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DYNAMICS OF TWO STRONGLY COUPLED VAN DER POL OSCILLATORS

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Abstract—A perturbation method is used to study the steady state behavior of two Van der Pol oscillators with strong linear diffusive coupling. It is shown that a bifurcation occurs which results in a transition from phase-locked periodic motions to quasi-periodic motions as the coupling is decreased or the detuning is increased. The analytical results are compared with a numerically generated solution.

1. INTRODUCTION

The object of this work is to study the dynamics of two Van der Pol oscillators with linear diffusive coupling. This system of non-linear, non-conservative limit cycle oscillators is described by a system of two second order ordinary differential equations:

$$\ddot{\mathbf{x}} + \mathbf{x} - \epsilon(\mathbf{I} - \chi^2)\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad \mathbf{A} = \alpha\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\Delta \end{pmatrix}, \quad \epsilon \ll 1.$$

Dots represent differentiation with respect to time, α is the diffusive coupling parameter, and Δ is related to the difference in natural frequency of the two uncoupled oscillators which will be referred to as the detuning parameter. In a previous work Rand and Holmes[1] studied this problem for small values of the coupling and detuning parameters. Here we extend their work by removing the assumption that α and Δ be small. The approach to the problem will be similar; we will use a perturbation analysis to reduce the study of the steady-state motions of these oscillators to the study of a system of algebraic equations.

Earlier work in this area has been done by several other investigators. Minorsky[2] has studied two Van der Pol oscillators with small coupling proportional to $\ddot{x}$. Hayashi and Kuramitsu[3] used an averaging method to study coupled Van der Pol and damped linear oscillators. Linkens[4, 5] has used the method of harmonic balance to study larger groups of Van der Pol oscillators with non-linear coupling. More in line with our approach is the work of Cohen and Neu[6] and Neu[7] who have used two-variable expansions to study phase-locking and rhythm splitting in pairs of weakly coupled limit cycle oscillators. More recently Neu[8, 9] has examined phase-locking of large populations of coupled oscillators using the two-variable expansion method to obtain approximate solutions to equations arising from a continuum model.

Problems concerning coupled non-linear oscillators frequently arise in biological applications. Many such examples can be found in the work of Pavlidis[10].

2. PERTURBATION ANALYSIS

We will use the two-variable expansion method to find an approximate solution of this system which is uniformly valid on a fixed finite time interval $0 < t < T = O(\epsilon^{-2})$. We shall neglect terms of order ϵ^2 throughout.

In order to obtain a zero order system that has a particularly simple solution (which will help to simplify the perturbation analysis), we first transform to the normal mode co-ordinates of the linear problem ($\epsilon = 0$).

We can write the co-ordinate transformation as

$$\mathbf{x} = \mathbf{R}\mathbf{y} \tag{2}$$

$$\mathbf{R} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{3}$$

where

$$a = [2(1 + s^2 + s\sqrt{(1 + s^2)})]^{-1/2}, \quad b = [2(1 + s^2 - s\sqrt{(1 + s^2)})]^{-1/2}$$

$$s = \frac{\Delta}{2\alpha}.$$

Note that $\mathbf{R}$ is an orthonormal matrix ($a^2 + b^2 = 1$) whose columns are the eigenvectors of $(\mathbf{I} - \mathbf{A})$ associated with the eigenvalues

$$\lambda_1 = 1 + \alpha(1 + s - \sqrt{(1 + s^2)}), \quad \lambda_2 = 1 + \alpha(1 + s + \sqrt{(1 + s^2)}). \tag{4}$$

Hence

$$\mathbf{R}^{-1} = \mathbf{R}^T.$$

Now we perform the transformation from $\mathbf{x}$ to $\mathbf{y}$ by substituting equation (2) into equation (1) and multiplying by $\mathbf{R}^{-1}$ on the left which gives

$$\ddot{\mathbf{y}} + \mathbf{I}\boldsymbol{\omega}\mathbf{y} = \epsilon\mathbf{R}^{-1}(\mathbf{I} - \chi^2)\mathbf{R}\dot{\mathbf{y}} \tag{5}$$

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \omega_i^2 = \lambda_i; \quad i = 1, 2.$$

Next we employ the two-variable expansion method by replacing the independent variable t (time) with two new independent variables ξ (stretched time) and η (slow time) as described by Cole[11] and Nayfeh[12]. Let:

$$\xi = \Omega t, \quad \Omega = 1 + \Omega_1\epsilon + \Omega_2\epsilon^2 + \,.. \tag{6}$$

$$\eta = \epsilon t. \tag{7}$$

Using the chain rule, derivatives with respect to t are now replaced by partial derivatives with respect to ξ and η.

$$\frac{\mathrm{d}}{\mathrm{d}t} = \Omega\frac{\partial}{\partial\xi} + \epsilon\frac{\partial}{\partial\eta}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \Omega^2\frac{\partial^2}{\partial\xi^2} + 2\epsilon\Omega\frac{\partial^2}{\partial\xi\partial\eta} + \epsilon^2\frac{\partial^2}{\partial\eta^2}. \tag{9}$$

Neglecting terms of $0(\epsilon^2)$, equation (5) becomes

$$(1 + 2\Omega_1\epsilon)\mathbf{y}_{\xi\xi} + 2\epsilon\mathbf{y}_{\xi\eta} + \mathbf{I}\boldsymbol{\omega}\mathbf{y} = \epsilon\mathbf{R}^{-1}(\mathbf{I} - \chi^2)\mathbf{R}\mathbf{y}_\xi \tag{10}$$

where alphabetic subscripts represent partial differentiation.

Next we expand the dependent variables $y_1(\xi, \eta)$ and $y_2(\xi, \eta)$ in power series in ϵ:

$$y_i(\xi, \eta) = y_{i0}(\xi, \eta) + \epsilon y_{i1}(\xi, \eta) + 0(\epsilon^2); \quad i = 1, 2. \tag{11}$$

Substituting equation (11) into equation (10), and equating coefficients of like powers of ϵ, we obtain the four equations

$$y_{10\xi\xi} + \omega_1{}^2 y_{10} = 0 \tag{12}$$

$$y_{20\xi\xi} + \omega_2{}^2 y_{20} = 0 \tag{13}$$

$$\begin{aligned} y_{11\xi\xi} + \omega_1{}^2 y_{11} = &-2\Omega_1 y_{10\xi\xi} - 2y_{10\xi\eta} \\ &+ [a^2(1 - a^2 y_{10}{}^2 - 2ab y_{10} y_{20} - b^2 y_{20}{}^2 \\ &+ b^2(1 - b^2 y_{10}{}^2 + 2ab y_{10} y_{20} - a^2 y_{20}{}^2)] y_{10\xi} \\ &+ [ab(1 - a^2 y_{10}{}^2 - 2ab y_{10} y_{20} - b^2{}_{20}{}^2) \\ &- ab(1 - b^2 y_{10}{}^2 + 2ab y_{10} y_{20} - a^2 y_{20}{}^2)] y_{20\xi} \end{aligned} \tag{14}$$

$$\begin{aligned} y_{21\xi\xi} + \omega_2{}^2 y_{21} = &-2\Omega_1 y_{20\xi\xi} - 2y_{20\xi\eta} \\ &+ [ab(1 - a^2 y_{10}{}^2 - 2ab y_{10} y_{20} - b^2 y_{20}{}^2) \\ &- ab(1 - b^2 y_{10}{}^2 + 2ab y_{10} y_{20} - a^2 y_{20}{}^2)] y_{10\xi} \\ &+ [b^2(1 - a^2 y_{10}{}^2 - 2ab y_{10} y_{20} - b^2 y_{20}{}^2) \\ &+ a^2(1 - b^2 y_{10}{}^2 + 2ab y_{10} y_{20} - a^2 y_{20}{}^2)] y_{20\xi}. \end{aligned} \tag{15}$$

Equations (12) and (13) have the solution

$$y_{10} = A(\eta) \sin \omega_1 \xi + B(\eta) \cos \omega_1 \xi \tag{16}$$

$$y_{20} = C(\eta) \sin \omega_2 \xi + D(\eta) \cos \omega_2 \xi. \tag{17}$$

We substitute (16) and (17) into (14) and (15) and require that $y_{11}(\xi, \eta)$ and $y_{21}(\xi, \eta)$ be uniformly valid for all $\xi > 0$. In order that y_{11} and y_{21} have no secular (resonance) terms, the coefficients of $\sin \omega_1 \xi$ and $\cos \omega_1 \xi$ on the right-hand side of (14) and the coefficients of $\sin \omega_2 \xi$ and $\cos \omega_2 \xi$ on the right-hand side of (15) must vanish, giving the equations:

$$8\Omega_1 \omega_1 A + 8B' - 4B + (a^4 + b^4) B(A^2 + B^2) + 4a^2 b^2 B(C^2 + D^2) = 0 \tag{18}$$

$$8\Omega_1 \omega_1 B - 8A' + 4A - (a^4 + b^4) A(A^2 + B^2) - 4a^2 b^2 A(C^2 + D^2) = 0 \tag{19}$$

$$8\Omega_1 \omega_2 C + 8D' - 4D + (a^4 + b^4) D(C^2 + D^2) + 4a^2 b^2 D(A^2 + B^2) = 0 \tag{20}$$

$$8\Omega_1 \omega_2 D - 8C' + 4C - (a^4 + b^4) C(C^2 + D^2) - 4a^2 b^2 C(A^2 + B^2) = 0 \tag{21}$$

where primes denote differentiation with respect to η.

Here we have excluded the special 3 : 1 resonance case[15] corresponding to $\omega_2 = 3\omega_1$ for which additional terms appear in equations (18)–(21).

Next we transform to polar co-ordinates,

$$y_{i0} = \rho_i(\eta) \cos [\omega_i \xi - \theta_i(\eta)], \quad i = 1, 2; \tag{22}$$

i.e.,

$$A(\eta) = \rho_1(\eta) \sin \theta_1(\eta), \quad B(\eta) = \rho_1(\eta) \cos \theta_1(\eta) \tag{23}$$

$$C(\eta) = \rho_2(\eta) \sin \theta_2(\eta), \quad D(\eta) = \rho_2(\eta) \cos \theta_2(\eta) \tag{24}$$

and replace (18)–(21) with (18)A + (19)B, (20)C + (21)D, (18)B–(19)A, and (20)D–(21)C, giving equations for the slowly varying amplitudes, $\rho_i(\eta)$ and phases, $\theta_i(\eta)$:

$$\rho_1{}^2\theta_1' - \Omega_1\omega_1\rho_1{}^2 = 0 \tag{25}$$

$$\rho_2{}^2\theta_2' - \Omega_1\omega_2\rho_2{}^2 = 0 \tag{26}$$

$$8\rho_1\rho_1' - 4\rho_1{}^2 + (a^4 + b^4)\rho_1{}^4 + 4a^2b^2\rho_1{}^2\rho_2{}^2 = 0 \tag{27}$$

$$8\rho_2\rho_2' - 4\rho_2{}^2 + (a^4 + b^4)\rho_2{}^4 + 4a^2b^2\rho_1{}^2\rho_2{}^2 = 0. \tag{28}$$

We now examine this system for equilibria which will correspond to steady-state solutions of the system (1). Up to this point Ω_1 has not been specified, but we must now take $\Omega_1 = 0$ in order to find non-trivial solutions which satisfy $\theta_1' = \theta_2' = 0$. We also simplify our notation by noting that ρ_1 and ρ_2 appear only to the second power and thus let:

$$\rho_1{}^2 = P, \quad \rho_2{}^2 = Q. \tag{29}$$

Equations (25) and (26) are now identically satisfied and the two remaining equations, (27) and (28), become:

$$P' = P(1 - gP - hQ) = F(P, Q) \tag{30}$$

$$Q' = Q(1 - hP - gQ) = G(P, Q) \tag{31}$$

where

$$g = \frac{1}{4}(a^4 + b^4), \quad h = a^2b^2. \tag{32}$$

This system of equations has four equilibrium points, where $F(P, Q) = G(P, Q) = 0$.

(i) $$P = Q = 0.$$

(ii) $$P = 0, \quad Q = \frac{1}{g} = \frac{8(1+s^2)}{1+2s^2}.$$

(iii) $$P = \frac{1}{g} = \frac{8(1+s^2)}{1+2s^2}, \quad Q = 0.$$

(iv) $$P = Q = \frac{1}{g+h} = \frac{8(1+s^2)}{3+2s^2}.$$

For the special case of $s^2 = (1/2)$, $g = h$ and there is a line of nonisolated equilibrium points where $P + Q = 6$ (see Fig. 2).

3. STABILITY AND SIGNIFICANCE OF EQUILIBRIA

Here we transform back to the original co-ordinates (x_1, x_2) to see what kind of motions correspond to the P, Q equilibria found in Section 2 and examine the linearized form of equations (30) and (31) to determine the stability of these motions.

The transformation is given by [cf. equations (29), (22), (3), (2)]:

$$x_{10} = a\sqrt{P}\cos(\omega_1\xi - \theta_1) - b\sqrt{Q}\cos(\omega_2\xi - \theta_2) \tag{33}$$

$$x_{20} = b\sqrt{P}\cos(\omega_1\xi - \theta_1) + a\sqrt{Q}\cos(\omega_2\xi - \theta_2). \tag{34}$$

The stability of the motion corresponds to that of the related equilibrium point in P, Q space and is determined by the trace and determinant of the linearization matrix

$$\mathbf{J} = \begin{pmatrix} \frac{\partial F}{\partial P} & \frac{\partial F}{\partial Q} \\ \frac{\partial G}{\partial P} & \frac{\partial G}{\partial Q} \end{pmatrix} \tag{35}$$

evaluated at that equilibrium point as developed in Minorsky[2] (see Fig. 1).

The first equilibrium, $P = Q = 0$, corresponds to the motion $x_1 = x_2 = 0$, for which

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{36}$$

$$\mathrm{Det}(\mathbf{J}) = 1 > 0, \quad \mathrm{Tr}(\mathbf{J}) = 2 > 0. \tag{37}$$

Hence this state where both oscillators are turned off is unstable for all parameter values. Note that the zero motion of a single uncoupled Van der Pol oscillator is also unstable and this property is preserved under the present coupling.

The next two equilibria $P = (1/g)$, $Q = 0$ and $P = 0$, $Q = (1/g)$ correspond to the phase-locked motions

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \frac{1}{\sqrt{g}} \cos(\omega_1\xi - \theta_1) \tag{38}$$

and

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \frac{1}{\sqrt{g}} \cos(\omega_2\xi - \theta_2) \tag{39}$$

respectively. These periodic solutions are non-linear normal modes in the sense of

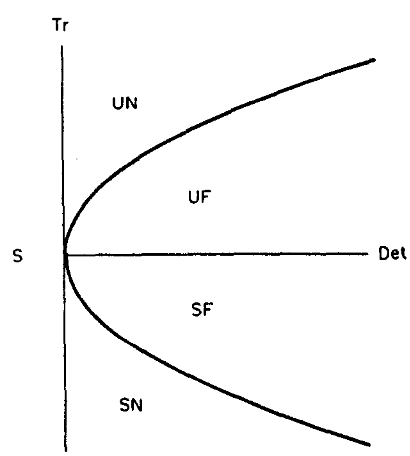

Fig. 1. Determination stability of equilibrium from the properties of the linearization matrix **J**. S = saddles (unstable); UN = unstable nodes; UF = unstable foci; SF = stable foci; SN = stable nodes. The node–focus transition curve is given by $Tr = 4$ Det.

Rosenberg[13]. The linearization matrix for both of these equilibrium points is

$$\mathbf{J} = \begin{pmatrix} 1 - \dfrac{h}{g} & 0 \\ -\dfrac{h}{g} & -1 \end{pmatrix} \tag{40}$$

$$\mathrm{Tr}(\mathbf{J}) = -\frac{h}{g} < 0. \tag{41}$$

(Note that h and g are positive.)

$$\mathrm{Det}(\mathbf{J}) = \frac{h}{g} - 1 = \frac{1}{4(1+s^2)} \cdot \frac{8(1+s^2)}{(1+2s^2)} - 1 = \frac{1-2s^2}{1+2s^2} \tag{42}$$

$$\mathrm{Tr}^2(\mathbf{J}) - 4 \cdot \mathrm{Det}(\mathbf{J}) = \frac{h^2}{g^2} - 4\frac{h}{g} + 4 = \left(\frac{h}{g} - 2\right)^2 \geq 0. \tag{43}$$

For $s^2 > (1/2)$, these points are stable nodes and the phase-locked motions are stable. These motions are unstable when $s^2 > (1/2)$ and the corresponding equilibria are saddles.

The last equilibrium point $P = Q = (1/(g+h))$ gives the motion:

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \frac{1}{\sqrt{(g+h)}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \cos(\omega_1 \xi - \theta_1) \\ \cos(\omega_2 \xi - \theta_2) \end{pmatrix} \tag{44}$$

$$\mathbf{J} = \frac{-1}{g+h} \begin{pmatrix} g & h \\ h & g \end{pmatrix} \tag{45}$$

$$\mathrm{Tr}(\mathbf{J}) = \frac{-2g}{g+h} < 0 \tag{46}$$

$$\mathrm{Det}(\mathbf{J}) = \frac{g^2 - h^2}{(g+h)^2} \tag{47}$$

$$\mathrm{Tr}^2(\mathbf{J}) - 4 \cdot \mathrm{Det}(\mathbf{J}) = \left(\frac{-2g}{g+h}\right)^2 - 4\frac{(g^2 - h^2)}{(g+h)^2} = \frac{4h^2}{g^2 + h^2} > 0. \tag{48}$$

This motion is, in general, quasi-periodic resulting from the summation of two periodic motions with noncommensurable frequencies. Such a motion is not periodic and there is no constant phase relation between the two oscillators. Here the stability is exactly the opposite of the previous case, stable for $s^2 > (1/2)$ and unstable for $s^2 < (1/2)$.

Using this information, we can construct a qualitative picture of the flow in the P, Q plane for various values of the parameter s as shown in Fig. 2, and can now understand the dynamic behavior of the system as follows. Recall that the parameter $s = (\Delta/2\alpha)$ represents the ratio of the detuning to the coupling. If the difference in the uncoupled frequencies is sufficiently small compared to the coupling, then almost all solutions tend towards a phase-locked periodic motion. As the parameter s is increased past the critical value $s^2 = (1/2)$, a bifurcation occurs which changes the stability of the non-trivial steady-state solutions. Such large values of the detuning (compared to the coupling) prevent phase-locking and the quasi-periodic motion results.

5. COMPARISON WITH NUMERICAL RESULTS

Having a set of analytical predictions at our disposal, a fourth order Runge–Kutta method was used tc numerically solve the original system of equations (1) to test these predictions.

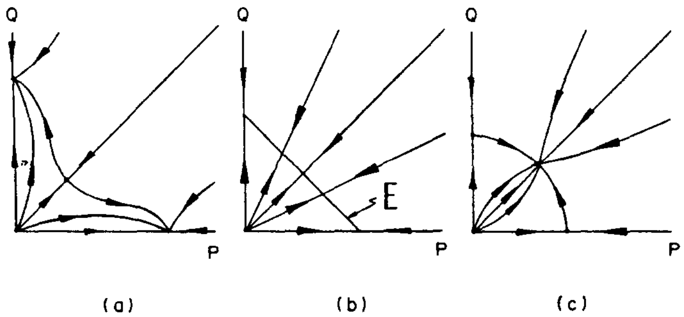

Fig. 2. Flow in the P, Q plane. (a) $s^2 < (1/2)$, (b) $s^2 = (1/2)$, (c) $s^2 > (1/2)$. The line E consists of nonisolated equilibria and is given by $P + Q = 6$.

The numerical experiments were begun by taking initial conditions close to the origin (i.e. at $t = 0$, $x_1 \ll 1$, $\dot{x}_1 \ll 1$, $x_2 \ll 1$, $\dot{x}_2 \ll 1$). For all such initial conditions and all values of s, the oscillation diverged from zero and approached some other solution. This confirmed that the solution with both oscillators 'turned off' is indeed unstable for all parameter values.

Next, the long term (after about 50 cycles) behavior of the solutions was examined over a range of values of s.

For $s^2 < (1/2)$, the perturbation analysis predicts that one mode should be 'turned off' (either $y_1 = 0(\epsilon)$ or $y_2 = 0(\epsilon)$ and we should be able to describe the solution by the amplitude and period of the active mode. To test this prediction, we performed a numerical integration of the system (1) and expressed the results in terms of the normal mode coordinates y_1, y_2 via the transformation (2). The amplitudes and periods of oscillation of y_1 and y_2 obtained by this method are given in Table 1 along with the analytic predictions. The percentage of the 'total amplitude' in the minor mode is plotted as a function of s in Fig. 3.

Both the amplitudes and periods show good quantitative agreement for values of $s < 0.2$ when we take $\epsilon = 0.1$. This confirms the existence of phase-locked solutions for small values of s.

Table 1. Comparison of analytical and numerical phase-locked solutions for $\epsilon = 0.1$*.

α	Δ	s^2	T	$\sim T$	A_1	$\sim A_1$	A_2	$\sim A_2$	r
0.1	0.00	0.0000	6.3	6.28	2.71	2.83	0.00	0.00	0.00
			6.3	6.28	0.00	0.00	2.71	2.83	0.00
0.1	0.01	0.0025	6.3	6.27	2.70	2.82	0.15	0.00	0.05
			5.7	5.72	0.15	0.00	2.67	2.82	0.05
0.1	0.05	0.0625	6.2	6.22	2.53	2.75	0.69	0.00	0.21
			5.7	5.67	0.71	0.00	2.50	2.75	0.22
0.1	0.10	0.2500	6.15	6.17	2.15	2.58	1.15	0.00	0.35
			5.6	5.59	1.2	0.00	2.1	2.58	0.38
1.0	0.00	0.0000	6.3	6.28	2.71	2.83	0.00	0.00	0.00
			6.3	6.28	0.00	0.00	2.71	2.83	0.00
1.0	0.05	0.0006	6.2	6.21	2.70	2.83	0.07	0.00	0.03
			3.6	3.63	0.06	0.00	2.44	2.83	0.02
1.0	0.10	0.0025	6.15	6.14	2.69	2.82	0.13	0.00	0.05
			3.6	3.60	0.12	0.00	2.43	2.82	0.05
1.0	0.25	0.016	5.95	5.94	2.65	2.80	0.33	0.00	0.11
			3.55	3.54	0.31	0.00	2.40	2.80	0.11

*A_1 and A_2 are the amplitude of oscillation of y_1 and y_2. T is the period of that oscillation. $\sim$ denotes analytical prediction. r is the fraction of the 'total amplitude' present in the minor mode, i.e.

$$r = \min\left(\frac{A_1}{A_1 + A_2}, \frac{A_2}{A_1 + A_2}\right).$$

The two lines of data for each set of parameter values correspond to different initial conditions.

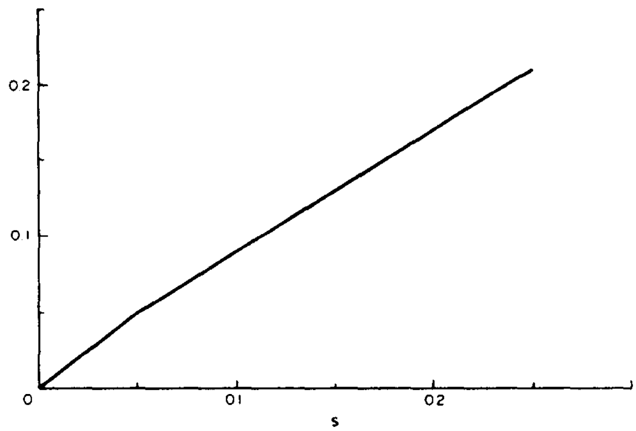

Fig. 3. Plot of fraction r of total amplitude present in minor mode versus parameter s from numerical integration data in Table 1. The plot is essentially the same for either mode and for either $\alpha = 0.1$ and $\alpha = 1$.

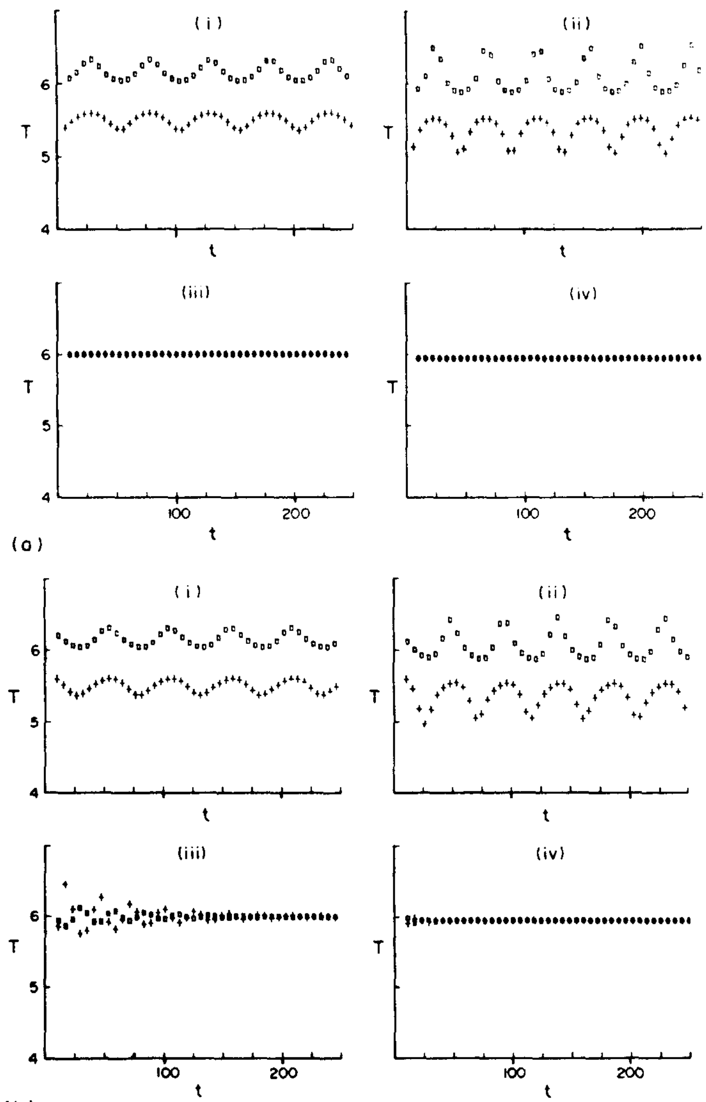

Fig. 4.(a) Theoretical predictions of period T versus time t for $\Delta = 0.25$. (i) $\alpha = 0.05$, $s^2 = 6.25$; (ii) $\alpha = 0.1$, $s^2 = 1.56$; (iii) $\alpha = 0.25$, $s^2 = 0.25$; (iv) $\alpha = 1$, $s^2 = 0.016$. In (i) and (ii) the upper and lower curves correspond to $x_1(t)$ and $x_2(t)$ respectively. (b) Numerical results for period T versus time t for $\Delta = 0.25$, $\epsilon = 0.1$. (i) $\alpha = 0.05$, $s^2 = 6.25$; (ii) $\alpha = 0.1$, $s^2 = 1.56$; (iii) $\alpha = 0.25$, $s^2 = 0.25$; (iv) $\alpha = 1$, $s^2 = 0.016$. In (i) and (ii) the upper and lower curves correspond to $x_1(t)$ and $x_2(t)$ respectively.

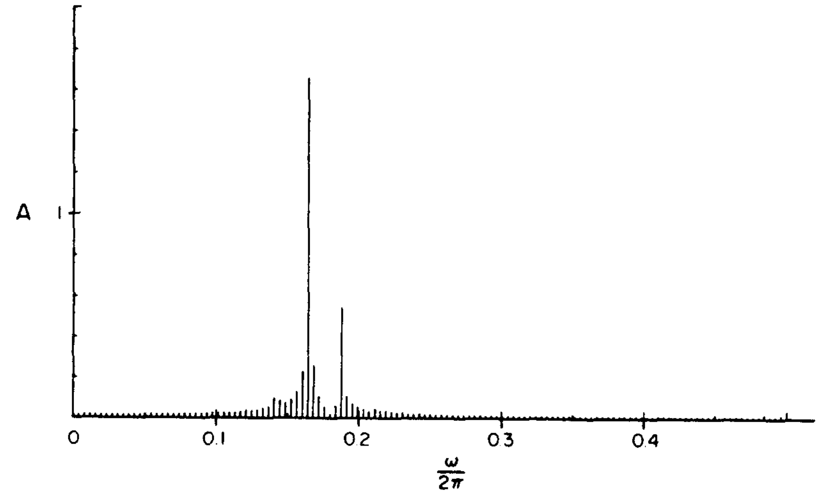

Fig. 5. Spectrum from fast Fourier transform of numerical solution for $\alpha = 0.1$, $\Delta = 0.25$, $\epsilon = 0.1$, $s^2 = 1.56$. A = amplitude, ω = frequency.

For larger values of s, the amplitude of the two modes become comparable and we can no longer describe the motion by a single period and amplitude. Instead two other methods are used to compare the numerical and analytical results.

The first method, due to Cohen *et al.*[14], is to plot the time between zero crossings with positive slope of each oscillator as a function of time. This is done for a sequence of parameter values where the detuning (Δ) is fixed and the coupling (α) is varied. The analytical predictions shown in Fig. 4(a) agree closely with the numerical results, given in Fig. 4(b), once the transients have died out. Also in Fig. 4, we can see the periodic change in 'period' referred to by Neu[9] as 'rhythm splitting'.

The second method of comparison employed a program which performed a fast Fourier transform on data produced by numerical integration. The output consists of a graph of the coefficients of the terms in the Fourier expansion of the numerically generated waveform.

The results of this procedure for a sample value of $s^2 > (1/2)$ are shown in Fig. 5 and, except for some small background contributions, agree very closely with the predictions of the perturbation analysis. The amplitudes of the two 'spikes' in the output are within 10% of the analytic values and the corresponding periods agree to within 1%.

One important difference between the numerical and analytic results should be noted however. Rather than the sharp transition from phase-locking to quasi-periodicity predicted by the perturbation analysis, the numerical results show a gradual change in the form of the solution. The motion slowly begins to pick up more than one main frequency component as s increases. This brings us back to the bifurcation which occurs in Fig. 2. The picture for $s^2 = (1/2)$ shows a line of non-isolated equilibrium points, which is not a structurally stable situation. A small perturbation of the flow could cause qualitative differences in the picture. The numerical results demonstrate that the prediction of this sharp transition is not correct and that there is something lost in neglecting terms small with respect to ϵ. It is expected that an analysis including these terms would show some smoother change occurring over a range of values of the parameter s.

6. CONCLUSIONS

By means of a two-variable expansion perturbation analysis we have found two types of approximate solution of the system of equations (1). The first is a periodic phase-locked solution, with phase difference 0 or π depending on initial conditions, which is stable for values of the parameter $s^2 < (1/2)$. The second is a quasi-periodic motion which is approached from almost all initial conditions if $s^2 > (1/2)$. Thus, we see that the loss of phase-locked oscillations occurs as the result of either increasing the detuning or decreasing the coupling in the system.

The analytical predictions agree closely with the numerical solutions of this system, except that the transition from phase-locked to quasi-periodic oscillations occurs gradually over a range of values of the parameter s in the numerically obtained solutions, in contrast to the sharp transition at $s^2 = (1/2)$ in the analytical solution.

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Résumé:

On utilise une méthode de perturbation pour étudier le comportement à l'état stationnaire de deux oscillateurs de Van der Pol avec un fort couplage diffusif linéaire. On montre qu'une séparation se produit qui résulte en une transition de mouvements périodiques à phases liées vers des mouvements quasi-périodiques lorsque le couplage décroit ou la désynchronisation s'accroit. On compare les résultats analytiques à une solution générée numériquement.

Zusammenfassung:

Eine Perturbationsmethode wird zur Untersuchung des stationären Verhaltens zweier Van-der-Pol-Schwinger mit stark linear diffusiver Kopplung verwendet. Es wird gezeigt, dass eine Verzweigung auftritt, welche einen Übergang von einer phasenfesten zu einer quasiperiodischen Bewegung bewirkt when die Kopplung verringert oder die Verstimmung vergrössert wird. Die analytischen Ergebnisse werden mit numerisch erzeugten Lösungen verglichen.