Journal of Sound and Vibration (1981) 77(4), 445–453

QUENCHING IN A SYSTEM OF VAN DER POL OSCILLATORS WITH NON-LINEAR COUPLING

Y. P. SINGH†

Department of Electrical Engineering, Indian Institute of Technology Kanpur, Kanpur—208016, India

(*Received* 14 *May* 1980, *and in revised form* 3 *February* 1981)

A perturbation analysis for non-linearly coupled van der Pol oscillators with harmonic forcing is presented. The disappearance of one frequency and hence the existence of single frequency oscillation due to the quenching effect is thus analytically demonstrated. The analytical results are confirmed by digital computer simulation. The effects of varying the forcing amplitude from small negative values to certain positive values are shown in the simulation results.

1. INTRODUCTION

In references [1–4] Mansour and Tondl investigated quenching in self-excited oscillators. The case of external excitation where there exists a well-known effect of synchronization had been studied by several authors for the case of a single oscillator [5]. In this paper quenching in a system of coupled van der Pol oscillators with non-linear coupling and harmonic forcing is analyzed by using the two time scale technique. The existence of oscillations at a single frequency and at two frequencies, without forcing, depends on the values of the non-linear coupling parameters. It is shown that the parameter space (r_2, t_2) can be divided into four regions for the case where there is an irrational frequency ratio $1:\sqrt{2}$. In region I there is stable oscillation for oscillator (1), in region II stable oscillation for oscillator (2), in region III stable oscillation for oscillators (1) and (2) independently, and in region IV there is stable oscillation for oscillators (1) and (2) simultaneously. This implies that the structure of oscillatory behaviour depends on the values of r_2 and t_2 in the parameter space (r_2, t_2).

It is shown further that external forcing may have a meaningful role in respect to the values of the parameters in region III since in this situation the coupled system may have a different oscillatory behaviour than in the unforced case. A forcing input of frequency ω (the linearized natural frequency of oscillator (1)) to oscillator (1) may cause the oscillation at $\sqrt{2}\omega$ (the linearized natural frequency of oscillator (2)) to be quenched. Similarly forcing the second oscillator at $\sqrt{2}\omega$ may cause the oscillations at ω to be quenched. The model for the forced case is therefore chosen so that the values of r_2 and t_2 lie in region III where oscillations at ω and $\sqrt{2}\omega$ both exist. The digital simulation results similarly are for the forced situation in region III, to provide corroboration of the analytical results.

† Presently with the Department of Electronics and Communication Engineering, University of Roorkee, Roorkee—247672, India.

2. ANALYSIS

2.1. GENERAL PERTURBATION ANALYSIS

Consider the system

$$\ddot{x}_1+\omega^2 x_1=\varepsilon(1-x_1^2-r_2x_2^2)\dot{x}_1+\varepsilon\bar{p}\cos\omega t,$$
$$\ddot{x}_2+2\omega^2 x_2=\varepsilon(1-x_2^2-t_2x_1^2)\dot{x}_2, \tag{1}$$

where $0<\varepsilon\ll 1$, $r_2>0$, $t_2>0$ and $\bar{p}$ is a real constant. The question of interest here is the possible existence of single frequency oscillations of this system. Accordingly the two time scale technique can be used to write the solution in the following form, as given in reference [6]:

$$x_i(t;\varepsilon)=\sum_{m=0}^{M}\varepsilon^m x_{im}(n,\xi)+O(\varepsilon^{M+1}),\ i=1,2. \tag{2}$$

Here $\xi=\varepsilon t$ and $\eta=(1+\varepsilon^2\sigma_2+\varepsilon^3\sigma_3+\cdots+\varepsilon^m\sigma_m)t$ are the slow and fast time scales, respectively, and the quantities x_{im} are functions of the two variables ξ and η. Their time derivatives are

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=\frac{\partial x_{i0}}{\partial\eta}+\varepsilon\left(\frac{\partial x_{i0}}{\partial\xi}+\frac{\partial x_{i1}}{\partial\eta}\right)+\varepsilon^2\left(\sigma_2\frac{\partial x_{i1}}{\partial\eta}+\frac{\partial x_{i1}}{\partial\xi}+\frac{\partial x_{i2}}{\partial\eta}\right)+\cdots,$$
$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2}=\frac{\partial^2x_{i0}}{\partial\eta^2}+\varepsilon\left(2\frac{\partial^2x_{i0}}{\partial\eta\,\partial\xi}+\frac{\partial^2x_{i1}}{\partial\eta^2}\right)$$
$$+\varepsilon^2\left(2\sigma_2\frac{\partial^2x_{i0}}{\partial\eta^2}+\frac{\partial^2x_{i0}}{\partial\xi^2}+\frac{\partial^2x_{i1}}{\partial\eta\,\partial\xi}+\frac{\partial^2x_{i2}}{\partial\eta^2}\right)+\cdots. \tag{3}$$

Substitution of equations (2) and (3) into equations (1) and collecting terms of like powers of ε leads to

$$\frac{\partial^2x_{10}}{\partial\eta^2}+\omega^2x_{10}=0,\qquad \frac{\partial^2x_{11}}{\partial\eta^2}+\omega^2x_{11}=-2\frac{\partial^2x_{10}}{\partial\eta\,\partial\xi}+(1-x_{10}^2-\sqrt{2}x_{20}^2)\frac{\partial x_{10}}{\partial\eta}+\bar{p}\cos\omega\eta, \tag{4, 5}$$

$$\frac{\partial^2x_{20}}{\partial\eta^2}+2\omega^2x_{20}=0,\qquad \frac{\partial^2x_{21}}{\partial\eta^2}+2\omega^2x_{21}=-2\frac{\partial^2x_{20}}{\partial\eta\,\partial\xi}+(1-x_{20}^2-t_2x_{10}^2)\frac{\partial x_{20}}{\partial\eta}. \tag{6, 7}$$

The solutions of equations (4) and (6) are now assumed to be of the forms

$$x_{10}=R_1\cos(\omega\eta+\varphi_1),\qquad x_{20}=R_2\cos(\sqrt{2}\omega\eta+\varphi_2). \tag{8}$$

Introducing the above solutions and their various derivatives into the right-hand sides of equations (5) and (7) leads to

$$\partial^2x_{11}/\partial\eta^2+\omega^2x_{11}=D_1\cos(\omega\eta+\varphi_1)+E_1\sin(\omega\eta+\varphi_1)+NST, \tag{9}$$
$$\partial^2x_{21}/\partial\eta^2+2\omega^2x_{21}=D_2\cos(\sqrt{2}\omega\eta+\varphi_2)+E_2\sin(\sqrt{2}\omega\eta+\varphi_2)+NST, \tag{10}$$

where

$$D_1=2\omega R_1\varphi_{1\xi}+\bar{p}\cos\varphi_1,\qquad E_1=2\omega R_{1\xi}-\omega(1-\tfrac{1}{4}R_1^2-\tfrac{1}{2}r_2R_2^2)R_1+\bar{p}\sin\varphi_1,$$
$$D_2=2R_2(\sqrt{2}\omega)\varphi_2,\qquad E_2=(\sqrt{2}\omega)2R_{2\xi}-(\sqrt{2}\omega)(1-\tfrac{1}{4}R_2^2-\tfrac{1}{2}t_2R_1^2)R_2, \tag{11}$$

and *NST* stands for non-secular terms [5]. For the solutions to be uniformly bounded, the secular terms in equations (9) and (10) must be equal to zero, so that

$$2R_{1\xi}=(1-\tfrac{1}{4}R_1^2-\tfrac{1}{2}r_2R_2^2)R_1-(\bar{p}/\omega)\sin\varphi_1, \tag{12}$$

$$2R_{2\xi}=(1-\tfrac{1}{4}R_2^2-\tfrac{1}{2}t_2R_1^2)R_2, \qquad 2\omega R_1\varphi_{1\xi}=-\bar{p}\cos\varphi_1, \qquad \phi_{2\xi}=0. \quad (13\text{–}15)$$

From equation (15), it is obvious that φ_2 is constant.

The steady state amplitudes and phases are obtained after substitution of $R_{1\xi}=0$, $R_{2\xi}=0$ and $\varphi_{1\xi}=0$ into equations (12)–(15). Then the following set of equations results:

$$(1-\tfrac{1}{4}\hat{R}_1^2-\tfrac{1}{2}r_2\hat{R}_2^2)\hat{R}_1-(\bar{p}/\omega)\sin\hat{\varphi}_1=0, \quad (16)$$

$$(1-\tfrac{1}{4}\hat{R}_2^2-\tfrac{1}{2}t_2\hat{R}_1^2)\hat{R}_2=0, \qquad \cos\hat{\varphi}_1=0 \text{ if } \hat{R}_1\neq 0. \quad (17, 18)$$

2.2. CASE 1: NO FORCING ($\bar{p}=0$)

When there is no forcing ($\bar{p}=0$), the steady state amplitudes are governed by (from equations (16) and (17))

$$(1-\tfrac{1}{4}\hat{R}_1^2-\tfrac{1}{2}r_2\hat{R}_2^2)\hat{R}_1=0, \qquad (1-\tfrac{1}{4}\hat{R}_2^2-\tfrac{1}{2}t_2\hat{R}_1^2)\hat{R}_2=0. \quad (19)$$

Solving for all possible equilibrium points of equations (19) leads to the following sets of equilibrium amplitudes: (i) $\hat{R}_1=0$, $\hat{R}_2=0$; (ii) $\hat{R}_1=0$, $\hat{R}_2=2$ (single mode oscillation $\sqrt{2}\omega$); (iii) $\hat{R}_1=2$, $\hat{R}_2=0$ (single mode oscillation ω); (iv) $\hat{R}_1\neq 0$, $\hat{R}_2\neq 0$ (simultaneous oscillations at ω and $\sqrt{2}\omega$), in which case

$$\hat{R}_1=2\sqrt{(1-2r_2)/(1-4r_2t_2)}, \qquad \hat{R}_2=2\sqrt{(1-2t_2)/(1-4r_2t_2)}.$$

Note for set (iv) that in particular when $r_2=\frac{1}{2}$ then $\hat{R}_1=0$ and $\hat{R}_2=2$, and when $t_2=\frac{1}{2}$ then $\hat{R}_2=0$ and $\hat{R}_1=2$. Further, for equal amplitude oscillations (i.e., $\hat{R}_1=\hat{R}_2$) then $r_2=t_2=r$ and the amplitudes are

$$\hat{R}_1=\hat{R}_2=2\sqrt{1/(1+2r)},$$

which obviously decreases with increasing r. Of the non-trivial solutions set (ii) exists everywhere in the (r_2, t_2) parameter plane and likewise set (iii) exists everywhere in the (r_2, t_2) parameter plane. For set (iv) the region of existence is shown as cross-hatched in Figure 1 in the (r_2, t_2) plane.

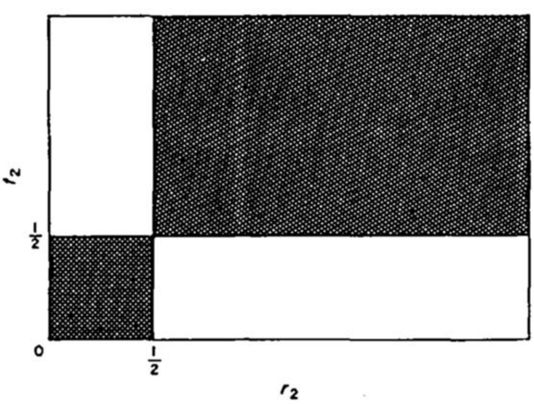

Figure 1. Existence region of equilibrium points satisfying $\hat{R}_1\neq 0$, $\hat{R}_2\neq 0$ in the (r_2, t_2) parameter plane.

The existence region of stationary amplitudes in the $(\hat{R}_1, \hat{R}_2)$ plane is shown in Figure 2. For set (iv) the location of the equilibrium point IV (see Figure 2) depends on r_2 and t_2. In any case the equilibrium point must be within or on the square with sides $\hat{R}_1=2$, $\hat{R}_2=2$, $\hat{R}_1=0$, $\hat{R}_2=0$.

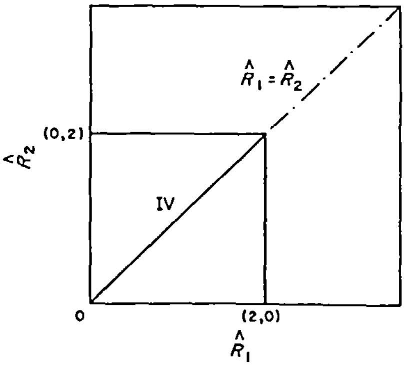

Figure 2. The existence region of stationary amplitudes in the $(\hat{R}_1, \hat{R}_2)$ plane.

To investigate the stability of the equilibrium points of amplitudes for the unforced case, the variational equations are written as

$$2\,\mathrm{d}\Delta R_1/\mathrm{d}\xi = (1 - \tfrac{3}{4}\hat{R}_1^2 - \tfrac{1}{2}r_2\hat{R}_2^2)\Delta R_1 - r_2\hat{R}_1\hat{R}_2\Delta R_2,$$
$$2\,\mathrm{d}\Delta R_2/\mathrm{d}\xi = (1 - \tfrac{3}{4}\hat{R}_2^2 - \tfrac{1}{2}t_2\hat{R}_1^2)\Delta R_2 - t_2\hat{R}_1\hat{R}_2\Delta R_1.$$

The characteristic equation of this system of equations is

$$m^2 - m(1 - \tfrac{3}{4}\hat{R}_1^2 - \tfrac{1}{2}r_2\hat{R}_2^2 + 1 - \tfrac{3}{4}\hat{R}_2^2 - \tfrac{1}{2}t_2\hat{R}_1^2)$$
$$+ (1 - \tfrac{3}{4}\hat{R}_1^2 - \tfrac{1}{2}r_2\hat{R}_2^2)(1 - \tfrac{3}{4}\hat{R}_2^2 - \tfrac{1}{2}t_2\hat{R}_1^2) - r_2t_2\hat{R}_1^2\hat{R}_2^2 = 0,$$

which may be written as $m^2 + \alpha_2 m + \alpha_1 = 0$, where

$$\alpha_2 = -[2 - (\tfrac{3}{4} + t_2)\hat{R}_1^2 - (\tfrac{3}{4} + r_2)\hat{R}_2^2],$$
$$\alpha_1 = (1 - \tfrac{3}{4}\hat{R}_1^2 - \tfrac{1}{2}r_2\hat{R}_2^2)(1 - \tfrac{3}{4}\hat{R}_2^2 - \tfrac{1}{2}t_2\hat{R}_1^2) - r_2t_2\hat{R}_1^2\hat{R}_2^2.$$

The stability conditions are $\alpha_1 > 0$ and $\alpha_2 > 0$. Evaluating α_1 and α_2 at $\hat{R}_1 = 0$ and $\hat{R}_2 = 0$ gives $\alpha_1 = 2$ and $\alpha_2 = -2$, so that $\hat{R}_1 = 0$, $\hat{R}_2 = 0$ is unstable. Evaluating α_1 and α_2 at $\hat{R}_1 = 0$ and $\hat{R}_2 = 0$ gives $\alpha_2 = (1 + 2t_2) > 0$ and $\alpha_1 = 2(2t_2 - 1)$, so that $\alpha_1 > 0$ if $t_2 > \frac{1}{2}$; hence for $\hat{R}_1 = 2$, $\hat{R}_2 = 0$ to be stable $t_2 > \frac{1}{2}$. Similarly for $\hat{R}_1 = 0$ and $\hat{R}_2 = 2$ one has $\alpha_2 = (1 + 2t_2) > 0$ and $\alpha_1 = 2(2r_2 - 1) > 0$, so that $\alpha_1 > 0$ if $r_2 > \frac{1}{2}$; hence for $\hat{R}_1 = 0$, $\hat{R}_2 = 2$ to be stable $r_2 > \frac{1}{2}$. Finally, for

$$\hat{R}_1 = 2\sqrt{(1 - 2r_2)/(1 - 4r_2t_2)}, \qquad \hat{R}_2 = 2\sqrt{(1 - 2t_2)/(1 - 4r_2t_2)},$$
$$\alpha_2 = 4[1 - (r_2 + t_2)]/[1 - 4r_2t_2], \qquad \alpha_1 = (\hat{R}_1^2\hat{R}_2^2/4)(1 - 4r_2t_2),$$

and so in order that the equilibrium point $\hat{R}_1 \neq 0$, $\hat{R}_2 \neq 0$ (i.e., the simultaneous oscillation at ω and $\sqrt{2}\omega$) is stable one must have both $r_2t_2 < \frac{1}{4}$ and $1 > (r_2 + t_2)$. The results are summarized in Figure 3.

It is convenient to use P_1, P_2 and P_3 to denote the periodic solutions of the unforced system of equations (1); these are defined as follows: $P_1 : \hat{R}_1 \neq 0$, $\hat{R}_2 = 0$, single frequency oscillation at ω; $P_2 : \hat{R}_1 = 0$, $\hat{R}_2 \neq 0$, single frequency oscillation at $\sqrt{2}\omega$; $P_3 : \hat{R}_1 \neq 0$, $\hat{R}_2 \neq 0$, simultaneous oscillation at ω and $\sqrt{2}\omega$. Note that for $r_2 + t_2 > 1$ instability is predicted for P_3 and for $r_2 + t_2 = 1$ there is a zero eigenvalue. This is an example of a critical case and Lyapunov's first method based on the first order terms to predict stability is insufficient. The (r_2, t_2) plane shown in Figure 3 illustrates the stability properties of the different cases previously outlined. The regions I–IV are defined as follows: region

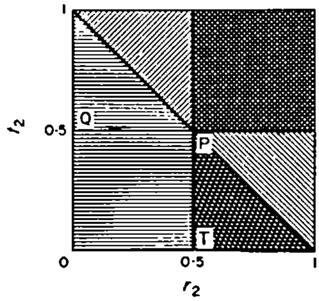

Figure 3. Behaviour of oscillations in the (r_2, t_2) plane. ▨, $\hat{R}_1=2$, $\hat{R}_2=0$; ▧, $\hat{R}_1=0$, $\hat{R}_2=2$; ▤, $\hat{R}_1\neq 0$, $\hat{R}_2\neq 0$. The line $r_2=t_2$ corresponds to $\hat{R}_1=\hat{R}_2$.

I: bounded by $r_2=0$, $t_2=\frac{1}{2}$, and $r_2=\frac{1}{2}$; region II: bounded by $t_2=0$, $t_2=\frac{1}{2}$, and $r_2=\frac{1}{2}$; region III: bounded by $t_2=\frac{1}{2}$, $r_2=\frac{1}{2}$; region IV: bounded by $t_2=\frac{1}{2}$, $r_2=\frac{1}{2}$, $t_2=0$, $r_2=0$. Note that these regions are open in the sense that the boundary lines are not included, from stability considerations. Again some of the regions have only two or three sides because specification of r_2 and/or t_2 is necessary to make them bounded on all sides. OU defines the equal amplitude locus and the amplitude is progressively decreasing from O onwards. In the following analyses for forced situations it is assumed that the r_2 and t_2 values are those in region (iii).

2.3. CASE 2: FORCING ($\bar{p}=0$) AT FREQUENCY ω

When forcing at frequency ω is applied to the first oscillator then from equation (18) $\hat{\phi}_1=\pm\pi/2$, so that $\hat{R}_1\neq 0$, $\hat{R}_2=0$ leads to

$$(1-\tfrac{1}{4}\hat{R}_1^2)\hat{R}_1=(\bar{p}/\omega)\sin\hat{\phi}_1. \tag{20}$$

From equation (20) it is obvious that $\hat{R}_1<2$ for $\hat{\phi}_1=\pi/2$ and $\hat{R}_1>2$ for $\hat{\phi}_1=-\pi/2$ if $\bar{p}>0$.

The variational equations for stability analysis in this case are

$$2\,\mathrm{d}\Delta R_1/\mathrm{d}\xi=(1-\tfrac{3}{4}\hat{R}_1^2-\tfrac{1}{2}r_2\hat{R}_2^2)\Delta R_1-r_2\hat{R}_1\hat{R}_2\Delta R_2-(\bar{p}/\omega)\cos\hat{\phi}_1\Delta\phi_1,$$

$$2\,\mathrm{d}\Delta R_2/\mathrm{d}\xi=(1-\tfrac{3}{4}\hat{R}_2^2-\tfrac{1}{2}t_2\hat{R}_1^2)\Delta R_2-t_2\hat{R}_1\hat{R}_2\Delta R_1,$$

$$2\omega\,\mathrm{d}\Delta\varphi_1/\mathrm{d}\xi=(\bar{p}/\hat{R}_1)\sin\hat{\varphi}_1\Delta\varphi_1+(\bar{p}/\hat{R}_1^2)\Delta R_1\cos\hat{\varphi}_1.$$

Evaluation of these variational equations at $\hat{R}_1\neq 0$, $\hat{R}_2=0$ and $\hat{\varphi}_1=\pm\pi/2$ leads to $2\,\mathrm{d}\Delta R_1/\mathrm{d}\xi=(1-\frac{3}{4}\hat{R}_1^2)\Delta R_1$, so that $\hat{R}_1$ is stable if $\hat{R}_1^2>\frac{4}{3}$,

$$2\,\mathrm{d}\Delta R_2/\mathrm{d}\xi=(1-\tfrac{1}{2}t_2\hat{R}_1^2)\Delta R_2, \tag{21}$$

so that $\hat{R}_2$ is stable if $\hat{R}_1^2>2/t_2$, and finally

$$2\omega\,\mathrm{d}\Delta\varphi_1/\mathrm{d}\xi=\pm(\bar{p}/\omega)\Delta\varphi_1, \tag{22}$$

from which it is clear that $\hat{\varphi}_1=\pi/2$ will give rise to instability in the phase of the solution, so that the only possible asymptotically stable phase is $\hat{\varphi}_1=-\pi/2$ if $\bar{p}>0$. From these results the following condition must be satisfied for stability of $R_1\neq 0$, $R_2=0$: $\hat{R}_1^2>\max(4/3, 2/t_2)$.

From equation (21) for $\hat{\varphi}_1=-\pi/2$ one has $(1-\hat{R}_1^2/4)\hat{R}_1=-\bar{p}/\omega$, or

$$(\tfrac{1}{4}\hat{R}_1^2-1)\hat{R}_1=\bar{p}/\omega. \tag{23}$$

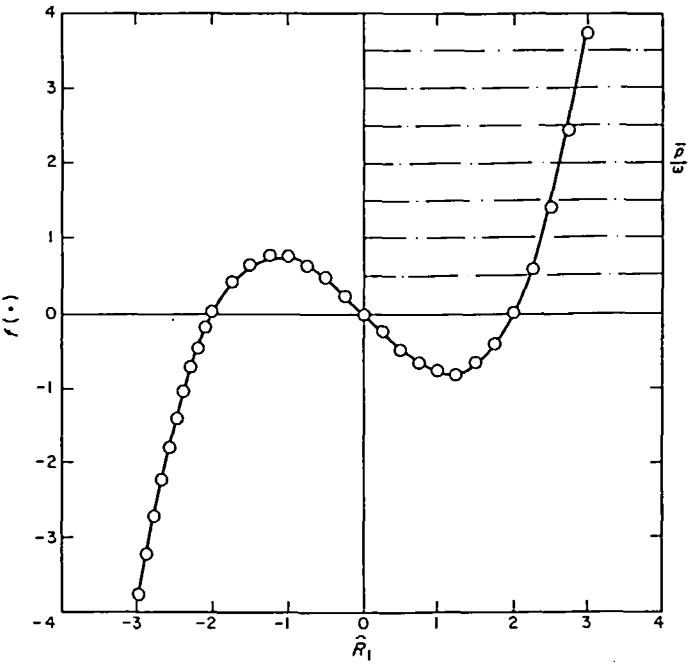

Figure 4. Stationary amplitude curve, $f(\hat{R}_1)=(-1+\hat{R}_1/4)\hat{R}_1$; $f(\circ)$, stationary amplitude curve; $\bar{p}$, forcing amplitude; ω, forcing frequency.

A plot of the expression on the left-hand side of equation (23) is shown on Figure 4. The intersections of the curve with lines parallel to the R_1-axis define the stationary amplitude. The ordinates of these parallel lines are proportional to the forcing amplitude (i.e., $\bar{p}/\omega$). It is obvious from Figure 4 that the only admissible stable oscillations have amplitudes greater than 2.

2.4. CASE 3: FORCING ($\bar{p}\neq 0$) AT FREQUENCY $\sqrt{2}\omega$

The amplitudes and phase equations for the case when the second oscillator is forced by a term $\bar{p}\sin\omega_2 t$ ($\omega_2=\sqrt{2}\omega$) are

$$2R_{1\xi}=(1-\tfrac{1}{4}R_1^2-\tfrac{1}{2}r_2R_2^2)R_1, \tag{24}$$

$$2R_{2\xi}=(1-\tfrac{1}{4}R_2^2-\tfrac{1}{2}t_2R_1^2)R_2-(\bar{p}/\omega_2)\sin\phi_2, \tag{25}$$

$$\phi_{1\xi}=0,\; 2\omega_2R_2\phi_{2\xi}=-\bar{p}\cos\phi_2. \tag{26, 27}$$

From equation (26) it is obvious that ϕ_1 is constant. The steady state amplitudes are obtained by solving

$$(1-\tfrac{1}{4}\hat{R}_1^2-\tfrac{1}{2}r_2\hat{R}_2^2)\hat{R}_1=0, \tag{28}$$

$$(1-\tfrac{1}{4}\hat{R}_2^2-\tfrac{1}{2}t_2\hat{R}_1^2)\hat{R}_2-(\bar{p}/\omega_2)\sin\hat{\phi}_2=0, \qquad \cos\hat{\phi}_2=0. \tag{29, 30}$$

From equations (28) and (29)

$$-(2t_2-1)(\hat{R}_1^2/4)+(2r_2-1)(\hat{R}_2^2/4)\hat{R}_1\hat{R}_2-(\hat{R}_1\bar{p}/\omega_2)\sin\hat{\phi}_2=0.$$

Now from equation (30), $\hat{\varphi}_2=\pm\pi/2$ in steady state. Since $\bar{p}\neq 0$, so $\hat{R}_2\neq 0$ from equation (29). Hence $\hat{R}_2=0$, $\hat{R}_1\neq 0$ is not a possible periodic solution, and the only periodic oscillation can be $\hat{R}_1=0$, $\hat{R}_2\neq 0$. The stationary amplitude of the single frequency

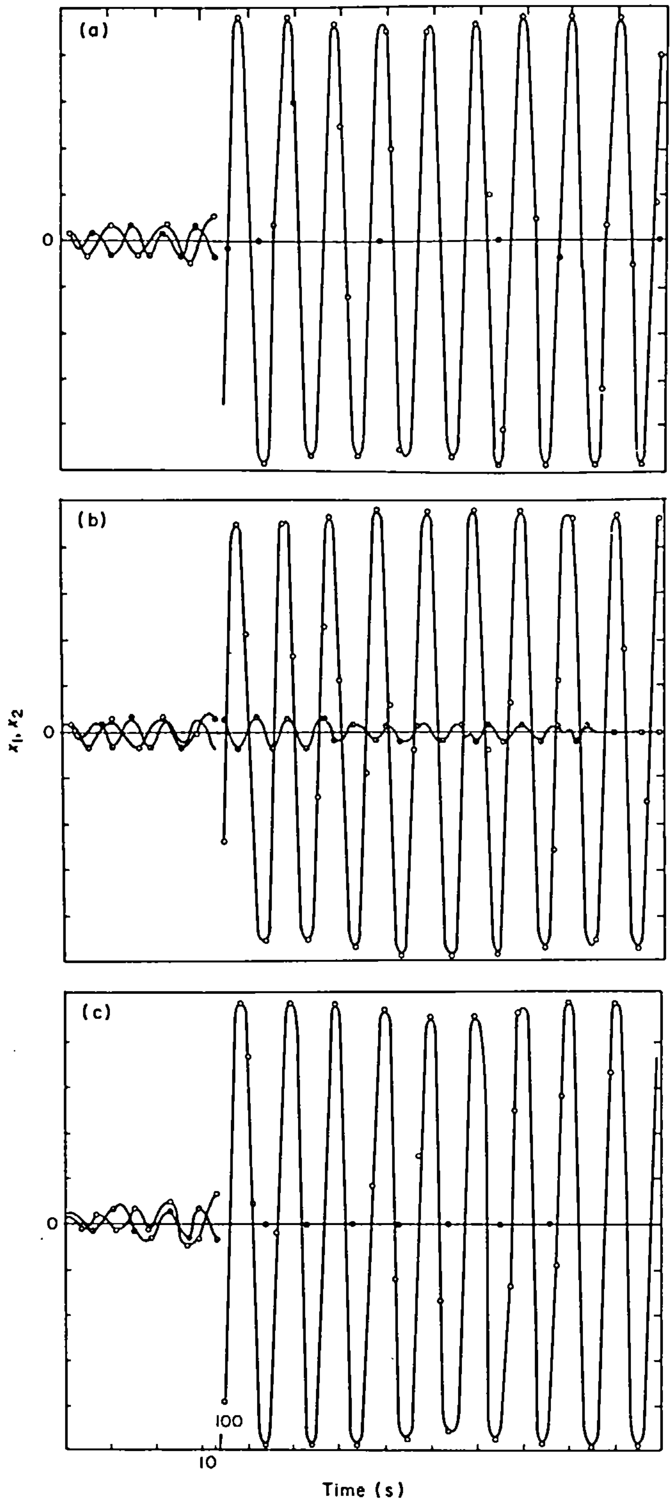

Figure 5. Time plots for forced mutually coupled van der Pol system. $S_1 = S_2 = 0{\cdot}1$, $r_2 = 1{\cdot}0$, $t_2 = 1{\cdot}0$, $\omega_1 = 2{\cdot}0$, $\omega_2 = 2{\cdot}828$; $x_1(0) = \dot{x}_1(0) = 0{\cdot}1$; $x_2(0) = \dot{x}_2(0) = 0{\cdot}0$. Horizontal scale 3·03 units/div. (a) $\bar{p} = 0{\cdot}5$, vertical scale 0·428 units/div; (b) $\bar{p} = 0{\cdot}25$, vertical scale 0·439 units/div; (c) $\bar{p} = 0{\cdot}75$, vertical scale 0·412 units/div.

oscillation at ω_2 is now obtained from $(1-\frac{1}{4}\hat{R}_2^2)\hat{R}_2-(\bar{p}/\omega_2)\sin\hat{\varphi}_2=0$, or

$$(1-\hat{R}_2^2/4)\hat{R}_2=(\bar{p}/\omega_2)\sin\hat{\varphi}_2. \tag{31}$$

From equation (30) it is obvious that $\hat{R}_2<2$ for $\hat{\varphi}_2=\pi/2$ and $\hat{R}_2>2$ for $\hat{\varphi}_2=-\pi/2$.

The variational equations for the stability analysis are

$$2\,\mathrm{d}\Delta R_1/\mathrm{d}\xi=(1-\tfrac{3}{4}\hat{R}_1^2-\tfrac{1}{2}r_2\hat{R}_2^2)\Delta R_1-r_2\hat{R}_1\hat{R}_2\Delta R_2,$$

$$2\,\mathrm{d}\Delta R_2/\mathrm{d}\xi=(1-\tfrac{3}{4}\hat{R}_2^2-\tfrac{1}{2}t_2\hat{R}_1^2)\Delta R_1-t_2\hat{R}_1\hat{R}_2\Delta R_1-(\bar{p}/\omega_2)\sin\hat{\varphi}_2\Delta\varphi_2,$$

$$2\omega_2\,\mathrm{d}\Delta\varphi_2/\mathrm{d}\xi=(\bar{p}/\hat{R}_2)\sin\hat{\varphi}_2\Delta\varphi_2+(\bar{p}/\hat{R}_2^2)\Delta R_2\cos\hat{\varphi}_2.$$

Evaluating these equations at $\hat{R}_1=0$, $\hat{R}_2\neq 0$ and $\hat{\varphi}_2=\pm\pi/2$ leads to

$$2\,\mathrm{d}\Delta R_1/\mathrm{d}\xi=(1-\tfrac{1}{2}r_2\hat{R}_2^2)\Delta R_1, \tag{32}$$

$$2\,\mathrm{d}\Delta R_2/\mathrm{d}\xi=(1-\tfrac{3}{4}\hat{R}_2^2)\Delta R_2\pm(\bar{p}/\omega_2)\Delta\varphi_2, \tag{33}$$

$$2\omega_2\,\mathrm{d}\Delta\varphi_2/\mathrm{d}\xi=\pm(\bar{p}/\hat{R}_2)\Delta\varphi_2. \tag{34}$$

It is clear from equation (34) that $+\pi/2$ will give rise to instability in the phase of the solution, and so the only possible asymptotically stable phase is $\hat{\varphi}_2=-\pi/2$. From equations (32) and (34) the stability of $\hat{R}_1=0$, $\hat{R}_2\neq 0$ is guaranteed if $\hat{R}_2^2>r_2/2$ and $R_2^2>4/3$, or

$$\hat{R}_2^2>\max(4/3,\ r_2/2). \tag{35}$$

Now the condition on $\bar{p}$ can be obtained so that the periodic solution at $\omega_2=\sqrt{2}\omega$ can be asymptotically stable as in Case 2. The condition for existence of $\hat{R}_2$ in its stable phase is $R_2>2{\cdot}0$. Substitution of the minimum of $\hat{R}_2$ into equation (25) for the stable phase leads to $(\bar{p}/\omega_2)>0$.

3. RESULTS OF DIGITAL SIMULATION

The time plots of the system (1) have been studied through digital computer simulation and the results for Case 2 are given in Figure 5. The plots show the disappearance of the oscillation at $\sqrt{2}\omega$ as the forcing amplitude is varied from a very low value to very high values. Further, the plots show that the oscillation at ω has an amplitude greater than 2, which agrees with the analytical results of Figure 4. The analytical findings for Case 3 can also be easily verified by digital computer simulation as for Case 2.

4. CONCLUSION

Quenching of a self-excited oscillation in a non-linearly coupled system of non-linear van der Pol oscillators has been predicted by applying the two time scale technique. The presence of a forcing term with the frequency of the forced system leads to the suppression of the other self-oscillations in the coupled system under the conditions specified in the analysis.

ACKNOWLEDGMENT

I should like to express my indebtedness to late Dr R. Subramanian for many useful discussions and suggestions.

REFERENCES

1. W. M. MANSOUR 1972 *Journal of Sound and Vibration* **25**, 395–405. Quenching of limit cycles of a van der Pol oscillator.
2. A. TONDL 1975 *Journal of Sound and Vibration* **42**, 251–260. Quenching of self-excited vibrations: equilibrium aspects.
3. A. TONDL 1975 *Journal of Sound and Vibration* **42**, 261–271. Quenching of self-excited vibrations: one- and two-frequency vibrations.
4. A. TONDL 1976 *Journal of Sound and Vibration* **45**, 285–294. Quenching of self-excited vibrations: effect of dry friction.
5. C. HAYASHI 1964 *Nonlinear Oscillations in Physical Systems*. New York: McGraw-Hill Book Company.
6. A. NAYFEH 1973 *Perturbation Methods*. New York: John Wiley and Sons.