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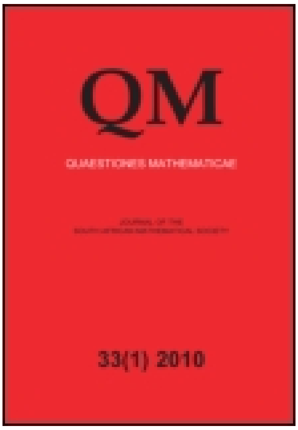

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ANALYSIS OF THE LIMIT CYCLE OF A GENERALISED VAN DER POL EQUATION BY A TIME TRANSFORMATION METHOD

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ANALYSIS OF THE LIMIT CYCLE OF A GENERALISED VAN DER POL EQUATION BY A TIME TRANSFORMATION METHOD

G.M. Moremedi and D.P. Mason

Dedicated to the memory of Siegfried Grässer

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Abstract. The properties of the limit cycle of a generalised van der Pol equation of the form $\ddot{u} + u = \varepsilon(1 - u^{2n})\dot{u}$, where ε is small and n is any positive integer, are investigated by applying a time transformation perturbation method due to Burton. It is found that as n increases the amplitude of the limit cycle oscillation decreases and its period increases. The time transformation solution is compared with the solution derived using the method of multiple scales and with a numerical solution. It is found that, to first order in ε, the time transformation solution for the limit cycle agrees better with the numerical solution than the multiple scales solution. Both perturbation solutions give the same result for the period of the limit cycle to second order in ε. The accuracy of the time transformation solution decreases as n increases.

1991 *Mathematics Subject Classification.* 34C05, 34E15.

1. Introduction. In this paper we will analyse the limit cycle of a generalised van der Pol equation of the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + u = \varepsilon(1 - u^{2n})\,\frac{\mathrm{d}u}{\mathrm{d}t}, \tag{1.1}$$

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where ε is a small positive parameter and n is any positive integer. When $n = 1$, (1.1) reduces to the van der Pol equation which has important applications, for instance, in electronic circuit theory. The van der Pol equation has been investigated extensively because it is one of the simplest equations which describes self-excited or self-sustained systems with one degree of freedom. The generalisation (1.1) is suitable for studying the effect on the limit cycle of increasing the degree of nonlinearity of the damping term, which may be done by increasing the value of the positive integer n.

The limit cycle of equation (1.1) has been investigated by Minorsky [14] and by Moremedi *et al.* [15]. Minorsky showed, by applying the stroboscopic method to (1.1), that the stationary amplitude, r_0^*, is given by

$$r_0^* = \left(\frac{\int_0^{2\pi} \sin^2 \psi_0 \mathrm{d}\psi_0}{\int_0^{2\pi} \sin^2 \psi_0 \cos^{2n} \psi_0 \mathrm{d}\psi_0} \right)^{\frac{1}{2n}} \tag{1.2}$$

and that when $n = 1, 2$ and 3, $r_0^* = 2$, 1.68 and 1.53 respectively. Minorsky also considered n as a rational number in which case u^{2n} in (1.1) has to be replaced by $| u^{2n} |$. For $n = \frac{1}{2}$, he found that $r_0^* = 2.36$. Minorsky concluded that for $n > 1$, $r_0^* < 2$ and for $0 < n < 1$, $r_0^* > 2$, where $r_0^* = 2$ is the value of the stationary amplitude for the van der Pol equation. Moremedi *et al.* [15] applied the derivative expansion version of the method of multiple scales and calculated the limit cycle correct to first order in ε and the period of the limit cycle correct to second order in ε for all values of the positive integer n. They investigated the manner in which the phase portrait of the limit cycle and its period depend on n. In the method of multiple scales the limit cycle is obtained by first calculating the phase trajectory for arbitrary initial conditions and then by letting $t \to \infty$ in its amplitude. In this paper we will apply the time transformation perturbation method, which was introduced by Burton [3] for calculating limit cycles, to equation (1.1). In this method the limit cycle is obtained directly by looking for a steady-state solution in the form of a Fourier series. We will investigate how the properties of the limit cycle depend on n and we will compare the results obtained using the time transformation method with those derived using

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the method of multiple scales and numerical methods. Burton [3] applied the time transformation method to determine the limit cycle of the van der Pol equation and found that for $0 < \epsilon \lesssim 2$, the results obtained by the time transformation method were more accurate than those derived by standard perturbation methods such as the generalised method of averaging.

The time transformation method was introduced by Burton [3] to analyse the limit cycles of autonomous nonlinear oscillators governed by a second order ordinary differential equation with a small parameter. It has been extended by Burton [4] to nonlinear damped oscillators, by Burton and Hamdan [5] to nonlinear autonomous conservative oscillators and by Hamdan and Burton [11] to forced nonlinear undamped oscillators. The method consists in transforming to a new time $T(t)$ such that in the domain of T the motion is simple harmonic. A steady-state solution in the form of a Fourier series is derived which gives a perturbation solution for the limit cycle directly. Unlike the method of multiple scales, the time transformation method does not give other trajectories in the phase plane and it does not provide information on the rate at which the system point in the phase plane tends to the limit cycle. This is not a disadvantage if it is the limit cycle only that is required.

The limit cycle of the van der Pol equation has been studied recently by several authors for values of ε which are small [8], [16], [17], [6], moderate [3], [7] and large [9], [1]. Other generalisations of the van der Pol equation, besides equation (1.1), have also been considered. The equation

$$\frac{d^2u}{dt^2} + u^{2n+1} = \varepsilon(1 - u^{2n+2})\frac{du}{dt} + \varepsilon a \cos\omega t\,, \tag{1.3}$$

where n is a positive integer and $a > 0$ and $\omega > 0$ are constants independent of ε, has been analysed mathematically by Obi [19]. Equation (1.3) reduces to the free van der Pol equation when $n = 0$ and $a = 0$. Obi obtained results on the number of periodic oscillations of (1.3) and showed that when $a = 0$ it has exactly one periodic oscillation. Nguyen [18] has considered a generalised van der Pol equation of the form

$$\frac{d^2u}{dt^2} + \omega^2 u = \varepsilon\left[1 - (u + q\cos\nu t)^2\right]\frac{du}{dt}\,, \tag{1.4}$$

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where ω, q and ν are constants independent of ε; when $q = 0$, (1.4) reduces to the van der Pol equation. Nguyen determined the stationary values of the amplitude of the oscillation when $\nu = \omega$ and $\nu = 3\omega$ and investigated the stability of the stationary oscillations. Holmes and Rand [12] have considered a generalisation of the van der Pol equation of the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \beta u + \delta u^3 = -(\alpha + \gamma u^2)\,\frac{\mathrm{d}u}{\mathrm{d}t}\,, \tag{1.5}$$

where α, β, γ and δ are constants. Equation (1.5) corresponds to the van der Pol equation when $\alpha < 0$, $\beta > 0$, $\gamma > 0$ and $\delta = 0$. Holmes and Rand examined the qualitative behaviour of the nonlinear oscillations governed by (1.5) and investigated the presence of local and global bifurcations.

An outline of the paper is as follows. The time transformation method is briefly described in Section 2. In Section 3 the quantities required in the time transformation method to calculate the limit cycle of (1.1) to first order in ε and the period of the limit cycle to second order in ε for all positive integers n are derived. The period of the limit cycle is calculated to second order in ε because the first order term in ε is found to vanish. In Section 4 the dependence of the properties of the limit cycle on n are investigated and the accuracy of the perturbation solutions for the limit cycle derived using the time transformation method and the method of multiple scales is compared with the aid of the numerical solution. The period of the limit cycle is considered in Section 5. Finally, concluding remarks are made in Section 6.

2. The time transformation method. Consider nonlinear oscillations governed by the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + u = -\varepsilon f\left(u,\ \frac{\mathrm{d}u}{\mathrm{d}t}\right)\,, \tag{2.1}$$

where the damping $\varepsilon f\left(u,\ \frac{\mathrm{d}u}{\mathrm{d}t}\right)$ depends on the amplitude and produces a limit cycle in the phase plane $\left(u,\ \frac{\mathrm{d}u}{\mathrm{d}t}\right)$. We suppose that the oscillation has a single maximum and a single minimum during one cycle of motion. We look for a one-to-one transformation between the time t and a new time T such that in the domain of T the motion is simple harmonic motion. The solution in the T domain is therefore

$$u = a\cos T\ , \tag{2.2}$$

where the constant a is the amplitude of the limit cycle and the period of the limit cycle is 2π in T.

We first rewrite (2.1) with T as independent variable. We have

$$F^2\frac{\mathrm{d}^2u}{\mathrm{d}T^2} + F\,\frac{\mathrm{d}F}{\mathrm{d}T}\,\frac{\mathrm{d}u}{\mathrm{d}T} + u = -\varepsilon f\left(u,\,F\,\frac{\mathrm{d}u}{\mathrm{d}T}\right)\ , \tag{2.3}$$

where

$$F(T) = \frac{\mathrm{d}T}{\mathrm{d}t}\ . \tag{2.4}$$

Equation (2.2) for u is then substituted into (2.3) to give the following first order ordinary differential equation for $F(T)$:

$$\sin T\,F\,\frac{\mathrm{d}F}{\mathrm{d}T} - (1-F^2)\cos T = \frac{\varepsilon}{a}\,f(a\cos T, -aF\sin T)\ . \tag{2.5}$$

The perturbation procedure consists in expanding $F(T)$ and a in power series in ε according to

$$F(T) = 1 + \epsilon F_1(T) + \epsilon^2 F_2(T) + \mathrm{O}(\epsilon^3), \tag{2.6}$$

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \mathrm{O}(\epsilon^3), \tag{2.7}$$

as $\varepsilon \to 0$, where O is the Landau symbol (Nayfeh [16]) and we have put $F_0(T) = 1$ because when $\varepsilon = 0$ the motion is simple harmonic motion in the t domain and $F(T) = 1$. This perturbation procedure is different from usual perturbation methods in that the dependent variable u is not expanded in a power series in ε. Equations (2.6) and (2.7) are substituted into (2.5) and by equating the coefficients of like powers of ε a set of first order ordinary differential equations for $F_i(T)$, $i \geq 1$, is obtained. To obtain a limit cycle solution each $F_i(T)$ must be periodic with period π or 2π depending on whether $f\left(u,\ \frac{\mathrm{d}u}{\mathrm{d}t}\right)$ contains odd or even nonlinearities,

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respectively. The constants a_i in the expansion (2.7) of the amplitude are chosen to ensure that $F_i(T)$ has the required periodicity.

The limit cycle in the phase plane $\left(u, \frac{du}{dt}\right)$ is given by

$$u = a\cos T\,, \tag{2.8}$$

$$\frac{du}{dt} = -aF(T)\sin T\,, \tag{2.9}$$

and by letting T vary from 0 to 2π. To obtain the limit cycle it is not necessary to determine $T(t)$. It is sufficient to know $F(T)$ defined by (2.4). Correct to first order in ε the limit cycle is given by

$$u = (a_0 + \epsilon a_1)\cos T + \ O\ (\epsilon^2), \tag{2.10}$$

$$\frac{du}{dt} = -[a_0 + \epsilon(a_1 + a_0 F_1(T))]\sin T + \ O(\epsilon^2), \tag{2.11}$$

as $\varepsilon \to 0$.

The time transformation, $t = t(T)$, and the period of the limit cycle, τ, can also be expressed in terms of $F(T)$. For, it follows directly form (2.4) that if we choose $T = 0$ when $t = 0$, then

$$t = \int_0^T \frac{dT'}{F(T')}\,. \tag{2.12}$$

Expanded correct to second order in ε, (2.12) is

$$t = \int_0^T \left[1 - \epsilon F_1(T') + \epsilon^2(F_1^2(T') - F_2(T')) + O(\epsilon^3)\right] dT', \tag{2.13}$$

as $\varepsilon \to 0$. The period of the limit cycle is

$$\tau = \int_0^{2\pi} \frac{dT}{F(T)} \tag{2.14}$$

and (2.14), expanded correct to second order in ε, is

$$\tau = \int_0^{2\pi} \left[1 - \epsilon F_1(T) + \epsilon^2(F_1^2(T) - F_2(T)) + \ O(\epsilon^3)\right] dT, \tag{2.15}$$

as $\varepsilon \to 0$.

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3. Perturbation solution. We will now use the time transformation method to calculate the limit cycle of the generalised van der Pol equation (1.1) correct to first order in ε and the period of the limit cycle correct to second order in ε for all values of the positive integer n.

By comparing equation (1.1) with the standard equation (2.1) it follows that

$$f(a\cos T,\ -aF\sin T) = (1 - a^{2n}\cos^{2n} T)Fa\sin T \tag{3.1}$$

and therefore (2.5) for $F(T)$ becomes

$$\sin TF\,\frac{\mathrm{d}F}{\mathrm{d}T} - (1 - F^2)\cos T = \varepsilon(1 - a^{2n}\cos^{2n} T)F\sin T\,. \tag{3.2}$$

When $\varepsilon = 0$, (3.2) is satisfied by the zero order term, $F_0 = 1$, in the perturbation expansion (2.6), as required. We substitute the perturbation expansions (2.6) and (2.7) into (3.2) and equate the coefficients of like powers of ε. There are no terms independent of ε.

(i) *First order in* ε

The differential equation for $F_1(T)$ is

$$\sin T\,\frac{\mathrm{d}F_1}{\mathrm{d}T} + 2\cos T\,F_1 = (1 - a_0^{2n}\cos^{2n} T)\sin T\,. \tag{3.3}$$

Now, it can be verified with the aid of the identity [10]

$$\cos^{2n} T = \frac{1}{2^{2n}}\left[\binom{2n}{n} + 2\sum_{k=1}^{n}\binom{2n}{n-k}\cos 2kT\right], \tag{3.4}$$

where

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}\,,\quad \binom{n}{0} = 1\,, \tag{3.5}$$

that

$$\cos^{2n} T\sin T = \frac{(2n)!}{2^{2n}}\sum_{k=0}^{n}\frac{(2k+1)}{(n-k)!(n+k+1)!}\sin(2k+1)T\,. \tag{3.6}$$

By using (3.6), (3.3) becomes

$$\text{(3.7)}\quad \sin T\,\frac{\mathrm{d}F}{\mathrm{d}T}+2\cos T F_1=\left[1-\frac{(2n)!}{n!(n+1)!}\left(\frac{a_0}{2}\right)^{2n}\right]\sin T$$
$$-(2n)!\left(\frac{a_0}{2}\right)^{2n}\sum_{k=1}^{n}\frac{(2k+1)}{(n-k)!(n+k+1)!}\sin(2k+1)T\,.$$

We look for a steady-state solution for $F_1(T)$ by considering a Fourier series of period π:

$$\text{(3.8)}\qquad F_1(T)=\sum_{k=0}^{\infty}A_k^{(1)}(n)\cos 2kT+\sum_{k=1}^{\infty}B_k^{(1)}(n)\sin 2kT\,,$$

where the Fourier coefficients, $A_k^{(1)}(n)$ and $B_k^{(1)}(n)$, are constants to be determined. If (3.8) is substituted into (3.7) then (3.7) takes the form

$$2A_0^{(1)}(n)\cos T+\sum_{k=1}^{\infty}\left[(k+1)A_k^{(1)}(n)-k\,A_{k+1}^{(1)}(n)\right]\cos(2k+1)T$$
$$+\sum_{k=1}^{\infty}\left[(k+1)B_k^{(1)}(n)-k\,B_{k+1}^{(1)}(n)\right]\sin(2k+1)T$$

$$\text{(3.9)}\qquad =\left[1-\frac{(2n)!}{n!(n+1)!}\left(\frac{a_0}{2}\right)^{2n}\right]\sin T$$
$$-(2n)!\left(\frac{a_0}{2}\right)^{2n}\sum_{k=1}^{n}\frac{(2k+1)}{(n-k)!(n+k+1)!}\sin(2k+1)T\,.$$

By considering the coefficients of $\cos(2k+1)T$ for $k\geq 0$ in (3.9) it follows that

$$\text{(3.10)}\qquad A_k^{(1)}(n)=0\,,\qquad k\geq 0\,.$$

Also, the coefficient of $\sin T$ in (3.9) must vanish. Thus

$$\text{(3.11)}\qquad a_0=2\left(\frac{n!(n+1)!}{(2n)!}\right)^{\frac{1}{2n}}\,.$$

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Equation (3.11) for a_0 agrees with Minorsky's [14] result (1.2) for the stationary amplitude r_0^*. For, by using the identity (3.4) applied to $\cos^{2n}\psi_0$ and $\cos^{2(n+1)}\psi_0$, (1.2) is readily integrated and it can be verified that $r_0^* = a_0$. Lastly, by equating the coefficients of $\sin(2k+1)T$ for $k \geq 1$ we find that

(3.12)
$$B_{k+1}^{(1)}(n) - \frac{(k+1)}{k} B_k^{(1)}(n) = \frac{(2k+1)n!(n+1)!}{k(n-k)!(n+k+1)!}, \qquad 1 \leq k \leq n,$$

$$B_k^{(1)}(n) = 0, \qquad k \geq n+1. \tag{3.13}$$

Equation (3.12) is a first order linear inhomogeneous difference equation for $B_k^{(1)}(n)$. We solve (3.12) subject to the boundary condition

$$B_{n+1}^{(1)}(n) = 0. \tag{3.14}$$

Now, the summing factor of the difference equation (3.12) is [2]

$$\left[\prod_{j=1}^{k} \frac{(j+1)}{j}\right]^{-1} = \frac{1}{k+1}. \tag{3.15}$$

By multiplying both sides of (3.12) by $(k+1)^{-1}$ we obtain

(3.16)
$$\frac{B_{k+1}^{(1)}(n)}{k+1} - \frac{B_k^{(1)}(n)}{k} = \frac{(2k+1)n!(n+1)!}{k(k+1)(n-k)!(n+k+1)!}, \qquad 1 \leq k \leq n.$$

In (3.16), $\frac{1}{k} B_k^{(1)}(n)$ appears as an exact discrete derivative. By summing both sides of (3.16) from 1 to $k-1$ we obtain

(3.17)
$$\frac{1}{k} B_k^{(1)}(n) = B_1^{(1)}(n) + n!(n+1)! \sum_{j=1}^{k-1} \frac{(2j+1)}{j(j+1)(n-j)!(n+j+1)!},$$
$$1 \leq k \leq n+1.$$

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By imposing the boundary condition (3.14) on (3.17) we find that

$$B_1^{(1)}(n) = -n!(n+1)!\sum_{j=1}^{n} \frac{(2j+1)}{j(j+1)(n-j)!(n+j+1)!} . \tag{3.18}$$

Thus (3.17) becomes

$$B_k^{(1)}(n) = -kn!(n+1)!\sum_{j=k}^{n} \frac{(2j+1)}{j(j+1)(n-j)!(n+j+1)!} , \quad 1 \le k \le n . \tag{3.19}$$

The steady-state solution of (3.3) is therefore

$$F_1(T) = \sum_{k=1}^{n} B_k^{(1)}(n) \sin 2kT , \tag{3.20}$$

where $B_k^{(1)}(n)$ for $1 \le k \le n$ is given by (3.19).

(ii) *Second order in* ε

We see from the perturbation expansions (2.10) and (2.11) that to calculate the limit cycle to first order in ε it remains to obtain a_1 which was not determined from the first order equation (3.3). Also, from the perturbation expansion (2.15) and from (3.20) for $F_1(T)$ it follows that the first order in ε contribution to the period τ vanishes because $F_1(T)$ is periodic with period π. The order ε^2 contribution to τ is obtained from the non-periodic terms in $F_1^2(T)$ and $F_2(T)$. To obtain τ correct to order ε^2 it therefore remains to calculate the non-periodic terms in $F_2(T)$.

The coefficient a_1 and the non-periodic terms in $F_2(T)$ are derived by considering the differential equation for $F_2(T)$ which is obtained by substituting (2.6) and (2.7) into (3.2) and equating the coefficients of ε^2:

(3.21)

$$\sin T \,\frac{dF_2}{dT} + 2\cos T\, F_2 = -2na_0^{2n-1}a_1 \cos^{2n} T \sin T - F_1 a_0^{2n} \cos^{2n} T \sin T - F_1 \frac{dF_1}{dT} \sin T - F_1^2 \cos T + F_1 \sin T .$$

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Consider a steady-state solution for $F_2(T)$ of the form

$$F_2(T) = \sum_{k=0}^{\infty} A_k^{(2)}(n) \cos 2kT + \sum_{k=1}^{\infty} B_k^{(2)}(n) \sin 2kT \,, \tag{3.22}$$

where the Fourier coefficients, $A_k^{(2)}(n)$ and $B_k^{(2)}(n)$, are constants. If (3.22) is substituted into (3.21) then (3.21) becomes

$$\begin{aligned} 2A_0^{(2)}(n) \cos T &+ \sum_{k=1}^{\infty} \left[(k+1)A_k^{(2)}(n) - kA_{k+1}^{(2)}(n)\right] \cos(2k+1)T \\ &+ \sum_{k=1}^{n} \left[(k+1)B_k^{(2)}(n) - kB_{k+1}^{(2)}(n)\right] \sin(2k+1)T \end{aligned}$$

$$\begin{aligned} = -2na_0^{2n-1}a_1 \cos^{2n} T \sin T - F_1 a_0^{2n} \cos^{2n} T \sin T \\ - F_1 \frac{\mathrm{d}F_1}{\mathrm{d}T} \sin T - F_1^2 \cos T + F_1 \sin T \,. \end{aligned} \tag{3.23}$$

Now, by using (3.6) and (3.11) for a_0 it follows that

$$2na_0^{2n-1}a_1 \cos^{2n} T \sin T = 2n \frac{a_1}{a_0} \sum_{k=0}^{n} C_k(n) \sin(2k+1)T \tag{3.24}$$

where

$$C_k(n) = \frac{(2k+1)n!(n+1)!}{(n-k)!(n+k+1)!} \,. \tag{3.25}$$

Thus to obtain a_1 it is sufficient to determine the coefficient of $\sin T$ on the right hand side of (3.23). Also, the only non-periodic term in (3.22) is $A_0^{(2)}(n)$ which is obtained by determining the coefficient of $\cos T$ on the right hand side of (3.23). We therefore pick out the coefficients of $\sin T$ and $\cos T$ from the terms on the right hand side of (3.23). We have

$$\begin{aligned} F_1 a_0^{2n} \cos^{2n} T \sin T \\ = \frac{1}{2}\left[B_1^{(1)}(n) + \sum_{k=1}^{n}\left(B_k^{(1)}(n) + B_{k+1}^{(1)}(n)\right)C_k(n)\right] \cos T \\ + \text{ terms of the form } \cos 3T, \cos 5T, \dots, \cos(4n+1)T, \end{aligned} \tag{3.26}$$

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$$(3.27)\quad F_1\,\frac{\mathrm{d}F_1}{\mathrm{d}T}\,\sin T = -\frac{1}{2}\left[\sum_{k=1}^{n} B_k^{(1)}(n)B_{k+1}^{(1)}(n)\right]\cos T + \text{ terms of the form } \cos 3T, \cos 5T, \ldots, \cos(4n+1)T,$$

$$(3.28)\quad F_1^2\cos T = \frac{1}{2}\left[\sum_{k=1}^{n}\left(\left(B_k^{(1)}(n)\right)^2 + B_k^{(1)}(n)B_{k+1}^{(1)}(n)\right)\right]\cos T + \text{ terms of the form } \cos 3T, \cos 5T, \ldots, \cos(4n+1)T,$$

$$(3.29)\quad F_1\sin T = \frac{1}{2}\,B_1^{(1)}(n)\cos T + \text{ terms of the form } \cos 3T, \cos 5T, \ldots, \cos(2n+1)T\,.$$

Thus (3.23) takes the form

$$(3.30)\quad 2A_0^{(2)}(n)\cos T + \sum_{k=1}^{\infty}\left[(k+1)A_k^{(2)}(n) - kA_{k+1}^{(2)}(n)\right]\cos(2k+1)T + \sum_{k=1}^{\infty}\left[(k+1)B_k^{(2)}(n) - kB_{k+1}^{(2)}(n)\right]\sin(2k+1)T$$

$$= -2n\,\frac{a_1}{a_0}\sum_{k=0}^{n} C_k(n)\sin(2k+1)T - \frac{1}{2}\left[\sum_{k=1}^{n}\left(\left(B_k^{(1)}(n)\right)^2 + C_k(n)B_{k+1}^{(1)}(n) + C_k(n)B_k^{(1)}(n)\right)\right]\cos T + \sum_{k=1}^{2n} D_k(n)\cos(2k+1)T\,,$$

where the coefficients $D_k(n)$ do not need to be calculated to obtain a_1 and $A_0^{(2)}(n)$.

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By considering the coefficient of $\sin T$ in (3.30) it follows that

$$a_1 = 0 \,. \tag{3.31}$$

Also, by equating the coefficients of $\cos T$ in (3.30) we obtain
(3.32)

$$A_0^{(2)}(n) = -\frac{1}{4}\sum_{k=1}^{n}\Big((B_k^{(1)}(n))^2 + C_k(n)B_{k+1}^{(1)}(n) + C_k(n)B_k^{(1)}(n)\Big) \,.$$

This completes the derivation of the quantities required to calculate the limit cycle to first order in ε and the period of the limit cycle correct to second order in ε. The results will be combined in Sections 4 and 5 where the properties of the limit cycle and the period will be considered.

Finally, the general form of $F_2(T)$ can be deduced from (3.30). By considering the coefficients of $\cos(2k+1)T$ for $k \geq 2n+1$ and the coefficients of $\sin(2k+1)T$ for $k \geq 1$ we find, respectively, that

$$A_k^{(2)}(n) = 0 \,, \qquad k \geq 2n+1 \,, \tag{3.33}$$

$$B_k^{(2)}(n) = 0 \,, \qquad k \geq 1 \,. \tag{3.34}$$

Thus from (3.22),

$$F_2(T) = \sum_{k=0}^{2n} A_k^{(2)}(n)\cos 2kT \,. \tag{3.35}$$

4. Limit cycle. In this section we will investigate the properties of the limit cycle in the phase plane $\left(u,\ \frac{\mathrm{d}u}{\mathrm{d}t}\right)$ of equation (1.1), correct to first order in ε.

The limit cycle is given by (2.10) and (2.11). On using $a_1 = 0$ and (3.20) for $F_1(T)$ we obtain

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(4.1)
$$u = a_0 \cos T + \mathrm{O}(\varepsilon^2),$$

(4.2)
$$\frac{\mathrm{d}u}{\mathrm{d}t} = -a_0 \sin T + \frac{\epsilon a_0}{2} \sum_{k=1}^{n} B_k^{(1)}(n)\big[\cos(2k+1)T - \cos(2k-1)T\big] + \mathrm{O}(\epsilon^2)\,,$$

as $\varepsilon \to 0$, where T varies from 0 to 2π and a_0 and $B_k^{(1)}(n)$ are given by (3.11) and (3.19) respectively. For the van der Pol equation, $n = 1$ and therefore $a_0 = 2$, $B_1^{(1)}(1) = -\frac{1}{2}$ and equations (4.1) and (4.2) reduce to [3]

$$u = 2\cos T + \mathrm{O}(\varepsilon^2)\,, \tag{4.3}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -2\sin T + \frac{\varepsilon}{2}(\cos T - \cos 3T) + \mathrm{O}(\varepsilon^2)\,, \tag{4.4}$$

as $\varepsilon \to 0$.

We will compare the time transformation representation, (4.1) and (4.2), of the limit cycle to order ε with that obtained using the derivative expansion version of the method of multiple scales [15] and with numerical solutions. We will first show that if (4.1) and (4.2) are rewritten in terms of the original time variable t and if the Fourier coefficients $B_k^{(1)}(n)$ are eliminated from the amplitude then the multiple scales representation of the limit cycle to first order in ε is obtained.

In order to rewrite (4.1) and (4.2) in terms of t, consider the time transformation $t(T)$ given by (2.13). If (3.20) for $F_1(T)$ is substituted into (2.13) then we obtain

$$T = t + \frac{\varepsilon}{2} \sum_{k=1}^{n} \frac{1}{k}\, B_k^{(1)}(n)(1 - \cos 2kT) + \mathrm{O}(\varepsilon^2)\,, \tag{4.5}$$

as $\varepsilon \to 0$ and if (4.5) is substituted into its own right-hand side we find that

$$T = t + \frac{\varepsilon}{2} \sum_{k=1}^{n} \frac{1}{k}\, B_k^{(1)}(n)(1 - \cos 2kt) + \mathrm{O}(\varepsilon^2)\,, \tag{4.6}$$

as $\varepsilon \to 0$. By substituting from (4.6) into (4.1) and by expanding in powers of ε it can be verified that

$$(4.7)\quad u(t;\varepsilon) = a_0 \left[\cos t - \varepsilon \left(\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} B_k^{(1)}(n) \right) \sin t \right] + \frac{\varepsilon a_0}{4} \sum_{k=1}^{n} \frac{1}{k} B_k^{(1)}(n)(\sin(2k+1)t - \sin(2k-1)t) + \mathrm{O}(\varepsilon^2)$$

as $\varepsilon \to 0$. But

$$(4.8)\quad \sum_{k=1}^{n} \frac{1}{k} B_k^{(1)}(n) \sin(2k-1)t = \sum_{k=1}^{n} \frac{1}{(k+1)} B_{k+1}^{(1)}(n) \sin(2k+1)t + B_1^{(1)}(n) \sin t\,,$$

where (3.14) was used. Thus (4.7) becomes

$$(4.9)\quad u(t;\varepsilon) = a_0 \left[\cos t - \frac{\varepsilon}{2} \left(\frac{1}{2} B_1^{(1)}(n) + \sum_{k=1}^{n} \frac{1}{k} B_k^{(1)}(n) \right) \sin t \right] - \frac{\varepsilon a_0}{4} \sum_{k=1}^{n} \frac{1}{k(k+1)} (k B_{k+1}^{(1)}(n) - (k+1) B_k^{(1)}(n)) \sin(2k+1)t + \mathrm{O}(\varepsilon^2)\,,$$

as $\varepsilon \to 0$. By using the difference equation (3.12) to rewrite the coefficient of $\sin(2k+1)t$ and using also (3.11) for a_0, equation (4.9) may be expressed in the form

$$(4.10)\quad u(t;\varepsilon) = a_0 \cos\phi - \frac{\varepsilon}{2} \left(\frac{a_0}{2}\right)^{2n+1} \sum_{k=1}^{n} \frac{(2k+1)}{k(k+1)(n+k+1)} \binom{2n}{n-k} \sin(2k+1)\phi + \mathrm{O}(\varepsilon^2)\,,$$

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as $\varepsilon \to 0$, where

$$\phi = t + \frac{\varepsilon}{2}\left(\frac{1}{2}B_1^1(n) + \sum_{k=1}^{n}\frac{1}{k}B_k^{(1)}(n)\right). \tag{4.11}$$

By performing a similar analysis on (4.2), or simply by differentiating (4.10) with respect to t, we obtain

$$\begin{aligned}\frac{du}{dt}(t;\varepsilon) &= -a_0\sin\phi \\ &- \frac{\varepsilon}{2}\left(\frac{a_0}{2}\right)^{2n+1}\sum_{k=1}^{n}\frac{(2k+1)^2}{k(k+1)(n+k+1)}\binom{2n}{n-k}\cos(2k+1)\phi \\ &+ O(\varepsilon^2),\end{aligned} \tag{4.12}$$

as $\varepsilon \to 0$. The limit cycle in the phase plane $\left(u, \frac{du}{dt}\right)$ is obtained by letting ϕ vary from 0 to 2π in (4.10) and (4.12). The representation (4.10) and (4.12) of the limit cycle is the same as that obtained using the method of multiple scales to first order in ε [15]. In deriving (4.10) and (4.12) from (4.1) and (4.2) the solution (3.19) for $B_k^{(1)}(n)$ was not used. All that was required was the difference equation (3.12) satisfied by $B_k^{(1)}(n)$ and the boundary condition (3.14). When $n = 1$, $a_0 = 2$ and (4.10) and (4.12) reduce to the result for the limit cycle of the van der Pol equation [16]:

$$u = 2\cos\phi - \frac{\varepsilon}{4}\sin 3\phi + O(\varepsilon^2), \tag{4.13}$$

$$\frac{du}{dt} = -2\sin\phi - \frac{3\varepsilon}{4}\cos 3\phi + O(\varepsilon^2), \tag{4.14}$$

as $\varepsilon \to 0$, where ϕ varies from 0 to 2π.

The two perturbation representations of the limit cycle, (4.1) and (4.2) and (4.10) and (4.12), were compared with the numerical solution in order to determine the accuracy of the two perturbation representations. First order ordinary differential equations with solutions consisting of rapidly and slowly varying components are referred to as stiff equations. As n or ε increases the first order system of two equations

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corresponding to equation (1.1) becomes stiff. Two numerical methods, the IMSL routines DIVPRK and DIVPAG, were used [13]. The routine DIVPRK, which is efficient for non-stiff systems, uses Runge-Kutta formulae of order five and six. The routine DIVPAG, which is efficient for stiff systems, uses the implicit Adams method up to order twelve and Geer's stiff method, which is a backward differentiation formula, up to order five. Both routines attempt to control the norm of the local error so as to keep the global error proportional to a tolerance specified by the user. The proportionality depends on the differential equation and on the range of integration. In order to estimate the global error we used the following procedure. For $n = 1, 5, 10$ and 15 and for suitable initial values of u and $\dot{u}$, one run with tolerance of 10^{-12} was performed using Geer's stiff method. The range of integration was $\Delta t = 10$, which corresponds to about one circuit of the limit cycle. These four runs were used as the high precision reference runs. Runs with weaker tolerance of 10^{-10} were then performed for the same values of n, the same set of initial conditions and the same range of integration using Geer's method, the Adams method and the routine DIVPRK. The difference of the results for u and $\dot{u}$ of the runs for the three methods with tolerance of 10^{-10} and the high precision run with tolerance 10^{-12} were then calculated for each value of n. In general, the differences increased slowly as n increased. For Geer's method and the Adams method, the differences were largely of order 10^{-9}. With the DIVPRK routine the differences were of order 10^{-11} for $n = 1$ and $n = 5$ and of order 10^{-10} for $n = 10$. It was not possible to implement the DIVPRK routine for $n = 15$ for the chosen initial values because of the increasing stiffness of the system. With a tolerance of 10^{-10}, an estimate of the global error using the DIVPAG routine is therefore of order 10^{-9}. For small values of n the routine DIVPRK is more accurate with a global error of order 10^{-11} or 10^{-10} but the routine becomes more difficult to implement as n increases. The differences in the results obtained using DIVPRK and DIVPAG were not apparent graphically and when the limit cycles were plotted using the two routines they coincided on the graphs. Only one was therefore plotted. The routine DIVPAG was more efficient and easier to implement than DIVPRK as the value of n increased.

The Fourier coefficients, $B_k^{(1)}(n)$, which occur in (4.2) were calculated using the solution (3.19). They are listed in Table 1 for $n = 1, 5, 10$

and 15. We see from Table 1 that the magnitude of $B_k^{(1)}(n)$ decreases steadily as k increases. The main contribution to the limit cycle comes from terms with values of $k \lesssim \frac{1}{2}\, n$, the most significant terms being those with small values of k. The rapid decrease in $|\; B_k^{(1)}(n) \;|$ as k approaches n is illustrated in Figure 1 where $\log_{10} |\; B_k^{(1)}(n) \;|$ is plotted against k for $n = 1,\ 5,\ 10$ and 15.

Table 1. The non-zero Fourier coefficients $B_k^{(1)}(n)$ for $n = 1, 5, 10$ and 15.

n	Non-zero Fourier Coefficient	n	Non-zero Fourier coefficient
1	$B_1^{(1)}(1) = -0.5$	15	$B_1^{(1)}(15) = -2.3807288952$
5	$B_1^{(1)}(5) = -1.4500000000$		$B_2^{(1)}(15) = -2.1143989932$
	$B_2^{(1)}(5) = -0.7571428571$		$B_3^{(1)}(15) = -1.4559191629$
	$B_3^{(1)}(5) = -0.2428571429$		$B_4^{(1)}(15) = -0.8455853916$
	$B_4^{(1)}(5) = -0.0460317460$		$B_5^{(1)}(15) = -0.4230808107$
	$B_5^{(1)}(5) = -0.0039682540$		$B_6^{(1)}(15) = -0.1830323701$
10	$B_1^{(1)}(10) = -2.0198773440$		$B_7^{(1)}(15) = -0.0681989278$
	$B_2^{(1)}(10) = -1.5397546898$		$B_8^{(1)}(15) = -0.0216948768$
	$B_3^{(1)}(10) = -0.8673243423$		$B_9^{(1)}(15) = -0.0058140591$
	$B_4^{(1)}(10) = -0.3872016872$		$B_{10}^{(1)}(15) = -0.0012881418$
	$B_5^{(1)}(10) = -0.1378482629$		$B_{11}^{(1)}(15) = -0.0002297168$
	$B_6^{(1)}(10) = -0.0384948385$		$B_{12}^{(1)}(15) = -0.0000316928$
	$B_7^{(1)}(10) = -0.0081459390$		$B_{13}^{(1)}(15) = -0.0000031747$
	$B_8^{(1)}(10) = -0.0012294895$		$B_{14}^{(1)}(15) = -0.0000002054$
	$B_9^{(1)}(10) = -0.0001179935$		$B_{15}^{(1)}(15) = -0.0000000065$
	$B_{10}^{(1)}(10) = -0.0000054125$		

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In Figure 2 the effect on the limit cycle of varying n for fixed ε is examined. The limit cycle is plotted for $\varepsilon = 0.4$ and $n = 1, 5, 10$ and 15. Three different solutions for the limit cycle are presented: the solution (4.1) and (4.2) which was derived using the time transformation method to first order in ε, (4.10) and (4.12) which is the solution obtained if the method of multiple scales is used to first order in ε and the numerical solution. We see from Figure 2 that the perturbation solutions become less accurate as n increases. Both perturbation solutions underestimate the velocity $\frac{du}{dt}$ by about the same amount over most of the range of the displacement but the time transformation solution is in better agreement with the numerical solution in the regions of maximum displacement. From (4.1) the maximum displacement correct to order ε in the time transformation solution is a_0 which decreases steadily as n increases; for $n = 1, 5, 10$ and 15 it follows from (3.19) that $a_0 = 2, 1.38, 1.23$ and 1.17, respectively. It can be verified with the aid of Stirling's formula, which states that when n is sufficiently large,

$$\ln n! \sim \ln n - n \,, \tag{4.15}$$

that $a_0 \to 1$ as $n \to \infty$ [15]. As n increases, the periodic oscillation which corresponds to the limit cycle becomes more jerky and its amplitude, a_0, decreases.

In Figures 3 and 4 the effect on the limit cycle of varying ε for fixed n is examined. In Figure 3 the two perturbation solutions to first order in ε and the numerical solution for the limit cycle are plotted for $n = 5$ and $\varepsilon = 0.2, 0.5, 0.8$ and 1.1. For this range of values of ε the time transformation solution is in better agreement with the numerical solution than is the multiple scales solution, especially near the points of maximum displacement. The time transformation solution is in good qualitative agreement with the numerical solution even for $\varepsilon = 1.1$ when we would expect a perturbation solution for small ε to cease to give useful results. Burton [3] has considered solutions to higher order in ε for the van der Pol equation, $n = 1$. He found that the time transformation solution to third order in ε is in good agreement with the numerical solution for values of ε as large as $\varepsilon = 2$. He also found that convergence of the time transformation method appeared to break down for values of ε greater than about $\varepsilon = 2$. In Figure 4 the limit cycles determined by the time transformation method to first order in ε are plotted on one graph for $n = 8$ and $\varepsilon = 0.001$, 0.5, 1.0, 1.5 and 2.0 and they are compared with the corresponding graph of the numerical solution. For $\varepsilon = 0.001$ the limit cycle is approximately circular with radius given independently of the initial conditions by (3.11). This compares with the case $\varepsilon = 0$ for which the motion is simple harmonic motion and the phase trajectory is circular with a radius determined by the initial conditions. For the perturbation solution the points of maximum displacement are independent of ε and are given by $u = \pm a_0 = \pm 1.270$. The points of maximum displacement in the numerical solution are approximately independent of ε and are given by $u = \pm 1.267$, ± 1.261, ± 1.253, ± 1.249 and ± 1.246 for $\varepsilon = 0.001$, 0.5, 1.0, 1.5 and 2.0 respectively. However, the magnitude of the maximum velocity, $\frac{du}{dt}$, steadily increases as ε increases. This effect is underestimated by the perturbation solution. The steady increase in the magnitude of the maximum velocity as ε increases compares with the steady decrease in the magnitude of the maximum displacement as n increases. The limit cycles generated by the time transformation method to first order in ε pass through the two points, $u = 0$, $\frac{du}{dt} = \mp a_0$, given by $T = \frac{\pi}{2}$ and $T = \frac{3\pi}{2}$, respectively. These two points correspond to the two small regions in the phase plane of the numerical solution in which the limit cycles intersect.

5. Period of the limit cycle. In this section we will derive the period of the limit cycle of (1.1) correct to second order in ε and we will investigate its properties.

The period of the limit cycle to second order in ε is given by (2.15). If (3.20) for $F_1(T)$ and (3.35) for $F_2(T)$ are substituted into (2.15) then (2.15) becomes

$$\begin{aligned}(5.1)\quad \tau = \int_0^{2\pi} \Big[1 - \varepsilon \sum_{k=1}^{n} B_k^{(1)}(n) \sin 2kT \\ + \varepsilon^2 \Big(\frac{1}{2} \sum_{k=1}^{n} \left(B_k^{(1)}(n)\right)^2 - A_0^{(2)}(n) + \text{periodic terms of the form} \\ \cos 2T,\ \cos 4T, \dots, \cos 4nT \Big) + \mathrm{O}(\varepsilon^3) \Big] \mathrm{d}t\ ,\end{aligned}$$

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as $\varepsilon \to 0$. If τ is expanded in a power series in ε according to

$$\tau = 2\pi\left(1 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + O(\varepsilon^3)\right) \tag{5.2}$$

as $\varepsilon \to 0$, then $\tau_1 = 0$ and

$$\tau_2 = \frac{1}{2}\sum_{k=1}^{n}\left(B_k^{(1)}(n)\right)^2 - A_0^{(2)}(n) . \tag{5.3}$$

By using (3.32) for $A_0^{(2)}(n)$, (5.3) becomes

$$\tau_2 = \frac{1}{4}\sum_{k=1}^{n}\left[3\left(B_k^{(1)}(n)\right)^2 + C_k(n)B_{k+1}^{(1)}(n) + C_k(n)B_k^{(1)}(n)\right] . \tag{5.4}$$

We can express τ_2 either entirely in terms of $B_k^{(1)}(n)$ or in terms of $C_k(n)$. We will consider both forms for τ_2.

We first express τ_2 entirely in terms of the Fourier coefficients $B_k^{(1)}(n)$. Now, by using (3.25) for $C_k(n)$, the difference equation (3.12) may be rewritten as

$$kB_{k+1}^{(1)}(n) - (k+1)B_k^{(1)}(n) = C_k(n) , \qquad 1 \le k \le n . \tag{5.5}$$

By using (5.5) to eliminate $C_k(n)$ from (5.4) we obtain

$$\tau_2 = \frac{1}{4}\sum_{k=1}^{n}\left[(2-k)\left(B_k^{(1)}(n)\right)^2 + k\left(B_{k+1}^{(1)}(n)\right)^2 - B_k^{(1)}(n)B_{k+1}^{(1)}(n)\right] . \tag{5.6}$$

But since $B_{n+1}^{(1)}(n) = 0$, it follows that

$$\sum_{k=1}^{n} k\left(B_{k+1}^{(1)}(n)\right)^2 = \sum_{k=0}^{n-1} k\left(B_{k+1}^{(1)}(n)\right)^2 = \sum_{k=1}^{n}(k-1)\left(B_k^{(1)}(n)\right)^2 , \tag{5.7}$$

and hence by substituting from (5.7) into (5.6), we obtain

$$\tau_2 = \frac{1}{4}\sum_{k=1}^{n} B_k^{(1)}(n)\left[B_k^{(1)}(n) - B_{k+1}^{(1)}(n)\right] . \tag{5.8}$$

Equation (5.8) gives τ_2 entirely in terms of the Fourier coefficients $B_k^{(1)}(n)$, which in turn are given by the solution (3.19). It is a convenient way to calculate τ_2 especially if the $B_k^{(1)}(n)$ have already been evaluated, for instance, to obtain the limit cycle. The Fourier coefficients occur as squares and products in (5.8). It therefore follows from Table 1 that the series will converge rapidly with only the Fourier coefficients with low values of k making a significant contribution.

We next eliminate $B_k^{(1)}(n)$ from τ_2 and obtain an expression for τ_2 in terms of $C_k(n)$. By squaring both sides of (5.5) it follows that

$$(5.9)\quad B_k^{(1)}(n)B_{k+1}^{(1)}(n) = \frac{k}{2(k+1)}\left(B_{k+1}^{(1)}(n)\right)^2 + \frac{(k+1)}{2k}\left(B_k^{(1)}(n)\right)^2 - \frac{1}{2k(k+1)}\left(C_k(n)\right)^2 .$$

By replacing $B_k^{(1)}(n)B_{k+1}^{(1)}(n)$ in (5.8) using (5.9) we obtain

$$(5.10)\quad \tau_2 = \frac{1}{8}\sum_{k=1}^{n}\left[\frac{(k-1)}{k}\left(B_k^{(1)}(n)\right)^2 - \frac{k}{(k+1)}\left(B_{k+1}^{(1)}(n)\right)^2 + \frac{1}{k(k+1)}\left(C_k(n)\right)^2\right] .$$

But

$$(5.11)\quad \sum_{k=1}^{n}\frac{(k-1)}{k}\left(B_k^{(1)}(n)\right)^2 = \sum_{k=0}^{n-1}\frac{k}{(k+1)}\left(B_{k+1}^{(1)}(n)\right)^2 = \sum_{k=1}^{n}\frac{k}{(k+1)}\left(B_{k+1}^{(1)}(n)\right)^2 ,$$

where we have used $B_{n+1}^{(1)}(n) = 0$. Thus (5.10) reduces to

$$(5.12)\quad \tau_2 = \frac{1}{8}\sum_{k=1}^{n}\frac{\left(C_k(n)\right)^2}{k(k+1)} .$$

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In deriving (5.12) from (5.8) the solution (3.19) for $B_k^{(1)}(n)$ was not used. All that was required was the difference equation (3.12) satisfied by $B_k^{(1)}(n)$ and the boundary condition (3.14). Equation (5.12) enables the period to be calculated without first having to evaluate the Fourier coefficients $B_k^{(1)}(n)$.

By using (5.8) and (5.12) for τ_2 and (3.25) for $C_k(n)$, the following two equivalent expressions for the period of the limit cycle are obtained:

(5.13) $$\tau = 2\pi\left[1+\frac{\varepsilon^2}{4}\sum_{k=1}^{n} B_k^{(1)}(n)(B_k^{(1)}(n)-B_{k+1}^{(1)}(n))+O(\varepsilon^3)\right] ,$$

and

(5.14)
$$\tau = 2\pi\left[1+\varepsilon^2\,\frac{(n!(n+1)!)^2}{8}\sum_{k=1}^{n}\frac{1}{k(k+1)}\left(\frac{2k+1}{(n-k)!(n+k+1)!}\right)^2 +O(\varepsilon^3)\right] ,$$

as $\varepsilon \to 0$. Equation (5.14) is exactly the same as the result which is derived using the method of multiple scales [15].

In Figure 5 the period of the limit cycle correct to second order in ε, given by (5.13) or equivalently (5.14), and the period calculated from the numerical solution are plotted against n for $\varepsilon = 0.4$. We see from Figure 5 that the period of the limit cycle increases as n increases. The perturbation solution for τ to second order in ε overestimates the numerical solution and becomes less accurate as n increases. For $n = 1$, 5, 10 and 15 and $\varepsilon = 0.4$ the relative error of the perturbation solution is 0.05%, 1.11%, 3.50% and 6.16%, respectively.

In Figure 6 the perturbation solution for the period of the limit cycle correct to second order in ε, given by (5.13) or (5.14), and the period calculated from the numerical solution are plotted against ε for $0 < \varepsilon \leq 0.9$ and $n = 8$. For $\varepsilon = 0$, the motion is simple harmonic motion with period 2π. For small values of $\varepsilon > 0$ the period of the limit cycle is approximately 2π. We see from Figure 6 that for the range of values of ε considered, the period of the limit cycle increases as ε increases. The perturbation solution overestimates the numerical solution and, as was to be expected, the perturbation solution becomes less accurate as ε increases. For $n = 8$, the relative error of the perturbation solution when $\varepsilon = 0.1, 0.3, 0.5, 0.7$ and 0.9 is 0.02%, 1.14%, 5.92%, 17.72% and 44.71%, respectively.

6. Concluding remarks. We found that the solution for the limit cycle to first order in ε derived using the time transformation method was in better quantitative agreement with the numerical solution, especially near the points of maximum displacement, than that which is obtained using the method of multiple scales. It gave good qualitative results for the limit cycle even for values of ε greater than unity. Both methods gave exactly the same result for the period of the limit cycle to second order in ε. The time transformation method gave the limit cycle quickly and directly. However, because we looked for a steady-state solution, it did not give information on the rate at which a system point in the phase plane tends to the limit cycle. This may be obtained by using, for example, the method of multiple scales [15].

We found that the period of the limit cycle increased as either n or ε increased. The shape of the limit cycle also changed when either n or ε increased. As n increased the magnitude of the maximum displacement in the limit cycle decreased. Correct to first order in ε the maximum displacement is a_0, given by (3.11), and a_0 decreases from 2 to 1 as n increases from 1 to ∞. In comparison, as ε increased we found that the magnitude of the maximum velocity in the limit cycle increased. These two effects tended to make the limit cycle less circular as either n or ε increased. The accuracy of the time transformation solution decreased as n increased and, since it is a perturbation expansion in powers of ε, its accuracy also decreased as ε increased.

Even when a numerical solution can be obtained the conclusions deduced from a perturbation solution are still important. They can help in the interpretation and understanding of the more accurate numerical results. A perturbation solution shows analytically, albeit approximately, how the solution depends on the parameters of the differential equation which, in the problem considered here, are n and ε. The solution (3.11) for a_0 shows clearly how the size of the limit cycle depends on n for small values of ε. The perturbation solution (5.14) for the period shows the dependence of the period on n and ε. The period is independent

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of first order terms in ε which explains why in Figure 6 it changes only slowly with ε for small values of ε.

We applied the time transformation method only to first order in ε to calculate the limit cycle and only to second order in ε to calculate the period. For given small values of n it would not be too difficult to calculate the solution to higher orders in ε and this has been done by Burton [3] for $n = 1$. However, the results presented here are valid for all values of the positive integer n and could therefore be used to examine how the properties of the limit cycle and the accuracy of the time transformation method depend on n.

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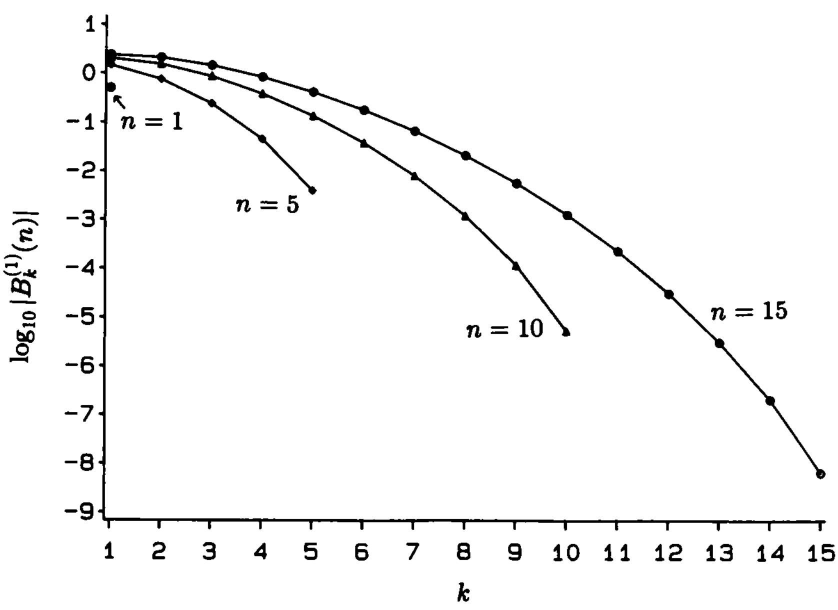

Figure 1. $\text{Log}_{10}\ |\ B_k^{(1)}(n)\ |$ plotted against k for $1 \le k \le n$ where $n = 1$, 5, 10 and 15. $B_k^{(1)}(n)$ is given by Eq. (3.19) for $1 \le k \le n$ and $B_k^{(1)}(n) = 0$ for $k \ge n + 1$.

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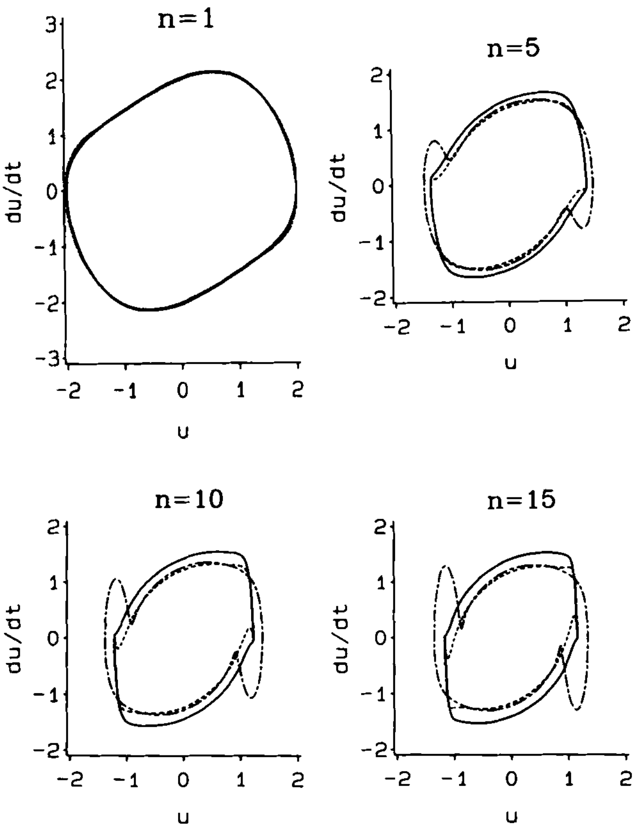

Figure 2. Limit cycles in the phase plane for $\varepsilon = 0.4$ and $n = 1,\ 5,\ 10$ and 15: numerical solution (————), time transformation solution to first order in ε(– – – – –) and method of multiple scales solution to first order in ε(— - — - — -).

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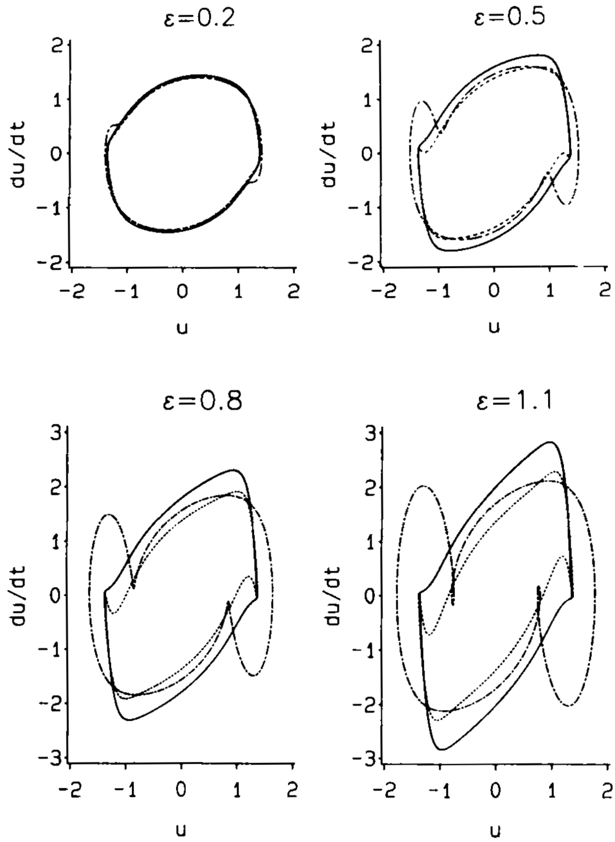

Figure 3. Limit cycles in the phase plane for $n = 5$ and $\varepsilon = 0.2,\ 0.5,\ 0.8$ and 1.1: numerical solution (————), time transformation solution to first order in ε(– – – – –) and method of multiple scales solution to first order in ε(— - — - — -).

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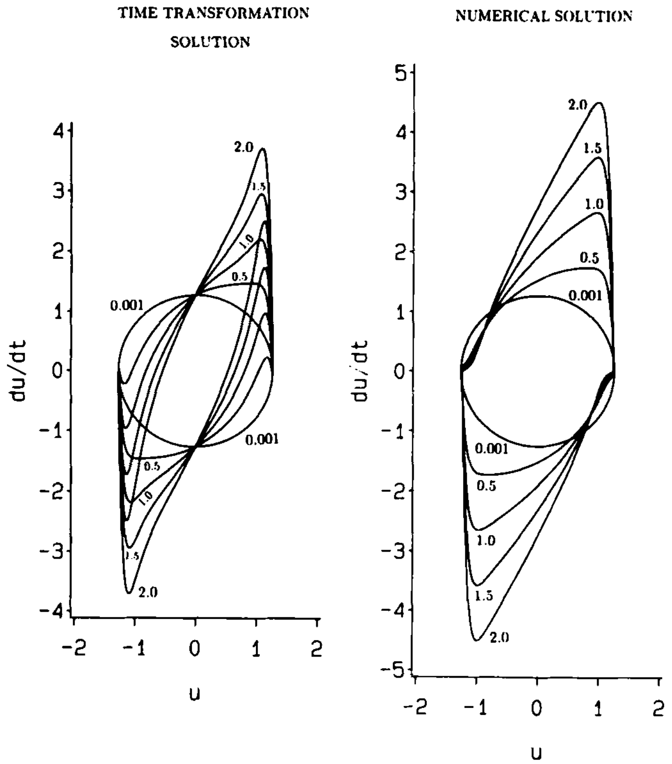

Figure 4. The time transformation solution to first order in ε and the numerical solution for the limit cycles in the phase plane for $n = 8$ and $\varepsilon = 0.001$, 0.5, 1.0, 1.5 and 2.0.

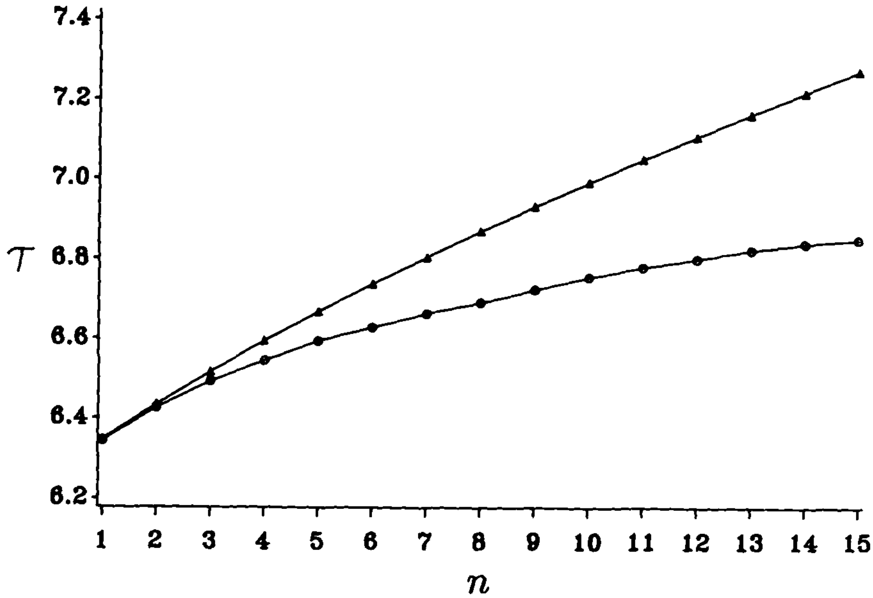

Figure 5. Period of the limit cycle, τ, plotted against n for $\varepsilon = 0.4$: numerical solution (o—o—o—o) and perturbation solution to second order in $\varepsilon(\Delta—\Delta—\Delta—\Delta)$.

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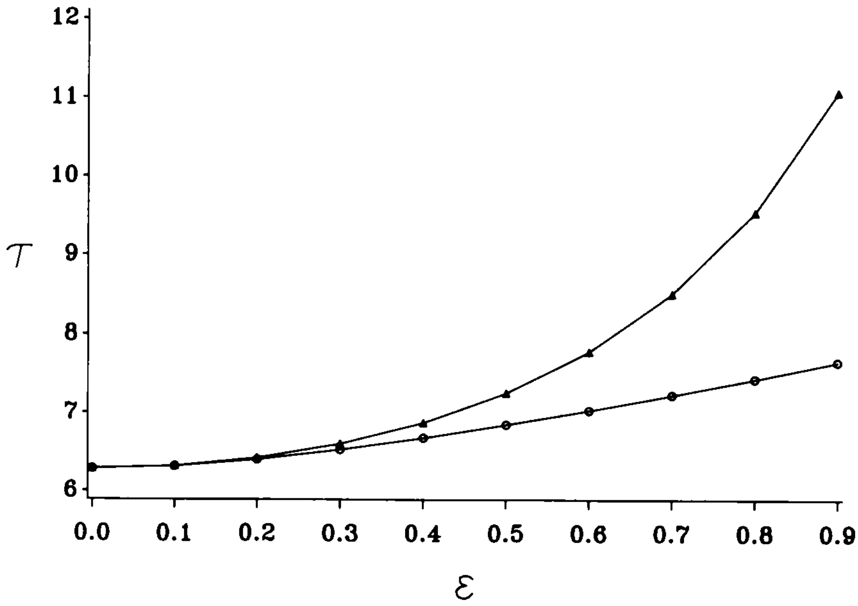

Figure 6. Period of the limit cycle, τ, plotted against ε for $n = 8$: numerical solution (o—o—o—o) and perturbation solution to second order in ε(Δ—Δ—Δ—Δ).

REFERENCES

1. BAVINCK, H. AND GRASMAN, J., The method of matched asymptotic expansions for the periodic solution of the van der Pol equation, *Int. J. Non-Linear Mech.* **9** (1974), 421–434.
2. BENDER, C.M. AND ORSZAG, S.A., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York, 1978 chap. 2.
3. BURTON, T.D., Non-linear oscillator limit cycle analysis using a time transformation approach, *Int. J. Non-Linear Mech.* **17** (1982), 7–19.
4. BURTON, T.D., On the amplitude decay of strongly non-linear damped oscillators, *J. Sound and Vibration* **87** (1983), 535–541.

5. BURTON, T.D. AND HAMDAN, M.N., Analysis of non-linear autonomous conservative oscillators by a time transformation method, *J. Sound and Vibration* **87** (1983), 543–554.
6. CHEN, S H., CHEUNG, Y.K. AND LAU, S.L., On perturbation procedure for limit cycle analysis, *Int. J. Non-Linear Mech.* **26** (1991), 125–133.
7. DADFAR, M.B., GEER, J. AND ANDERSEN, C.M., Perturbation analysis of the limit cycle of the free van der Pol equation, *SIAM J Appl. Math.* **44** (1984), 881–895.
8. DAVIS, R.T. AND ALFRIEND, K.T., Solutions to van der Pol's equation using a perturbation method, *Int. J. Non-Linear Mech.* **2** (1967), 153–162.
9. DORODNICYN, A.A., Asymptotic solution of the van der Pol equation, *Prikl. Mat. Mekh.* **11** (1947), 313–328; *Am. Math. Soc. transl. ser. 1* **4** (1962), 1.
10. GRADSHTEYN, I.S. AND RYZHIK, I.M., *Table of Integrals Series and Products*, Academic, New York, 1980, p. 25.
11. HAMDAN, M.N. AND BURTON, T.D., Analysis of forced non-linear undamped oscillators by a time transformation method, *J. Sound and Vibration* **110** (1986), 223–232.
12. HOLMES, P. AND RAND, D., Phase portraits and bifurcations of the non-linear oscillator: $\ddot{x} + (\alpha + \gamma x^2)\dot{x} + \beta x + \delta x^3 = 0$, *Int. J. Non-Linear Mech.* **15** (1980), 449–458.
13. IMSL MATH/LIBRARY, *FORTRAN Subroutines for Mathematical Applications*, 1987, pp. 633–651.
14. MINORSKY N., *Nonlinear Oscillations*, D van Nostrand, Princeton, 1962, pp. 557–559.
15. MOREMEDI, G.M., MASON, D.P. AND GORRINGE, V., On the limit cycle of a generalised van der Pol equation, *Int. J. Non-Linear Mech.* **28** (1993), 237–250.
16. NAYFEH, A.H., *Perturbation Methods*, Wiley, New York, 1973, p. 8, chaps 5 and 6.
17. NAYFEH, A.H., *Introduction to Perturbation Techniques*, Wiley, New York, 1981, chap. 6.
18. NGUYEN, V.D., Some properties of the generalised van der Pol equation, *J. Tech. Phys. (Poland)* **17** (1976), 435–440.
19. OBI, C., Analytical theory of nonlinear oscillations. IV: The periodic oscillations of the equation $\ddot{x} - \varepsilon(1 - x^{2n+2})x + x^{2n+1} = \varepsilon a \cos\omega t$, *SIAM J. Appl. Math.* **31** (1976), 345–357.

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