Int. J. Non-Linear Mechanics. Vol. 12, pp. 307–314. Pergamon Press 1977. Printed in Great Britain

COMPUTATION OF FUNCTIONALS OVER DISCONTINUOUS SAMPLE PATHS OF A STOCHASTIC VAN DER POL OSCILLATOR

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(*Received* 3 *February* 1976. *Revised* 9 *February* 1977)

Abstract—This paper deals with a random van der Pol oscillator. It is assumed that the oscillator is subjected to two different kinds of perturbation. The first kind of perturbation is represented by the standard Wiener process and the second kind by a homogeneous process with independent increments, finite second order moments, mean zero and no continuous sample functions. In order to measure quantitatively the stochastic stability of the oscillator, two functionals are defined over its phase plane sample paths. It is shown that each of these functionals is a solution to a corresponding partial integro-differential equation. A numerical procedure for the solution of these equations, is suggested, and its efficiency and applicability are demonstrated with examples.

1. INTRODUCTION

In the investigation of self-excited, mutually coupled oscillator systems with external excitation, one is often led to the forced van der Pol equation. For details see references [1, 2].

Let a random van der Pol oscillator be given by

$$\frac{d^2y(t)}{dt^2} - b(1+\sigma_2 z_2(t))(1-ay^2)\frac{dy(t)}{dt} + \omega_0^2(1+\sigma_1 z_1(t))y(t) = K\frac{d\eta(t)}{dt} \tag{1}$$

where $a, b, \sigma_1, \sigma_2, \omega_0^2$ and K are given positive numbers and $\{(z_1(t), z_2(t)), t \geqslant 0\}$ is a vector of independent Gaussian white noises with $Ez_i(t) = 0, t \geqslant 0, i = i, 2$, and $E\{z_i(t)z_i(s)\} = \delta(t-s)$, $t, s \geqslant 0$, $i = 1, 2$. $\{\eta(t), t \geqslant 0\}$ is a homogeneous process with independent increments, finite second order moments, zero mean and no continuous sample functions. More details on $\{\eta(t), t \geqslant 0\}$ are given in the next section.

By introducing the state variables $x_1 \triangleq y$ and $x_2 \triangleq dx_1/dt$, the system given by equations (1) is represented equivalently by

$$\left.\begin{aligned} dx_1 &= x_2\,dt \quad t > 0 \\ dx_2 &= [-\omega_0^2 x_1 + b(1-ax_1^2)x_2]\,dt - \omega_0^2\sigma_1 x_1\,dW_1 + b\sigma_2(1-ax_1^2)x_2\,dW_2 + K\,d\eta, \quad t > 0 \end{aligned}\right\} \tag{2}$$

where $\{(W_1(t), W_2(t)), t \geqslant 0\}$ is a vector of independent Wiener processes with

$$E[W_i(t) - W_i(s)] = 0, \quad t, s \geqslant 0, \quad i = 1, 2 \tag{3}$$

$$E[(W_i(t) - W_i(s))^2] = |t-s|, \quad t, s \geqslant 0, \quad i = 1, 2 \tag{4}$$

For several reasons (see [3–5]) it follows that equations (2) do not properly represent the system given by equations (1). Using a similar procedure to that described in references [4, 5], it can be shown that the system given by equations (1) is more adequately represented by the following set of stochastic differential equations

$$\left.\begin{aligned} dx_1 &= x_2\,dt \\ dx_2 &= F(x_1, x_2)\,dt - \omega_0^2\sigma_1 x_1\,dW_1 + b\sigma_2(1-ax_1^2)x_2\,dW_2 + K\,d\eta \end{aligned}\right\} \quad t > 0 \tag{5}$$

where

$$F(x_1, x_2) \triangleq -\omega_0^2 x_1 + b(1-ax_1^2)x_2 + b^2\sigma_2^2(1-ax_1^2)^2 x_2/2 \tag{6}$$

Equations (5) determine a stochastic process $\{\boldsymbol{\Omega}(t) \triangleq (\Omega_1(t), \Omega_2(t)), t \geqslant 0\}$, with right continuous sample paths.

The effects of noise on the operation of non-linear oscillators, including van der Pol type oscillators, have been investigated intensively, see for example references [6–11], and the references cited there.

A special problem of interest in the study of oscillators subjected to random disturbances is that of determining the probability of the time taken for the random response process to pass out of a prescribed safe domain for the first time starting from a given initial value. This is called the first-passage time problem. It has attracted a great deal of attention in recent years because of its application to the reliability of vibrational systems subjected to stochastic influences. For example see [8] or [12] and the references cited there.

In this paper, an entirely different approach in the study of the noise's effect on the oscillator operation, is taken. The novelty of this study is due to two factors. First, a random perturbation, $K\,d\eta$, with no continuous sample functions, is added to the noise's sources. Second, the discussion is concentrated on the behaviour of $\{\mathbf{\Omega}(t), t \geqslant 0\}$ only over a bounded domain in the (x_1, x_2)-plane.

Denote by S the following open domain in R^2

$$S \triangleq \{(x_1, x_2): |x_1| + |x_2| < 1\} \tag{7}$$

and let S^c denote its complement. Also, let $\tau(x_1, x_2)$ denote the first time at which $\mathbf{\Omega}(t)$ enters the set S^c, when the initial conditions are $\mathbf{\Omega}(0) = (x_1, x_2) \varepsilon S$.

In order to study quantatively the stochastic stability of the random process $\{\mathbf{\Omega}(t), t \geqslant 0\}$, the following functionals are computed here

$$V(x_1, x_2) \triangleq E\left[\int_0^{\tau(x_1, x_2)} ds | \mathbf{\Omega}(0) = (x_1, x_2)\right], \quad (x_1, x_2) \varepsilon S \tag{8}$$

$$U(x_1, x_2) \triangleq E\left[\int_0^{\tau(x_1, x_2)} (\Omega_1^2(s) + \Omega_2^2(s))\, ds | \mathbf{\Omega}(0) = (x_1, x_2)\right], \quad (x_1, x_2) \varepsilon S \tag{9}$$

The values of V constitute a weak stability property of the oscillator's operation, since they indicate the expected value of the time $\{\mathbf{\Omega}(t), t \geqslant 0\}$, is included in S. In the same manner the values of U can also serve as a measure for the oscillator's operation. In general, stochastic stability analysis provides us with bounds or limit theorems on V and U while the method proposed in the paper, which enables us to obtain the values of V and U, provides us with tools for performing a more quantitative analysis.

Let $k(x_1, x_2), (x_1, x_2) \varepsilon S$, be a given function. In the next section, a partial integro-differential equation is derived. A solution to this equation, whenever it exists, is shown there to be equal to

$$E\left[\int_0^{\tau(x_1, x_2)} k(\Omega_1(s), \Omega_2(s))\, ds | \mathbf{\Omega}(0) = (x_1, x_2)\right], \quad (x_1, x_2) \varepsilon S$$

In section 3 a numerical procedure, for the solution of the partial integro-differential equation, is suggested. In section 4, the efficiency and applicability of the computational method, is demonstrated by the computation of the values of V and U over S, for various values of the parameters a, b, k and ω_0^2. Also, the computations are carried for two different classes of jump distributions of the process $\{\eta(t), t \geqslant 0\}$.

2. DERIVATION OF THE PARTIAL INTEGRO-DIFFERENTIAL EQUATION

Let $\mathscr{B}$ denote the σ-algebra of Borel sets of R^2. It is assumed here that

$$\eta(t) = \iint u_1 \tilde{v}(t, du_1\, du_2), \quad t \geqslant 0 \tag{10}$$

where, for any $A \varepsilon \mathscr{B}$

$$\tilde{v}(t, A) = v(t, A) - t\pi(A), \quad t \geqslant 0 \tag{11}$$

and $v(t, A)$ is a Poisson measure in R^2 with

$$Ev(t, A) = t\pi(A), \quad t \geqslant 0 \tag{12}$$

For more details on $\{v(t, A), t \geqslant 0, A\varepsilon\mathscr{B}\}$ see references [13, 14]. The process $\{\eta(t), t \geqslant 0\}$ as given by equations (10)–(12), is a homogeneous process with independent increments, finite second order moments, mean zero and no continuous sample paths [13].

It is assumed here that

$$\int\pi(\mathrm{d}u_1\,\mathrm{d}u_2) = \pi_0(\mathrm{d}u_1) \tag{13}$$

where $\pi_0(A)$, $A\varepsilon\mathscr{B}_0$, is a probability measure on $\mathscr{B}_0$.* Also, it is assumed that

$$m_1 \triangleq \int u_1\pi_0(\mathrm{d}u_1) = 0 \tag{14}$$

Let $g(x_1, x_2)$, $(x_1, x_2)\varepsilon R^2$, be bounded and twice continuously differentiable on S. From the previous assumptions and from the generalized Ito formula [13], it can be shown that

$$E[g(\Omega_1(t), \Omega_2(t))|\boldsymbol{\Omega}(0) = (x_1, x_2)] = g(x_1, x_2) + E\left[\int_0^t (Lg)(\Omega_1(s), \Omega_2(s))\,\mathrm{d}s|\boldsymbol{\Omega}(0) = (x_1, x_2)\right], \quad (x_1, x_2)\varepsilon S \tag{15}$$

where

$$(Lg)(x_1, x_2) = x_2\partial g(x_1, x_2)/\partial x_1 + F(x_1, x_2)\partial g(x_1, x_2)/\partial x_2 + (1/2)G(x_1, x_2) \times \partial^2 g(x_1, x_2)/\partial x_2^2 - g(x_1, x_2) + \int g(x_1, x_2 + Ku_1)\pi_0(\mathrm{d}u_1) \tag{16}$$

and

$$G(x_1, x_2) = \omega_0^4\sigma_1^2x_1^2 + b^2\sigma_2^2(1 - ax_1^2)^2x_2^2 \tag{17}$$

Denote

$$\tau(x_1, x_2) \triangleq \inf\{t:\boldsymbol{\Omega}(t)\varepsilon S^c \quad \text{and} \quad \boldsymbol{\Omega}(0) = (x_1, x_2)\varepsilon S\} \tag{18}$$

and

$$\tau_T \triangleq \min\{T, \tau(x_1, x_2)\} \tag{19}$$

and assume that g is a smooth solution to the following partial integro-differential equation

$$\left.\begin{aligned}(Lg)(x_1, x_2) &= -k(x_1, x_2), (x_1, x_2)\varepsilon S\\ g(x_1, x_2) &= 0, (x_1, x_2)\varepsilon S^c\end{aligned}\right\} \tag{20}$$

where $k(x_1, x_2)$, $(x_1, x_2)\varepsilon S$, is a given real continuous function.

Then, it follows from equations (15)–(20) that

$$E[g(\Omega_1(\tau_T), \Omega_2(\tau_T))|\boldsymbol{\Omega}(0) = (x_1, x_2)] = g(x_1, x_2) - E\left[\int_0^{\tau_T} k(\Omega_1(s), \Omega_2(s))\,\mathrm{d}s|\boldsymbol{\Omega}(0) = (x_1, x_2)\right], (x_1, x_2)\varepsilon S \tag{21}$$

By following the same arguments as in [13, pp. 304–306] it follows that

$$\lim_{T\to\infty} E[g(\Omega_1(\tau_T), \Omega_2(\tau_T))|\boldsymbol{\Omega}(0) = (x_1, x_2)] = E[g(\Omega_1(\tau(x_1, x_2)), \Omega_2(\tau(x_1, x_2)))|\boldsymbol{\Omega}(0) = (x_1, x_2)] = 0 \tag{22}$$

Hence

$$g(x_1, x_2) = E\left[\int_0^{\tau(x_1, x_2)} k(\Omega_1(s), \Omega_2(s))\,\mathrm{d}s|\boldsymbol{\Omega}(0) = (x_1, x_2)\right], (x_1, x_2)\varepsilon S \tag{23}$$

A straightforward conclusion of this section is given in the following theorem.

Theorem 1

Let V and U be solutions to the following equations correspondingly

$$\left.\begin{aligned}(LV)(x_1, x_2) &= -1, \quad (x_1, x_2)\varepsilon S\\ V(x_1, x_2) &= 0, \quad (x_1, x_2)\varepsilon S^c\end{aligned}\right\} \tag{24}$$

and

$$\left.\begin{aligned}(LU)(x_1, x_2) &= -(x_1^2 + x_2^2), \quad (x_1, x_2)\varepsilon S\\ U(x_1, x_2) &= 0, \quad (x_1, x_2)\varepsilon S^c\end{aligned}\right\} \tag{25}$$

* $\mathscr{B}_0$ is the σ-algebra of Borel sets of R.

then

$$V(x_1, x_2) = E\left[\int_0^{\tau(x_1,x_2)} \mathrm{d}s | \boldsymbol{\Omega}(0) = (x_1, x_2)\right], \quad (x_1, x_2) \varepsilon S \tag{8'}$$

and

$$U(x_1, x_2) = E\left[\int_0^{\tau(x_1,x_2)} (\Omega_1^2(s) + \Omega_2^2(s))\,\mathrm{d}s | \boldsymbol{\Omega}(0) = (x_1, x_2)\right], \quad (x_1, x_2) \varepsilon S \tag{9'}$$

3. THE NUMERICAL METHOD

In order to solve (24)–(25) the region S is first represented by a grid of points (x_{1j}, x_{2i}) where

$$\left.\begin{aligned} x_{1j} &= -1 + (j-1)h \\ x_{2i} &= -1 + (i-1)h \\ |x_{1j}| + |x_{2i}| &\leqslant 1 \end{aligned}\right\} \tag{26}$$

h being the mesh size along both axes. Each partial integro differential equation is then replaced by a finite difference scheme. The new set of equations is iterated until the difference between two consecutive iterations does not exceed a given tolerance ε.

The finite difference scheme suggested here is based upon the "up-wind" method [10, 15], and leads to fast convergence for all the cases under discussion. The usual central finite differences scheme was also applied for equations (24)–(25). However, this procedure was significantly slower than the upwind algorithm and was considered only for comparison purposes.

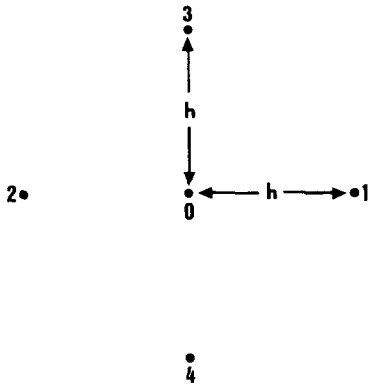

Fig. 1.

Let 0 be an internal grid point, with neighbouring points 1, 2, 3, 4 (Fig. 1). The region S is divided into four sub-regions I–IV in the following manner:

$$\begin{aligned} \mathrm{I} &\triangleq \{(x_1, x_2) \varepsilon S, \quad x_2 \geqslant 0, \quad F(x_1, x_2) \geqslant 0\} \\ \mathrm{II} &\triangleq \{(x_1, x_2) \varepsilon S, \quad x_2 < 0, \quad F(x_1, x_2) \geqslant 0\} \\ \mathrm{III} &\triangleq \{(x_1, x_2) \varepsilon S, \quad x_2 \geqslant 0, \quad F(x_1, x_2) < 0\} \\ \mathrm{IV} &\triangleq \{(x_1, x_2) \varepsilon S, \quad x_2 < 0, \quad F(x_1, x_2) < 0\} \end{aligned} \tag{27}$$

The finite differences equivalents of equations (24)–(25) based upon up-wind approach have their unique forms at each sub-region. Consider for instance equation (24) over I. It should be replaced by

$$x_{20}\frac{V_1 - V_0}{h} + F_0\frac{V_3 - V_0}{h} + \tfrac{1}{2}G_0\frac{V_3 + V_4 - 2V_0}{h^2} - V_0 + I_0 = -1 \tag{28}$$

where

$$(x_{10}, x_{20}) = (x_1, x_2)|_0, \quad F_0 = F|_0, \quad G_0 = G|_0, \quad I_0 = \int V(x_{10}, x_{20} + Ku)\pi_0(\mathrm{d}u)$$

Finally one gets

$$V_0 = \frac{(1/2)G_0(V_3+V_4)+h^2(1+I_0)+h(x_{20}V_1+F_0V_3)}{G_0+h^2+h(x_{20}+F_0)}, \quad (x_{10},x_{20})\varepsilon I \tag{29}$$

Thus the denominator is composed of strictly positive terms avoiding possible instability. Also, since F_0 appears in both numerator and denominator of (29), it should not cause any harm upon taking large values. The other expressions for V_0 are

$$V_0 = \frac{(1/2)G_0(V_3+V_4)+h^2(1+I_0)+h(-x_{20}V_2+F_0V_3)}{G_0+h^2+h(-x_{20}+F_0)}, \quad (x_{10},x_{20})\varepsilon \mathrm{II} \tag{30}$$

$$V_0 = \frac{(1/2)G_0(V_3+V_4)+h^2(1+I_0)+h(x_{20}V_1-F_0V_4)}{G_0+h^2+h(x_{20}-F_0)}, \quad (x_{10},x_{20})\varepsilon \mathrm{III} \tag{31}$$

$$V_0 = \frac{(1/2)G_0(V_3+V_4)+h^2(1+I_0)+h(-x_{20}V_2-F_0V_4)}{G_0+h^2+h(-x_{20}-F_0)}, \quad (x_{10},x_{20})\varepsilon \mathrm{IV} \tag{32}$$

Similar expressions can be derived for equation (25). One should simply insert $h^2(x_{10}^2+x_{20}^2+I_0)$ instead of $h^2(1+I_0)$ in the numerators of equations (29)–(32).

Equations (29)–(32) are iterated over the grid points using over-relaxation technique, i.e., the final new value of V_0 is taken as

$$V_0 = \omega\bar{V}_0+(1-\omega)\bar{\bar{V}}_0 \tag{33}$$

where $\bar{V}_0$ is calculated from equations (29)–(32), $\bar{\bar{V}}_0$ is the old value at the grid point and ω the over-relaxation factor. The use of a proper ω speeds up convergence by a significant margin.

4. RESULTS

Equations (24)–(25) were solved for $\sigma_1^2 = 1$, $\sigma_2^2 = 1$, $0.5 \leqslant a, b \leqslant 2$, $1 \leqslant \omega_0^2 \leqslant 50$. Two choices of the probability measure $\pi_0(du)$ were considered:

$$\pi_0(du) = \begin{cases} \dfrac{du}{2\alpha}, & |u| < \alpha \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

$$\pi_0(du) = \begin{cases} \left(\dfrac{1}{\alpha} - \dfrac{|u|}{\alpha^2}\right)du, & |u| < \alpha \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

In both cases, it is easily seen that $\int V(x_1, x_2+Ku)\pi_0(du)$ depends upon one number $\beta = K\alpha$. This parameter was taken between 0.5 and 3.

In order to evaluate the accuracy of the approximated solutions three mesh sizes, namely $h = 1/10, 1/20, 1/40$ were considered. However, most of the tabulated results were computed with $h = 1/20$. The tolerance was taken as $\varepsilon = 10^{-5}$, and for most cases the over-relaxation factor $\omega = 1.4$ served as an efficient accelerating coefficient.

Tables 1 and 2 consist of values of $V(x_1, x_2)$ computed for $\pi_0(du)$ as given in equation (34), for three mesh sizes.

Table 1. $a = b = 1, \omega_0^2 = 1, \beta = 0.5$

h	$V(0,-0.8)$	$V(0,-0.5)$	$V(0,0)$	$V(0.2,0)$	$V(0.5,0)$
1/10	0.175	0.604	2.27	1.44	0.547
1/20	0.176	0.611	2.28	1.49	0.575
1/40	0.176	0.615	2.29	1.52	0.590

Table 2. $a = b = 1, \omega_0^2 = 10, \beta = 0.5$

h	$V(0,-0.8)$	$V(0,-0.5)$	$V(0,0)$	$V(0.2,0)$	$V(0.5,0)$
1/10	0.130	0.385	1.80	0.164	0.00976
1/20	0.140	0.418	1.82	0.165	0.00983
1/40	0.145	0.439	1.86	0.161	0.00986

Generally, upon using an up-wind finite differences scheme, one expects to achieve an $0(h)$ type accuracy. However, for large ω_0^2, the solution V, U to equations (24)–(25) possess very large derivatives near the origin $(0,0)$. Consequently one needs meshes finer than $h = 1/40$ to provide the $0(h)$ type of error at this neighbourhood. The problem whose solution is presented in Table 2 was also solved by central finite differences. The values of $V(0,0)$ were 2.18, 1.99, 1.94 for $h = 1/10, 1/20, 1/40$ respectively. This certainly confirms the validity of the values given by Table 2, and the reliability of the up-wind method.

All computations were carried on a CDC-3600 computer and the average time needed for solving a problem was 5 sec, 1 min, 15 min, for $h = 1/10, 1/20, 1/40$ respectively.

Table 3 consists of values of $V(0,0)$, $\iint V\,dx_1\,dx_2$, $U(0,0)$, $\max U$ for various cases. The mesh size is $h = 1/40$, the over-relaxation factor is taken as 1.4, and $\beta = 0.5$. The table gives a strong indication that V, U are far more sensitive to changes in b, ω_0^2 than to a change in a. Also, for a small ω_0^2 and a relatively large b, $U(x_1, x_2)$ possesses a local minimum at the origin that turns into an absolute maximum for higher values of ω_0^2 or lower values of b. This is not the case for $\max V(x_1, x_2)$ that was always found to be located at the origin, and thus coincided with $V(0,0)$.

Table 3

			$LV = -1$				$LU = -(x_1^2 + x_2^2)$			
			π_0 of (34)		π_0 of (35)		π_0 of (34)		π_0 of (35)	
a	b	ω_0^2	$V(0,0)$	$\iint V$	$V(0,0)$	$\iint V$	$U(0,0)$	$\max U$	$U(0,0)$	$\max U$
2	2	1	1.75	0.67	2.24	0.74	0.064	0.130	0.078	0.139
2	2	4	1.69	0.34	2.13	0.36	0.057	0.060	0.067	0.067
2	1	1	2.31	1.17	3.13	1.33	0.160	0.195	0.211	0.224
2	1	4	2.10	0.55	2.69	0.59	0.119	0.124	0.143	0.143
2	0.5	1	3.04	1.78	4.43	2.14	0.279	0.314	0.396	0.397
2	0.5	4	2.50	0.76	3.30	0.84	0.172	0.179	0.213	0.213
1	2	1	1.74	0.62	2.23	0.68	0.062	0.109	0.075	0.116
1	2	4	1.69	0.33	2.12	0.35	0.057	0.060	0.066	0.066
1	1	1	2.28	1.11	3.08	1.26	0.153	0.184	0.199	0.200
1	1	4	2.09	0.54	2.68	0.59	0.118	0.123	0.141	0.141
1	0.5	1	3.00	1.73	4.36	2.07	0.271	0.306	0.382	0.383
1	0.5	4	2.49	0.75	3.29	0.84	0.172	0.178	0.212	0.212
0.5	2	1	1.74	0.60	2.22	0.66	0.061	0.101	0.074	0.107
0.5	2	4	1.69	0.33	2.12	0.35	0.056	0.059	0.066	0.066
0.5	1	1	2.27	1.08	3.05	1.23	0.150	0.179	0.194	0.207
0.5	1	4	2.09	0.54	2.67	0.59	0.117	0.122	0.140	0.140
0.5	0.5	1	2.98	1.70	4.32	2.05	0.267	0.301	0.375	0.376
0.5	0.5	4	2.49	0.75	3.29	0.83	0.171	0.177	0.211	0.211

If one lets K increase indefinitely, one should find that $V(x_1, x_2; K) \to V(x_1, x_2; \infty)$, $U(x_1, x_2; K) \to U(x_1, x_2; \infty)$ where $V(x_1, x_2; \infty)$, $U(x_1, x_2; \infty)$ are the solutions of equations (24) and (25) respectively excluding the integral terms. It is indeed the case as can be seen from Tables 4 and 5. These tables consist of representative values of $V(x_1, x_2)$ for $a = b = \sigma_1^2 = \sigma_2^2 = 1$, $h = 0.1$, $\omega = 1.4$, $\varepsilon = 10^{-5}$. The parameter K is replaced by $N = K\alpha/h$ that takes the values 5, 10, 20, 50, 100.

Table 4. $LV = -1, \omega_0^2 = 1, \pi_0$ of (34)

(x_1, x_2)	$V(\cdot,\cdot; 5)$	$V(\cdot,\cdot; 10)$	$V(\cdot,\cdot; 20)$	$V(\cdot,\cdot; 50)$	$V(\cdot,\cdot; 100)$	$V(\cdot,\cdot; \infty)$
$(-0.7, 0)$	0.146	0.141	0.139	0.138	0.137	0.136
$(-0.3, 0)$	1.134	0.888	0.747	0.680	0.661	0.642
$(0, 0)$	2.273	1.598	1.241	1.085	1.041	1.000
$(0, -0.7)$	0.292	0.268	0.229	0.199	0.191	0.184
$(0, -0.3)$	1.056	0.831	0.672	0.596	0.574	0.554

Table 5. $LV = -1, \omega_0^2 = 4, \pi_0$ of (34)

(x_1, x_2)	$V(\cdot,\cdot;5)$	$V(\cdot,\cdot;10)$	$V(\cdot,\cdot;20)$	$V(\cdot,\cdot;50)$	$V(\cdot,\cdot;100)$	$V(\cdot,\cdot;\infty)$
$(-0.7, 0)$	0.011	0.011	0.011	0.011	0.011	0.011
$(-0.3, 0)$	0.289	0.266	0.245	0.234	0.231	0.228
$(0, 0)$	2.054	1.530	1.220	1.079	1.038	1.000
$(0, -0.7)$	0.263	0.249	0.215	0.194	0.187	0.181
$(0, -0.3)$	0.858	0.716	0.598	0.538	0.521	0.505

5. SUMMARY

The algorithm suggested in this work for solving equations (24)–(25) provides sufficient accuracy for small and moderate values of ω_0^2. For a large value of ω_0^2 one should use finer meshes near the origin to guarantee the same accuracy. However, the general characteristics of the solution might be deduced by using a rough mesh as well. The solutions $V(x_1, x_2)$ of equation (24) seem to possess an absolute maximum at the origin. This varies regarding the solutions $U(x_1, x_2)$ of equation (25). Here one still gets an absolute maximum at $(0, 0)$ provided that ω_0^2 is not "too small" and b is not "too large". Otherwise, $U(x_1, x_2)$ has a local minimum at the origin. In that case the absolute maximum is located at two symmetric points $(-\delta, 0), (\delta, 0), 0 < \delta < 1$ as was confirmed throughout the numerical work. Finally it should be noted that both $V(x_1, x_2), U(x_1, x_2)$ as expected are quite sensitive to changes in b, ω_0^2 but not to a change in a.

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Résumé:

Cet article traite d'un oscillateur de van der Pol aléatoire. On suppose que l'oscillateur est soumis à deux sortes de perturbation différentes. La première sorte est représentée par le processus standard de Wiener et la seconde par un processus homogène avec des incréments indépendants, des moments du second ordre finis, une valeur moyenne nulle et aucune fonction échantillon continue. En vue d'étudier le comportement dynamique de l'oscillateur on définit deux fonctionelles sur ses chemins echantillon dans le plan de phase. On montre que chacune de ces fonctionelles est une solution d'une équation intégro-différentielle partielle correspondante. On suggère une méthode numérique pour résoudre ces équations et on démontre son efficacité et son applicabilité avec des exemples.

Zusammenfassung:

Diese Arbeit behandelt Zufallsschwinger der van der Polschen Art. Es wird angenommen, dass der Schwinger zwei verschiedenen Arten von Perturbationen ausgesetzt ist. Die erste Perturbationsart wird durch den normalen Weiner-Prozess dargestellt und die zweite Art durch einen homogenen Prozess mit unabhängigen Inkrementen, endlichen Momenten zweiter Ordnung, verschwindendem Mittel und nichtkoninuierlichen Probenfunktionen. Um die stochastische Stabilität des Schwingers quantitative zu messen, werden zwei Funktionale über den Probenweg in der Phasenebene definiert. Es wird gezeigt, dass beide Funktionale Lösungen einer entsprechenden partiellen Integraldifferentialgleichung sind. Ein numerisches Verfahren fur die Lösung dieser Gleichungen wird vorgeschlagen und dessen Wirksamkeit und Verwondbarkeit werden an Hand von Beispielen demonstriert.