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MODE ANALYSIS OF A SYSTEM OF MUTUALLY COUPLED VAN DER POL OSCILLATORS WITH COUPLING DELAY

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Abstract—A system of mutually coupled van der Pol oscillators containing fifth-order conductance characteristic, with the coupling delay, are analyzed by using the non-linear mode analysis. In particular, it has been demonstrated that zero state, two single modes, and one double mode are stable only for sufficiently small τ.

The analytical results have been verified by using the digital simulation.

1. INTRODUCTION

The study of systems of mutually coupled non-linear oscillators is of interest to several fields of engineering, statistical mechanics, electronics, and biological science, and has been done experimentally and theoretically by several authors [1–3].

Recently, Datardina and Linkens [9] have investigated a coupled system of van der Pol oscillators with fifth-power non-linear characteristics for modeling the myoelectrical activity of the human large intestine.

In this paper we consider the effects of coupling delay [1, 10–12] on the processes of interaction of oscillators in the system discussed in [9] and theoretically investigate the properties of such a system.

The structure and stability of modes are clarified by using the non-linear mode analysis of Endo and Mori [4–6] and these results are compared with the results obtained by the digital computer study (digital simulation).

2. MODE ANALYSIS

The system under consideration is described by the differential–difference equations

$$\begin{aligned} &\ddot{x}_1' + \xi(s_1 - s_3 x_1'^2 + s_5 x_1'^4)\dot{x}_1' + \omega^2 x_1' - k x_{2\tau'}' = 0 \\ &\ddot{x}_2' + \xi(s_1 - s_3 x_2'^2 + s_5 x_2'^4)\dot{x}_2' + \omega^2 x_2' - k x_{1\tau'}' = 0 \end{aligned} \tag{1}$$

where k is the coupling factor (positive constant), $\dot{} \equiv (\mathrm{d}/\mathrm{d}t')$, $\ddot{} \equiv (d^2/dt'^2)$, $x_{1\tau'}' = x_1'(t' - \tau')$, $x_{2\tau'}' = x_2'(t' - \tau')$, and τ' is the delay time (positive constant).

The normalized equations of equation (1) are obtained by changing the amplitude and time as in

$$x_N' = \sqrt[4]{\left(\frac{s_1}{s_5}\right)} x_N (N = 1, 2),$$

$$\tau' = \tau/\omega, \tag{2}$$

and

$$t' = t/\omega.$$

By defining

$$\epsilon = \xi s_1/\omega, \quad \alpha = k/\omega^2$$

and

$$\beta = \frac{s_3}{\sqrt{(s_1 s_5)}} \cdot (\epsilon, \alpha \ll 1) \tag{3}$$

so

$$\begin{aligned} &\ddot{x}_1 + \epsilon(1 - \beta x_1^2 + x_1^4)\dot{x}_1 + x_1 - \alpha x_{2\tau} = 0 \\ &\ddot{x}_2 + \epsilon(1 - \beta x_2^2 + x_2^4)\dot{x}_2 + x_2 - \alpha x_{1\tau} = 0 \end{aligned} \tag{4}$$

where

$$\dot{} \equiv (\mathrm{d}/\mathrm{d}t), \ \ddot{} \equiv (\mathrm{d}^2/\mathrm{d}t^2), \ x_{1\tau} = x_1(t-\tau)$$

and

$$x_{2\tau} = x_2(t-\tau).$$

Let the vectors $\mathbf{x}$, $\mathbf{x}_\tau$, $\mathbf{X}$ and $\mathbf{Z}$ be

$$\begin{aligned} \mathbf{x} &= [x_1, x_2]^T \\ \mathbf{x}_\tau &= [x_{1\tau}, x_{2\tau}]^T \\ \mathbf{X} &= [x_1^3, x_2^3]^T \\ \mathbf{Z} &= [x_1^5, x_2^5]^T \end{aligned}$$

then (4) can be written in the matrix-vector notation as follows:

$$\ddot{\mathbf{x}} + \epsilon\dot{\mathbf{x}} - \frac{\epsilon}{3}\beta\dot{\mathbf{X}} + \frac{1}{5}\epsilon\dot{\mathbf{Z}} + \mathbf{x} - \alpha B\mathbf{x}_\tau = 0 \tag{5a}$$

where B is real and symmetric as

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5b}$$

If an orthogonal transformation is applied to (5a), then

$$\mathbf{x} = P\mathbf{y},\ P^TP = I(PP^T = I),\ I\text{: unit matrix}$$

$$\mathbf{y} = [y_1, y_2]^T,$$

and (6)

$$\mathbf{y}_\tau = [y_{1\tau}, y_{2\tau}]^T$$

and equation (5a) becomes

$$\ddot{\mathbf{y}} + \epsilon\dot{\mathbf{y}} - \frac{\epsilon}{3}\beta P^T\dot{\mathbf{X}} + \frac{1}{5}\epsilon P^T\dot{\mathbf{Z}} + \mathbf{y} = \alpha P^TBP\mathbf{y}_\tau. \tag{7}$$

Here B is easily diagonalized under the condition that matrix P is orthogonal, i.e., each column (or row) vector of P forms an orthonormal-system (see [6], [9])

$$P_{11} = P_{12} = P_{21} = \frac{1}{\sqrt{2}}$$

$$P_{22} = -\frac{1}{\sqrt{2}}. \tag{8}$$

Using these results, equation (7) can be rewritten as

$$\ddot{\mathbf{y}} + \epsilon\dot{\mathbf{y}} - \frac{\epsilon}{3}\beta P^T\dot{\mathbf{X}} + \frac{1}{5}\epsilon P^T\dot{\mathbf{Z}} + \mathbf{y} = \alpha B_\alpha \mathbf{y}_\tau \tag{9}$$

where

$$B_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In the above equation, the unperturbed equation (the equation with $\epsilon = 0$) is indicated by

$$\ddot{\mathbf{y}} + \mathbf{y} = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}\mathbf{y}_\tau \tag{10}$$

and the solutions (modes) of this equation are obtained as

$$y_1 = A_1 \sin(\omega_1 t + \theta_1) \tag{11a}$$

$$y_2 = A_2 \sin(\omega_2 t + \theta_2) \tag{11b}$$

$$1 - \omega_1^2 = \alpha \cos(\omega_1 \tau) \tag{11c}$$

$$1 - \omega_2^2 = -\alpha \cos(\omega_2 \tau). \tag{11d}$$

Next we will consider the perturbed equation of (9) (equation with $\epsilon \neq 0$) on the basis of these results. In order to accomplish this, the $P^T\dot{\mathbf{X}}$ and $P^T\dot{\mathbf{Z}}$ of (9) should be equivalently linearized in the $\mathbf{y}$ domain.

By the same way as in [6], we have the equivalent-linearized equations of (9)†:

$$\begin{aligned} \ddot{y}_1 + \epsilon\left\{1 - \frac{\beta}{8}(A_1^2 + 2A_2^2) + \frac{1}{32}(A_1^4 + 3A_2^4 + 6A_1^2A_2^2)\right\}\dot{y}_1 + y_1 &= \alpha y_{1\tau} \\ \ddot{y}_2 + \epsilon\left\{1 - \frac{\beta}{8}(2A_1^2 + A_2^2) + \frac{1}{32}(3A_1^4 + A_2^4 + 6A_1^2A_2^2)\right\}\dot{y}_2 + y_2 &= -\alpha y_{2\tau} \end{aligned} \tag{12}$$

Thus, the equivalent-linearized equations have been obtained as in equation (12). The averaged equations with regard to the amplitudes and the phases can then be derived from substituting the unperturbed solutions [equation (11)] into these equations, on condition that the amplitudes and the phases are slowly varying functions of time (compared to t). So, the averaged equations can be written as follows:

$$\dot{A}_1^2 = -\epsilon\left\{1 - \frac{\beta}{8}(A_1^2 + 2A_2^2) + \frac{1}{32}(A_1^4 + 3A_2^4 + 6A_1^2A_2^2)\right\}A_1^2 - \frac{\alpha}{\omega_1}\sin(\omega_1\tau)A_1^2 \tag{13a}$$

$$\dot{A}_2^2 = -\epsilon\left\{1 - \frac{\beta}{8}(2A_1^2 + A_2^2) + \frac{1}{32}(3A_1^4 + A_2^4 + 6A_1^2A_2^2)\right\}A_2^2 + \frac{\alpha}{\omega_2}\sin(\omega_2\tau)A_2^2 \tag{13b}$$

$$\dot{\theta}_1 = 0 \tag{13c}$$

$$\dot{\theta}_2 = 0. \tag{13d}$$

†In this paper the ratio of two frequencies (ω_1/ω_2) is assumed to be irrational (non-resonance). Strictly speaking, there may be some cases where this ratio can be rational for particular values of α and τ, but such cases are omitted.

From equation (13) it can be noted that the stability of modes is determined only by the amplitudes A_1 and A_2.

3. STABILITY PROBLEM

The stationary states of modes are determined by reducing all the first-order time derivatives in the averaged equations (13a) and (13b) to zero. The stability of a stationary state is then distinguished by introducing small disturbances around the stationary state and determining whether all the eigenvalues of the resultant Jacobian

$$J_{ij} = \frac{d}{dA_j^2}\left(\frac{d(A_i^2)}{dt}\right) \tag{14}$$

have negative real parts.

The elements of the Jacobian associated with (13a) and (13b) are:

$$\begin{aligned}
J_{11} &= -\epsilon\left\{1 - \frac{\beta}{4}(A_1^2 + A_2^2) + \frac{3}{32}(A_1^4 + A_2^4 + 4A_1^2A_2^2)\right\} - \frac{\alpha}{\omega_1}\sin(\omega_1\tau) \\
J_{12} &= -\epsilon\left\{-\frac{\beta}{4}A_1^2 + \frac{3}{16}(A_1^4 + A_1^2A_2^2)\right\} \\
J_{21} &= -\epsilon\left\{-\frac{\beta}{4}A_2^2 + \frac{3}{16}(A_1^2A_2^2 + A_2^4)\right\} \\
J_{22} &= -\epsilon\left\{1 - \frac{\beta}{4}(A_1^2 + A_2^2) + \frac{3}{32}(A_1^4 + A_2^4 + 4A_1^2A_2^2)\right\} + \frac{\alpha}{\omega_2}\sin(\omega_2\tau).
\end{aligned} \tag{15}$$

(a) *Stability of the zero state*

For the zero state to exist $A_1 = A_2 = 0$. Hence, from equation (15) the zero state is stable if the following inequalities are simultaneously satisfied

$$\begin{aligned}
-\epsilon - \frac{\alpha}{\omega_1}\sin(\omega_1\tau) < 0, \\
-\epsilon + \frac{\alpha}{\omega_2}\sin(\omega_2\tau) < 0.
\end{aligned} \tag{16}$$

(b) *Stability of the single modes*

(i) First we assume $A_1 \neq 0$ and $A_2 = 0$, then from equation (13a) the stationary value of A_1 is given by

$$-\epsilon\left(1 - \frac{\beta}{8}A_1^2 + \frac{1}{32}A_1^4\right) - \frac{\alpha}{\omega_1}\sin(\omega_1\tau) = 0. \tag{17}$$

It is apparent that this mode is stable provided

$$-\epsilon\left(-\frac{\beta}{8} + \frac{A_1^2}{16}\right) < 0 \tag{18a}$$

and

$$-\epsilon\left(1 - \frac{\beta}{4}A_1^2 + \frac{3}{32}A_1^4\right) + \frac{\alpha}{\omega_2}\sin(\omega_2\tau) < 0. \tag{18b}$$

The spatial dependence for mode ω_1 (in-phase single mode) can be obtained from equations (6), (8), (11a), and (11c) by setting $\theta_1 = 0$ as

$$\begin{aligned}
x_1 = \frac{1}{\sqrt{2}}A_1\sin(\omega_1 t), \\
x_2 = \frac{1}{\sqrt{2}}A_1\sin(\omega_1 t)
\end{aligned} \tag{19}$$

(ii) Next for $A_1 = 0$ and $A_2 \neq 0$, A_2 is then given by

$$-\epsilon\left(1-\frac{\beta}{8}A_2^2+\frac{1}{32}A_2^4\right)+\frac{\alpha}{\omega_2}\sin(\omega_2\tau)=0 \tag{20}$$

and this mode is stable if

$$\begin{aligned} -\epsilon\left(1-\frac{\beta}{4}A_2^2+\frac{3}{32}A_2^4\right)-\frac{\alpha}{\omega_1}\sin(\omega_1\tau)<0, \\ -\epsilon\left(-\frac{\beta}{8}+\frac{1}{16}A_2^2\right)<0. \end{aligned} \tag{21}$$

In the same manner as that described for mode ω_1, the spatial dependence for mode ω_2 (anti-phase single mode) is

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}A_2\sin(\omega_2 t), \\ x_2 &= -\frac{1}{\sqrt{2}}A_2\sin(\omega_2 t). \end{aligned} \tag{22}$$

(c) *Double mode stability*

For the double mode to be stable, both A_1 and A_2 must exist. Hence, these values are given by

$$\begin{aligned} -\epsilon\left\{1-\frac{\beta}{8}(A_1^2+2A_2^2)+\frac{1}{32}(A_1^4+3A_2^4+6A_1^2A_2^2)\right\}-\frac{\alpha}{\omega_1}\sin(\omega_1\tau)=0 \\ -\epsilon\left\{1-\frac{\beta}{8}(2A_1^2+A_2^2)+\frac{1}{32}(3A_1^4+A_2^4+6A_1^2A_2^2)\right\}+\frac{\alpha}{\omega_2}\sin(\omega_2\tau)=0. \end{aligned} \tag{23}$$

The characteristic equation for the double mode is

$$\lambda^2-(J_{11}+J_{22})\lambda+(J_{11}J_{22}-J_{12}J_{21})=0. \tag{24}$$

where J_{11}, J_{12}, J_{21} and J_{22} are given in equation (15). For a stable double mode to exist, it is necessary and sufficient to fulfil the conditions

$$\begin{aligned} J_{11}+J_{22}<0 \\ J_{11}J_{22}-J_{12}J_{21}>0. \end{aligned} \tag{25}$$

The spatial dependence is shown for the double mode

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}A_1\sin(\omega_1 t+\theta_1)+\frac{1}{\sqrt{2}}A_2\sin(\omega_2 t+\theta_2), \\ x_2 &= \frac{1}{\sqrt{2}}A_1\sin(\omega_1 t+\theta_1)-\frac{1}{\sqrt{2}}A_2\sin(\omega_2 t+\theta_2) \end{aligned} \tag{26}$$

θ_1 and θ_2 are arbitrary [see equations (6), (8), and (11) again].

Thus the four modes in the system under consideration have been analyzed, and their expressions (two single modes and one double mode) are given in equations (19), (22), and (26).

4. NUMERICAL RESULTS

Consider the case in which the parameters of equation (4) are given by

$$\epsilon=0.1, \quad \alpha=0.15, \quad \text{and} \quad \beta=3.4.$$

First we survey the stability of the zero state. The result is shown in Fig. 1. Hence, it is possible to consider that in the regions of the unstable zero state, each oscillator in the system described by equation (4) behaves as a 'soft' oscillator. (On the other hand, in the stable regions, each oscillator behaves as a 'hard oscillator.) This result agrees very well with the result obtained from the digital simulation.‡

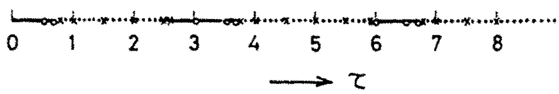

Fig. 1. Zero state. Solid line, analytical results (stable); dashed line, analytical results (unstable); open circles, digital simulation (stable); crosses, digital simulation (unstable).

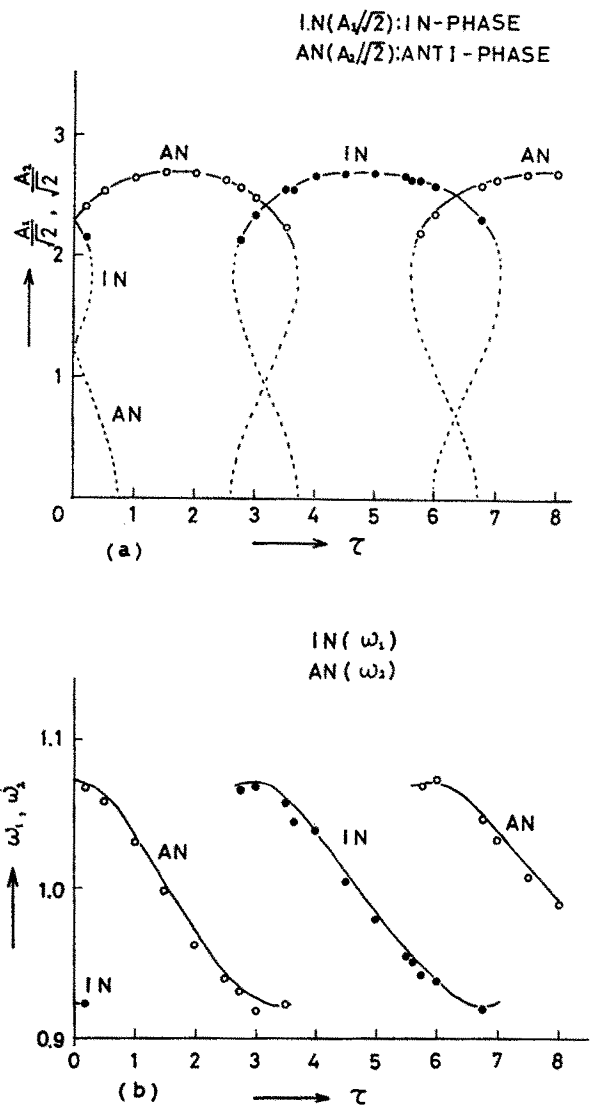

Fig. 2. (a) Amplitudes of single modes; (b) Frequencies of single modes. Solid line, analytical results (stable); dashed line, analytical results (unstable); solid circles, digital simulation (in-phase); open circles, digital simulation (anti-phase).

‡Numerical integrations of equations (4) were performed with respect to time with the modified fourth-order Runge-Kutta method [13].

Next the results of the analysis for the single modes are given in Fig. 2. These results have been checked against the digital simulation and good correlation between analytical and simulated results is shown in these figures.

Finally the double modes are analyzed. The results obtained are presented in Figs 3–6. From these results, it is proved that the double modes occur only for sufficiently small values of τ.

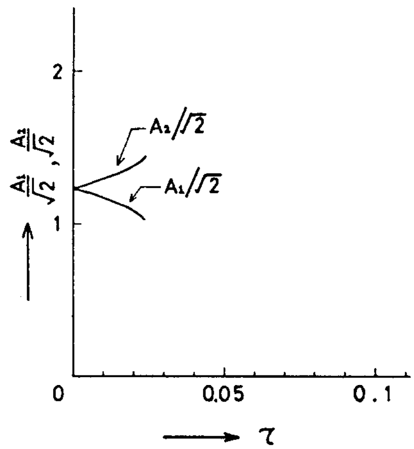

Fig. 3. Amplitudes of double modes. For this range of τ, the frequencies (ω_1, ω_2) of each mode are nearly equal to those at $\tau = 0$. Amplitudes of unstable double modes are not shown (analytical result).

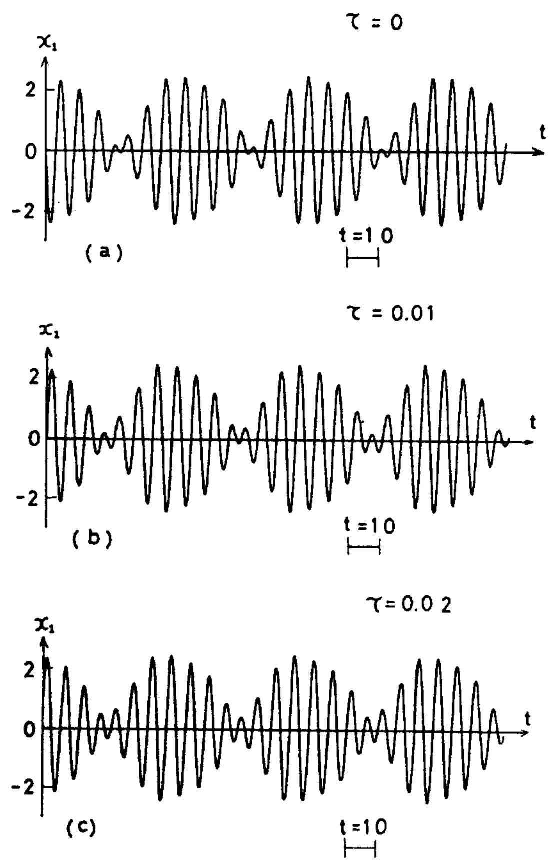

Fig. 4. For caption see overleaf.

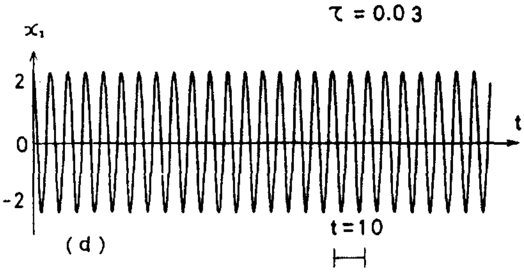

Fig. 4. Waveforms obtained by digital simulation.

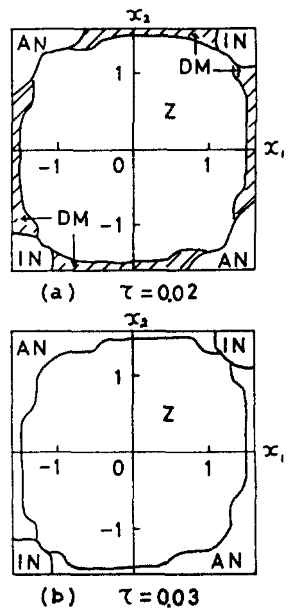

Fig. 5. Regions of attraction for stable modes [9]. Z, zero state; DM, double mode, x_1 and x_2 represent the initial conditions on the oscillator outputs ($x_1 = 0$, $x_2 = 0$ for $-\tau \leq t < 0$); (digital simulation).

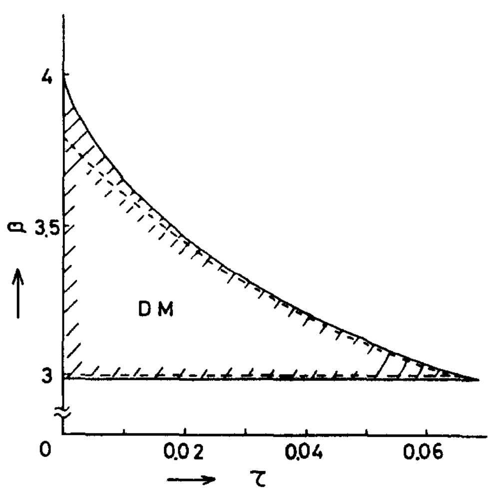

Fig. 6. Stable region of β for a double mode in terms of τ. Solid line shows analytical boundary; dotted line shows the boundary obtained from the digital simulation.

5. CONCLUSIONS

The properties of modes on a system of mutually coupled van der Pol oscillators containing fifth-order conductance characteristic, with the coupling delay, have been investigated by using the non-linear mode analysis and the results obtained have been checked against the digital simulation. From this investigation, we draw the following conclusions.

(1) The regions of stable zero state and those of unstable zero state alternate in the space of system parameters and delay.

(2) The regions in which the anti-phase single mode is stable and those in which the in-phase single mode is stable alternate in the space of system parameters and delay, and yet two single modes are stable in some intervals containing

$$\tau = \pi(n-1); \quad n = 1, 2, 3, \ldots.$$

(3) When the double modes occur, they are stable only for sufficiently small values of τ.

In this paper the two-oscillator case was considered. Both theoretical and experimental studies for the cases of more than two oscillators will be reported at a later date.

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Résumé:

En utilisant l'analyse de mode non linéaire on étudie un système d'oscillateurs de Van der Pol mutuellement couplés contenant une caractéristique de conductance du cinquième ordre avec retard de couplage. En particulier on a démontré que l'état zéro, deux modes simples et un mode double sont stables seulement pour un suffisamment petit.

On a vérifié les résultats analytiques avec une simulation numerique.

Zusammenfassung:

Unter Verwendung der nichtlinearen Schwingungsform-Analyse wird ein System miteinander gekoppelter van-der-Polscher Schwinger mit Kopplungsverzug und Leitungscharakteristiken der fuenften Ordnung untersucht. Im besonderen wird demonstriert, dass der Grundzustand, zwei Einzelschwingungsformen und eine Doppelschwingungsform nur fuer genuegend kleine Werte fuer τ stabil sind. Die analytischen ERgebnisse wurden mit Hilfe einer digitalen Simultaion bestaetigt.