(t),\zeta_2(t)),\, t\geqslant 0\}$ a vector of independent Gaussian white noises with $E\zeta_i(t)=0$, $t\geqslant 0$, $i=1,2$, and $E\{\zeta_i(t),\zeta_i(s)\}=\sigma_i\,\delta(t-s)$, $t,s\geqslant 0$, $i=1,2$ where σ_i, $i=1,2$ are given positive numbers.

By introducing the state variables $x_1 \triangleq Y$ and $x_2 \triangleq \mathrm{d}x_1/\mathrm{d}t$, the system given by Eq. (1) is represented equivalently by

$$\mathrm{d}x_1 = x_2\,\mathrm{d}t, \quad t>0, \tag{2}$$

$$\mathrm{d}x_2 = Ku\,\mathrm{d}t - x_1\,\mathrm{d}W_1 + x_2(1-ax_1^2)\,\mathrm{d}W_2, \quad t>0, \tag{3}$$

where $\{(W_1(t),W_2(t)),\, t\geqslant 0\}$ is a vector of independent Wiener processes with

$$E[W_i(t)-W_i(s)] = 0, \quad t,s\geqslant 0, \quad i=1,2, \tag{4}$$

$$E[(W_i(t)-W_i(s))^2] = \sigma_i\,|t-s|, \quad t,s\geqslant 0, \quad i=1,2. \tag{5}$$

For several reasons [see Refs (**2**) and (**3**)] it follows that Eqs. (2) and (3) do not properly represent the system given by Eq. (1). Following (**3**) it can be shown that the system given by Eq. (1) is more adequately represented by the following set of stochastic differential equations

$$dx_1 = x_2\,dt, \quad t>0, \tag{6}$$

$$dx_2 = [Ku+\tfrac{1}{2}(1-ax_1^2)^2 x_2\sigma_2]\,dt - x_1\,dW_1 + x_2(1-ax_1^2)\,dW_2, \quad t>0. \tag{7}$$

Equations (6) and (7) determine, for a given function u, a stochastic process $\{\eta(t;\,u) = (\eta_1(t;\,u), \eta_2(t;\,u)), t\geqslant 0\}$.

It can be shown [see Ref. (**4**)] that if u is a bounded and sectionally continuous function on any finite interval of $\{t\geqslant 0\}$, then $\{\eta(t;\,u), t\geqslant 0\}$ is a strong Markov process.

Denote by S the following open domain in the (x_1, x_2)-plane

$$S \triangleq \{(x_1,x_2): |x_1|+|x_2|<1\}, \tag{8}$$

with boundary ∂S. Also, let $\tau(x_1,x_2;\,u)$ denote the first time that $\eta(t;\,u)$ hits the set ∂S, using the control function u and with initial conditions

$$[\eta_1(0;\,u), \eta_2(0;\,u)] = (x_1,x_2)\in S.$$

In this paper the following problem is treated: *Find a bounded, sectionally continuous function* $u^*(t) = u^*(\eta_1(t;\,u^*), \eta_2(t;\,u^*))$, $0\leqslant t\leqslant \tau(x_1,x_2;\,u^*)$, *such that the functional*

$$J(x_1,x_2;\,u) \triangleq E\left[\int_0^{\tau(x_1,x_2;u)} (\eta_1^2(s;\,u)+\eta_2^2(s;\,u)+\lambda u^2(s))\,ds \,|\, \eta(0;\,u) = (x_1,x_2)\right],$$

will be minimized. Here λ is a given positive number.

Denote

$$V(x_1,x_2) \triangleq \inf_{u\in U} J(x_1,x_2;\,u), \tag{10}$$

where U is the class of all the control functions of the form

$$u(t) = u(\eta_1(t;\,u), \eta_2(t;\,u)), \quad t\geqslant 0.$$

Then, by using the principle of optimality it can be shown, by using the procedure described in Ref. (**5**), that $V(x_1,x_2)$, whenever it exists, is a solution to the following nonlinear partial differential equation

$$x_2\frac{\partial V}{\partial x_1} + \tfrac{1}{2}(1-ax_1^2)^2 x_2\sigma_2\frac{\partial V}{\partial x_2} + \tfrac{1}{2}[x_1^2\sigma_1 + x_2^2(1-ax_1^2)^2\sigma_2]\frac{\partial^2 V}{\partial x_2^2} - \frac{K^2}{4\lambda}\left(\frac{\partial V}{\partial x_2}\right)^2 + x_1^2+x_2^2 = 0, \quad (x_1,x_2)\in S, \tag{11}$$

with boundary conditions

$$V(x_1,x_2) = 0, \quad (x_1,x_2)\in\partial S. \tag{12}$$

In that case, the control function $u^*\in U$ which minimizes the functional $J(x_1,x_2;\,u)$ is given by

$$u^*(x_1,x_2) = -(K/2\lambda)\,\partial V(x_1,x_2)/\partial x_2, \quad (x_1,x_2)\in S. \tag{13}$$

In order to implement the optimal feedback control law, $u^*(x_1, x_2)$, an efficient algorithm for the solution of Eqs. (11) and (12) has to be found. In this paper, a finite differences algorithm for the solution of (11) and (12) is suggested. This numerical procedure led to fast convergence for the cases that were physically relevant.

II. The Numerical Method

In order to solve Eqs. (11) and (12), (11) is replaced by an "up wind" finite differences scheme [see Ref. (**6**)]. The region S is divided into two subregions

$$\left.\begin{aligned} \mathrm{I} &\triangleq \{(x_1, x_2) \colon (x_1, x_2) \in S, x_2 \geqslant 0\} \\ \mathrm{II} &\triangleq \{(x_1, x_2) \colon (x_1, x_2) \in S, x_2 < 0\} \end{aligned}\right\} \tag{14}$$

and for each of them (11) is given its finite differences form.

Replace S by a Cartesian grid of points with a constant mesh size h. Let $0 \equiv (x_1, x_2)$ be an internal grid point with neighbouring points 1, 2, 3, 4 [Fig. 1]. Assuming $x_1^2 + x_2^2 > 0$ and

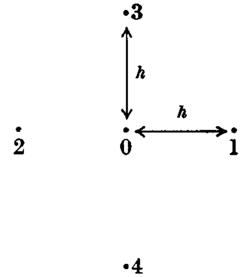

FIG. 1.

defining

$$P = \tfrac{1}{2}[x_1^2 \sigma_1 + x_2^2(1 - ax_1^2)^2 \sigma_2](V_3 + V_4), \tag{15}$$

$$Q = \tfrac{1}{2}[x_1^2 \sigma_1 + x_2^2(1 - ax_1^2) \sigma_2], \tag{16}$$

$$R = \frac{K^2}{8\lambda}(V_3 - V_4), \tag{17}$$

$$T = h^2(x_1^2 + x_2^2). \tag{18}$$

one gets the following equivalents for (11):

$(x_1, x_2) \in \mathrm{I}, \quad V_3 - V_4 \geqslant 0$:

$$V_0 = \frac{P + hx_2 V_1 + \frac{1}{2}h(1 - ax_1^2)^2 x_2 \sigma_2 V_3 + RV_4 + T}{Q + hx_2 + \frac{1}{2}h(1 - ax_1^2)^2 x_2 \sigma_2 + R}, \tag{19}$$

$(x_1, x_2) \in \mathrm{I}, \quad V_3 - V_4 < 0$:

$$V_0 = \frac{P + hx_2 V_1 + \frac{1}{2}h(1 - ax_1^2)^2 x_2 \sigma_2 V_3 - RV_3 + T}{Q + hx_2 + \frac{1}{2}h(1 - ax_1^2)^2 x_2 \sigma_2 - R}, \tag{20}$$

$(x_1, x_2) \in \mathrm{II}, \quad V_3 - V_4 \geqslant 0$:

$$V_0 = \frac{P - hx_2 V_2 - \frac{1}{2}h(1-ax_1^2)^2 x_2 \sigma_2 V_4 + RV_4 + T}{Q - hx_2 - \frac{1}{2}h(1-ax_1^2)^2 x_2 \sigma_2 + R}, \tag{21}$$

$(x_1, x_2) \in \mathrm{II}, \quad V_3 - V_4 < 0$:

$$V_0 = \frac{P - hx_2 V_2 - \frac{1}{2}h(1-ax_1^2)^2 x_2 \sigma_2 V_4 - RV_3 + T}{Q - hx_2 - \frac{1}{2}h(1-ax_1^2)^2 x_2 \sigma_2 - R}. \tag{22}$$

At the origin (11) reduces to $\partial V/\partial x_2 = 0$ which is replaced by

$$V_0 = V_3. \tag{23}$$

The set of Eqs. (19)–(23) is solved by iteration over the whole grid. The computation ends when the difference between two consecutive iterations does not exceed a given tolerance ε.

III. Results

The numerical scheme suggested in the previous section was found to converge for $0{\cdot}1 \leqslant \sigma_1,\ \sigma_2 \leqslant 0{\cdot}3$; $0{\cdot}1 \leqslant a \leqslant 20$. Since the finite differences (19)–(23) are of the first-order, the numerical and exact solutions are expected to differ from each other at the most by $O(h)$. This is in complete agreement with Tables I and II which contain the approximated numerical values of $V(x_1, x_2)$ at six representative grid points.

TABLE I

$\sigma_1 = \sigma_2 = 0{\cdot}1,\ \varepsilon = 10^{-5}$

a	h	$V(-0{\cdot}5, 0)$	$V(-0{\cdot}5, 0{\cdot}3)$	$V(0, 0)$	$V(0, 0{\cdot}3)$	$V(0, 0{\cdot}7)$	$V(0{\cdot}5, 0{\cdot}3)$
10	0·100	0·418	0·345	0·344	0·296	0·175	0·143
	0·050	0·419	0·396	0·353	0·302	0·177	0·146
	0·025	0·420	0·429	0·358	0·305	0·178	0·147
20	0·100	0·368	0·251	0·326	0·285	0·177	0·117
	0·050	0·356	0·280	0·331	0·289	0·179	0·115
	0·025	0·355	0·297	0·334	0·291	0·180	0·114
0·1	0·100	0·430	0·344	0·341	0·292	0·173	0·148
	0·050	0·433	0·392	0·349	0·297	0·174	0·152
	0·025	0·435	0·422	0·354	0·300	0·175	0·154

TABLE II

$\sigma_1 = \sigma_2 = 0{\cdot}2,\ \varepsilon = 10^{-5}$

a	h	$V(-0{\cdot}5, 0)$	$V(-0{\cdot}5, 0{\cdot}3)$	$V(0, 0)$	$V(0, 0{\cdot}3)$	$V(0, 0{\cdot}7)$	$V(0{\cdot}5, 0{\cdot}3)$
2	0·100	0·422	0·330	0·337	0·287	0·163	0·152
	0·050	0·431	0·371	0·346	0·292	0·165	0·156
	0·025	0·436	0·399	0·350	0·295	0·166	0·158

All computations were carried out on a CDC 3600 computer, and Table III demonstrates the speed of convergence.

Finally, it should be noted that if $V(x_1, x_2)$ solves Eq. (11), so does $V(-x_1, -x_2)$, and assuming uniqueness of the solution, we obtain

$$V(-x_1, -x_2) = V(x_1, x_2), \quad (x_1, x_2) \in S \tag{24}$$

as was confirmed by the numerical results.

TABLE III

Computation time

σ_1	σ_2	a	h	No. of grid points	No. of iterations	Time
0·1	0·1	4	0·100	221	25	3 sec
0·1	0·1	0·1	0·100	221	23	3 sec
0·2	0·2	20	0·100	221	33	4 sec
0·2	0·1	0·1	0·050	841	81	35 sec
0·1	0·2	0·1	0·050	841	75	32 sec
0·2	0·2	20	0·050	841	100	43 sec
0·1	0·1	10	0·025	3281	306	8·7 min
0·1	0·1	20	0·025	3281	286	8·2 min
0·2	0·2	2	0·025	3281	279	8·0 min

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