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Stochastic Averaging on Graphs: Noisy Duffing-van der Pol Equation[1]

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Abstract. The purpose of this work is to improve the understanding of the solution of the noisy *Duffing-van der Pol* equation. We achieve this by developing rigorous methods to replace, in some limiting regime, the original complicated system by a *simpler, constructive, and rational* approximation – a low-dimensional model of the dynamical system. To this end, we study the equations as a random perturbation of a two-dimensional Hamiltonian system. We achieve the model-reduction through *stochastic averaging* and the reduced Markov process takes its values on a graph with certain gluing conditions at the vertex of the graph. Examination of the reduced Markov process on the graph yields many important results, namely, mean exit times, probability density functions, and stochastic bifurcations.

INTRODUCTION

We examine the stochastic version of the *Duffing-van der Pol* equation,

$$\ddot{q} = (\alpha + \nu_2\xi_1)q + \beta\dot{q} + aq^3 + bq^2\dot{q} + \nu_1\xi_2 \tag{1}$$

where α and β are the bifurcation parameters, ξ_1 and ξ_2 are white noise processes with unit intensities. Equation (1) represents the generic normal form of the double zero co-dimension two bifurcation if one allows both α and β to be bifurcation parameters [4], and there are many physical problems whose dynamics are described by the above equation for some parameter values. For simplicity we consider only $\alpha \in \mathbb{R}^+$. This equation consists of both linear and nonlinear restoring and dissipative terms which allow one to model various mechanical and structural dynamics problems. In this model, a sustained oscillation arises from a balance between energy generation at low amplitude and energy dissipation at large amplitude.

In this paper we extend the work by Arnold *et al.* [1] and Liang and Sri Namachchivaya [7] to obtain analytical results, based on the work of Freidlin and Wentzell [2], for the noisy Duffing-van der Pol equation away from the trivial solution. Without loss of generality we have taken $a = -1.0, b = -1.0$ and the

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qualitative behavior is examined for all values of the unfolding parameters β and $\alpha \in \mathbb{R}^+$ by making use of the system Hamiltonian.
In the absence of dissipation and random perturbations, system (1) has multiple fixed points, one of which is connected to itself by a homoclinic orbit and the reduced Markov process takes its values on a graph [2]. The domain of the averaged generator involves a *gluing condition* at the vertex of the graph. The first passage time problem is studied for this averaged generator and the probability density function is obtained by solving the Fokker-Planck equation.

STATEMENT OF THE PROBLEM

Equation (1) can be put into two first order equations:

$$\dot{u} = v, \quad \dot{v} = \alpha u - u^3 + (\beta - u^2)v + u\nu_2\xi_1 + \nu_1\xi_2 \tag{2}$$

In order to examine the stochastic behavior away from the trivial solution, we write Eq. (2) as a weakly perturbed Hamiltonian system, where the Hamiltonian contains strong nonlinear terms. Here we use the standard rescalings [3]:

$$\alpha = \epsilon^2\bar{\alpha}, \quad \beta = \epsilon^2\bar{\beta}, \quad u = \epsilon x, \quad v = \epsilon^2 y, \quad \nu_2 = \epsilon^2\bar{\nu_2}, \quad \nu_1 = \epsilon^3\bar{\nu_1}, \quad \bar{t} = \epsilon t.$$

By omitting the bars and putting things in a proper mathematical setting, we have

$$d\hat{Z}_t^\varepsilon = \{\bar{\nabla}H(\hat{Z}_t^\varepsilon) + \varepsilon b(\hat{Z}_t^\varepsilon)\}dt + \sqrt{\varepsilon}\sigma(\hat{Z}_t^\varepsilon)dW_t, \quad \hat{Z}_0^\varepsilon = z \stackrel{\text{def}}{=} (x,y)^T \in \mathbb{R}^2 \tag{3}$$

In obtaining eq. (3), we make use of the fact that $\frac{1}{\sqrt{\epsilon}}W_{\varepsilon t}$ is also a standard Wiener process. The corresponding set of unperturbed deterministic equations are given by

$$\dot{\hat{Z}}_t = \bar{\nabla}H(\hat{Z}_t), \quad \hat{Z}_0 = z \in \mathbb{R}^2 \tag{4}$$

with the Hamiltonian H given by

$$H(x,y) \stackrel{\text{def}}{=} \frac{y^2}{2} - \alpha\frac{x^2}{2} + \frac{x^4}{4}, \quad \alpha \in \mathbb{R}^+ \tag{5}$$

The deterministic system (4) has three fixed points at (0,0), ($\sqrt{\alpha}$,0), ($-\sqrt{\alpha}$,0) with eigenvalues $\lambda_{(0,0)} = \pm\sqrt{\alpha}$, $\lambda_{(\pm\sqrt{\alpha},0)} = \pm i\sqrt{2\alpha}$ respectively. The periodic orbits for $H < 0$ encircle one fixed point (the oscillations), while for $H > 0$, we have periodic orbits which encircle all three fixed points (the rotations), and $H = 0$ corresponds to a homoclinic orbit connected at (0,0). The dissipative and the stochastic components in Eq. (3) are defined as

$$b(x,y) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ (\beta - x^2)y \end{pmatrix}, \quad \sigma(x,y) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ x\nu_2 & \nu_1 \end{pmatrix} \tag{6}$$

where ν_2 and ν_1 are the noise intensities. Thus, we shall asume that Eq. (4) is perturbed by a small intensity random diffusive motion described by $\mathcal{L}$

$$\mathcal{L}f(z) = \frac{1}{2}\sum_{i,j} a_{i,j}(z)\frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_i b_i(z)\frac{\partial f}{\partial z_i}(z), \qquad z \in \mathbb{R}^2 \tag{7}$$

where $a_{i,j}(z) \stackrel{\text{def}}{=} (\sigma(z)\sigma^T(z))_{ij}$. In order to make the time scales transparent, we deal with finite time intervals by defining the equations in slow time $t \to \varepsilon t$

$$dZ_t^\varepsilon = \{\frac{1}{\varepsilon}\bar{\nabla}H(Z_t^\varepsilon) + b(Z_t^\varepsilon)\}dt + \sigma(Z_t^\varepsilon)dW_t, \quad Z_0^\varepsilon = z \in \mathbb{R}^2 \tag{8}$$

Here, we have made use of the fact that the law of $\{Z_t^\varepsilon;\, t \geq 0\}$ is the same as the law of $\{\hat{Z}_{t/\varepsilon}^\varepsilon;\, t \geq 0\}$. It is clear that Z_t^ε defined in Eq. (8) is a Markov process on $\mathbb{R}^2$ whose generator is

$$\mathcal{L}^\varepsilon \stackrel{\text{def}}{=} \mathcal{L} + \frac{1}{\varepsilon}\bar{\nabla}H \tag{9}$$

with domain $\mathcal{D}(\mathcal{L}^\varepsilon) = \mathbf{C}_b^2(\mathbb{R}^2)$. The purpose of this work is to replace, in some limiting regime, Eq. (8) by a *simpler, constructive, and rational* approximation – a low-dimensional model of the dynamical system. We achieve the model reduction from a two-dimensional Markov process in Z_t^ε to a one-dimensional process in H through *stochastic averaging.*

STOCHASTIC AVERAGING ON GRAPHS

The Hamiltonian system (4) has multiple fixed points, one of which is connected to itself by a homoclinic orbit, and $H(Z_t^\varepsilon)$ no longer converges to a Markov process. The machinery of the classical analysis [6] has until recently yielded little in the way of reduction for stochastic systems with more complex Hamiltonian structure, such as in the case of noisy Duffing-van der Pol equations where H looks like a collection of non-degenerate inverted hills.

The first extension of averaging to include multiple fixed points with homoclinic orbits was given by Freidlin and Wentzell [2]. A clever treatment in a neighborhood of the separatrix where the unperturbed orbits have arbitrarily long periods was carried out. Here the reduced Markov process takes its values in a graph with certain gluing conditions defined at vertices connecting the edges of the graph. In this work, as in Freidlin and Wentzell [2], we consider the graph Γ, shown in Fig. 1, homeomorphic to the set of connected components of the level sets of the Hamiltonian. Thus, by introducing $¥ : \mathbb{R}^2 \to \Gamma$, the reduced Markov process takes its values on a graph and it was shown by Freidlin and Wentzell [2] that processes $¥(Z_t^\varepsilon)$ on Γ converge weakly to a diffusion process on Γ.

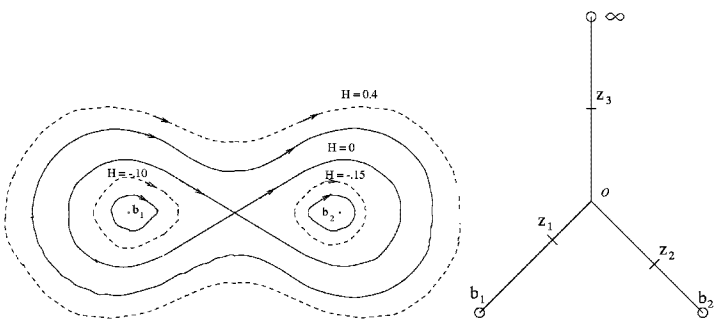
The usual averaging procedure relies upon an implicit assumption that the instantaneous frequency must be non-zero or the periods of oscillations or rotations

are finite. Hence, a subtle treatment is necessary in a neighborhood of the homoclinic orbit where the unperturbed orbits have arbitrarily long periods. In order to remedy this problem, two different models, one which is valid away from the homoclinic orbit, the other valid in a boundary layer about the homoclinic orbit are introduced. After some tedious analysis, a gluing condition which allows a unified treatment of the problem in the entire domain is obtained. The reader is referred to Freidlin and Wentzell [2] for details. We make use of these results to calculate the characteristics of the limiting process for the noisy Duffing-van der Pol equations. The complete set of equations for the averaged generator and the gluing conditions are presented. Let $(Z_t^\varepsilon; \mathbb{P}_z^\varepsilon)$ be the diffusion process on $\mathbb{R}^2$ corresponding to the differential operator (8). Here, $\mathbb{P}_z^\varepsilon$ is the law of Z_t^ε.

Then, the distributions of the processes $¥(Z_t^\varepsilon)$ in the space of continuous functions with values in $¥(\mathbb{R}^2)$ with respect to $\mathbb{P}_z^\varepsilon$ converge weakly as $\varepsilon \to 0$ to the probability measure $\mathbb{P}_¥$. Here $(¥(t), \mathbb{P}_¥)$ is the process on the graph defined by operators $\hat{\mathcal{L}}_i$ as

$$\hat{\mathcal{L}}_i f_i(H) = \bar{A}_i(H) f_i' + \frac{1}{2}\bar{\sigma}_i^2(H) f_i'' \tag{10}$$

on the three edges of the graph, where H is only a local coordinate in each edge, since it can take the same value for different trajectories. In evaluating the drift and diffusion coefficients for each edge we make use of Eq. (8). We change the time integral to the path integral with respect to the fast variable Z_t^ε while averaging over one period of the fast motion of Z_t^ε. This process effectively removes the fast



(a) (b)

FIGURE 1.

variable Z_t^ε and yields on each edge I_i of the graph

$$\bar{A}_i(H) = \frac{1}{T_i(H)} \int_0^{T_i(H)} [Q(x,H)G(x) + F(x)]\, dt = \frac{1}{T_i(H)} [B_i(H) + C_i(H)]$$

$$B_i(H) = 2 \int_{x_i^-}^{x_i^+} G(x)\sqrt{Q(x,H)}dx = \beta B_i^1(H) - B_i^2(H)$$

$$B_i^1(H) = 2 \int_{x_i^-}^{x_i^+} \sqrt{Q(x,H)}dx, \quad B_i^2(H) = 2 \int_{x_i^-}^{x_i^+} x^2\sqrt{Q(x,H)}dx$$

$$C_i(H) = 2 \int_{x_i^-}^{x_i^+} \frac{F(x)}{\sqrt{Q(x,H)}}dx, \quad T_i(H) = \int_0^T dt = 2 \int_{x_i^-}^{x_i^+} \frac{dx}{\sqrt{Q(x,H)}}$$

$$\bar{\sigma}_i^2(H) = \frac{1}{T_i(H)} \int_0^{T_i(H)} [2Q(x,H)F(x)]\, dt = \frac{1}{T_i(H)} \sigma_i^2(H)$$

$$\sigma_i^2(H) = \int_0^{T_i(H)} 2Q(x,H)F(x)dt = \nu_2^2 B_i^2(H) + \nu_1^2 B_i^1(H)$$

$$Q(x,H) = 2H + \alpha x^2 - \frac{x^4}{2}, \quad G(x) = \beta - x^2, \quad F(x) = \frac{1}{2}(\nu_1^2 + \nu_2^2 x^2)$$

where $x_i^{(\pm)}$ are the points where the periodic orbit intersects the $x-$axis, i.e., the points where $y = 0$. The gluing condition for the vertex $\mathcal{O}$ that corresponds to the saddle point is given by

$$\sum_{i=1}^{3} (\pm)\sigma_i^2(\mathcal{O}) f_i'(\mathcal{O}) = 0 \tag{11}$$

where $f_i'(\mathcal{O}) = \lim_{H \to H(\mathcal{O})} f_i'(H)$ for $(H,i) \in I_i$. The '+' sign is taken if the coordinate H on the edge I_i is greater that $H(\mathcal{O})$ and the '$-$' sign is taken otherwise. In addition the function $(H,i) \to f_i(H)$ is continuous on Γ. Thus, the domain of the averaged generator (10) is given by

$$\mathcal{D}(\hat{\mathcal{L}}_i) \stackrel{\text{def}}{=} \left\{ f_i \in \mathbf{C}(\Gamma) : \sum_{i=1}^{3} (\pm)\sigma_i^2(\mathcal{O}) f_i'(\mathcal{O}) = 0 \right\}$$

Making use of the relation $\frac{d\sigma_i^2(H)}{dH} = 2C_i(H)$, the generator (10) reduces to

$$\hat{\mathcal{L}}_i f_i(H) = \frac{1}{T_i(H)} \left(B_i(H) f_i'(H)\right) + \frac{1}{2} \frac{1}{T_i(H)} \left(\sigma_i^2(H) f_i'(H)\right)' \tag{12}$$

For different values of H, we have different path integrals (oscillations or rotations) and thus different drift $\bar{A}(H)$ and diffusion coefficients $\bar{\sigma}_{HH}^2(H)$. They are evaluated as follows:

1. $\alpha > 0, \quad H < 0$: The integrals are calculated along the paths which correspond to the "oscillations". Due to the Z_2 symmetry of the Hamiltonian we have two such edges and these are denoted by $i = 1, 2$

$$
\begin{aligned}
m^2 &= \frac{2\sqrt{\alpha^2+4H}}{\alpha+\sqrt{\alpha^2+4H}}, \quad H = \frac{\alpha^2(m^2-1)}{(2-m^2)^2} \\
T_i(H) &= 2(\frac{2-m^2}{\alpha})^{\frac{1}{2}}F(m), \quad C_i(H) = 2(\frac{\alpha}{2-m^2})^{\frac{1}{2}}\left[\nu_2^2 E(m) + \frac{\nu_1^2}{2\alpha}(2-m^2)F(m)\right] \\
B_i^1(H) &= \frac{4}{3}(\frac{\alpha}{2-m^2})^{\frac{3}{2}}\left[(2-m^2)E(m) - 2(1-m^2)F(m)\right] \qquad (13) \\
B_i^2(H) &= \frac{8}{15}(\frac{\alpha}{2-m^2})^{\frac{5}{2}}\left\{\left[6(m^2-1)+2(2-m^2)^2\right]E(m) + (2-m^2)(m^2-1)F(m)\right\}
\end{aligned}
$$

Where $F(m), E(m)$ are complete elliptic integrals of the first and the second kinds with the modulus m. $T_i(H)$ is the period of the periodic orbit.

2. $\alpha > 0, \quad H > 0$: In this case, the integrals are calculated along the paths which correspond to the "rotations".

$$
\begin{aligned}
m^2 &= \frac{1}{2}(1+\frac{\alpha}{\sqrt{\alpha^2+4H}}), \quad H = \frac{\alpha^2 m^2(1-m^2)}{(2m^2-1)^2}, \quad T_3(H) = 4(\frac{2m^2-1}{\alpha})^{\frac{1}{2}}F(m) \\
C_3(H) &= 2(\frac{\alpha}{2m^2-1})^{\frac{1}{2}}\left\{2\nu_2^2 E(m) + \left[-2\nu_2^2(1-m^2) + \frac{\nu_1^2}{\alpha}(2m^2-1)\right]F(m)\right\} \qquad (14) \\
B_3^1(H) &= \frac{8}{3}(\frac{\alpha}{2m^2-1})^{\frac{3}{2}}\left[(2m^2-1)E(m) + (1-m^2)F(m)\right] \\
B_3^2(H) &= \frac{16}{15}(\frac{\alpha}{2m^2-1})^{\frac{5}{2}}\left[2(m^4-m^2+1)E(m) + (-2+m^2)(1-m^2)F(m)\right]
\end{aligned}
$$

The H process can move from one region of closed trajectories to another. The aim of this paper is to consider the results pertaining to the ends of the edges where there is a homoclinic orbit connecting a saddle point and to examine the fate of the trajectories which leave the region where they originated. We are also interested in the qualitative changes in the probability densities p_α which are solutions of the *Fokker-Planck equation* (FPE) associated with the generator (10).

MEAN FIRST PASSAGE TIME

The scale and speed measures on each edge connecting the vertex are given by

$$S_i(H) \stackrel{\text{def}}{=} \int_{z_i}^{H} s_i(\eta)d\eta, \quad \text{and} \quad M_i(H) \stackrel{\text{def}}{=} \int_{z_i}^{H} m_i(\eta)d\eta$$

where

$$s_i(\eta) = \frac{1}{\sigma_i^2(\eta)} \exp\left\{-2\int_{z_i}^{\eta} \frac{B_i(\xi)}{\sigma_i^2(\xi)} d\xi\right\}, \quad m_i(\eta) = T_i(\eta) \exp\left\{2\int_{z_i}^{\eta} \frac{B_i(\xi)}{\sigma_i^2(\xi)} d\xi\right\}$$

and H is an interior point for each edge I_i. For $i = 3$, the lower and upper limits of integration are H and z_3 respectively. In order to examine the boundary behavior, we consider the quantities Σ_i and N_i. Here, Σ_i roughly measures the time to reach the left or right boundary starting from an interior point $H_i \in \Gamma_i \backslash \mathcal{O}$ while N_i measures the time to reach an interior point H_i starting from the boundary z_i or $\mathcal{O}$. For a thorough discussion on boundary classification, see [5]. Consider the vertex $\mathcal{O}$, then

$$\Sigma_i(\mathcal{O}) = \int_H^{H(\mathcal{O})} \{\int_z^{H(\mathcal{O})} \{m_i(y)dy\} s_i(z)\} dz,$$
$$N_i(\mathcal{O}) = \int_H^{H(\mathcal{O})} \{\int_z^{H(\mathcal{O})} \{s_i(y)dy\} m_i(z)\} dz$$

The ends of the edges I_i are fixed points of elliptic or saddle type. The Feller classification defines whether the end of an edge in the H coordinate is accessible from the inside and whether the inside is accessible from the end of an edge. This provides information as to whether a trajectory will reach the homoclinic orbit or equivalently the vertex $\mathcal{O}$. If the homoclinic orbit is accessible then the analysis needed to study the fate of the trajectory requires the "gluing" condition (11).

Based on $\Sigma_i(b_i)$ and $N_i(b_i)$, we find that the boundaries b_1 and b_2 are entrance boundaries while b_3 (which corresponds to $H \to \infty$) is a natural boundary. Since $\mathcal{D}'(h) = T(h)$, where $\mathcal{D}(h)$ is the area bounded by $E(h)$ ($E(h) \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 : H(x,y) = h\}$), it is clear that $S_i(H)$ and $M_i(H)$ are finite as $H \to H(\mathcal{O})$. Thus, according to the Feller classification, vertex $\mathcal{O}$ is accessible and the gluing condition is needed to solve the mean exit time problem.

Suppose that at time $t = 0$, the state of the system corresponds to some point defined by $H(0) = h$ within $\mathcal{D}$ which is the domain of attraction with boundary $\partial\mathcal{D}$. Here $\mathcal{D}$ is defined by the deterministic part and may have some physical significance. Let z_i be the points on the edge of the graph corresponding to the boundary $\partial\mathcal{D}$. We are interested in the time $\tau_c \stackrel{\text{def}}{=} \min\{\tau_i\}$ where $\tau_i = \inf\{t \geq 0; H(t) = z_i\}$ is the hitting time of the averaged process to the level z_i.

Define the mean time to reach either z_1, z_2, or z_3 by the function $u(h) \stackrel{\text{def}}{=} \mathbb{E}[\tau_c : H(0) = h]$. Since the averaged process is Markov with the generator (10) it follows from the classical theory of Markov processes

$$\hat{\mathcal{L}}_i u_i(h) = -1, \ (h,i) \in \Gamma \backslash \mathcal{O} \tag{15}$$

with the boundary and gluing conditions given by

$$u_i(z_i) = 0 \ \forall \ i, \ \text{ and } \ \sum_{i=1}^{3} (\pm)\sigma_i^2(\mathcal{O}) u_i'(\mathcal{O}) = 0$$

The generator (10) is defined for continuous functions $u(h), h \in \Gamma$. Hence, at the vertex $\mathcal{O}$ we have $u_1(\mathcal{O}) = u_2(\mathcal{O}) = u_3(\mathcal{O})$. Thus, we have six boundary conditions for determining uniquely the mean first passage time. The solution of Eq. (15) is given by

$$u_i(h) = -2\int_{z_i}^{h}\left[\int_{z_i}^{\eta} m_i(\xi)\,d\xi\right] s_i(\eta)\,d\eta + \beta_i\left[S_1(h) - S_1(z_1)\right] + \alpha_i, \quad i = 1,2$$

$$u_3(h) = -2\int_{h}^{z_3}\left[\int_{\eta}^{z_3} m_3(\xi)\,d\xi\right] s_3(\eta)\,d\eta + \beta_3\left[S_3(z_3) - S_3(h)\right] + \alpha_3 \tag{16}$$

Imposing the boundary conditions at $h = z_1, z_2$ and z_3 yields $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Applying the continuity and gluing conditions at the vertex $\mathcal{O}$, the constants $\beta_1, \beta_2, \beta_3$ can be determined for various values of the system parameters $\alpha, \beta, \nu_1, \nu_2$. The gluing conditions for the Duffing-van der Pol equation are obtained by taking

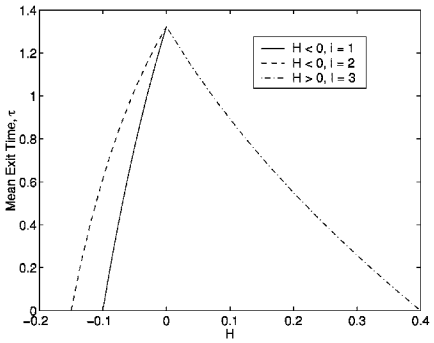


FIGURE 2. Variation of the mean exit time, τ, with the energy level H

appropriate limits of the drift and diffusion terms given by Eq.(13) and Eq.(14). The dissipation effect at the vertex is given by the standard Melnikov function associated with the homoclinic orbit

$$B_i(\mathcal{O}) = \beta B_i^1(\mathcal{O}) - B_i^2(\mathcal{O}) \tag{17}$$

$$B_i(\mathcal{O}) = \frac{2}{15}\sqrt{\alpha^3}\left\{5\beta - 4\alpha\right\}, \text{ for } i = 1,2, \text{ and } B_3(\mathcal{O}) = \frac{4}{15}\sqrt{\alpha^3}\left\{5\beta - 4\alpha\right\}$$

while the noise effects $\lim_{H\to H(\mathcal{O})}\sigma_i^2(H)$ at the vertex $\mathcal{O}$ are given by

$$\sigma_i^2(\mathcal{O}) = \frac{4}{15}\sqrt{\alpha^3}\left\{5\nu_1^2 + 4\nu_2^2\alpha\right\}, \text{ for } i = 1,2, \tag{18}$$

$$\text{and } \sigma_3^2(\mathcal{O}) = \frac{8}{15}\sqrt{\alpha^3}\left\{5\nu_1^2 + 4\nu_2^2\alpha\right\}$$

Now, taking the appropriate signs and making use of Eq.(??) the gluing condition (11) for the Duffing-van der Pol equation simplifies to

$$u_1'(\mathcal{O}) + u_2'(\mathcal{O}) - 2u_3'(\mathcal{O}) = 0, \tag{19}$$

In this work, the numerical calculations were performed with the aid of *Mathematica 3.0*. Figure 2 shows the variation of the mean exit time with H for $z_1 = -0.1$, $z_2 = -0.15$, $z_3 = 0.4$, $\alpha = 1$, $\beta = 1$, $\nu_1 = 1$, $\nu_2 = 1$.

STATIONARY DENSITY

In this section, we shall examine the stationary behavior of the Fokker-Planck equation associated with the generator (12). In order to obtain the FPE, let $E(h) \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 : H(x,y) = h\}$. The inner product with respect to the Lebesgue measure is given by the usual definition

$$\langle f(x,y), g(x,y) \rangle_{\mathbb{R}^2} \stackrel{\text{def}}{=} \int \int_{\mathcal{D}(h)} f(x,y), g(x,y) dx\, dy$$

where $\mathcal{D}(h)$ is the area bounded by $E(h)$. Making use of the relation $\mathcal{D}'(h) = T(h)$ (where the prime denotes differentiation with respect to the energy level, h), the appropriate inner product in the local coordinate H is given by

$$\langle f(H), g(H) \rangle_H \stackrel{\text{def}}{=} \int_H f(H) g(H) T(H) dH \tag{20}$$

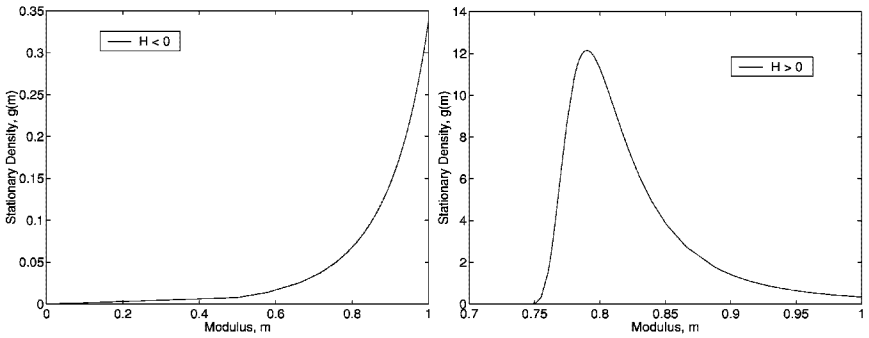
The adjoint equation, and the boundary conditions are determined by considering the inner product (20),

$$\sum_{i=1}^{3} (\pm) \int_0^{b_i} \widehat{\mathcal{L}} f_i(H) g_i(H) \mathcal{D}(z)'\, dH = \sum_{i=1}^{3} (\pm) \int_0^{b_i} \widehat{\mathcal{L}}^* g_i(H) f_i(H)\, dH$$
$$+ \sum_{i=1}^{3} (\pm)\, \frac{1}{2} \sigma_i^2(H) f_i'(H) g_i(H) \Big|_0^{b_i}$$
$$+ \sum_{i=1}^{3} (\pm) \left\{ B_i(H)) g_i(H) - \frac{1}{2} \sigma_i^2(H) g_i'(H) \right\} f_i(H) \Big|_0^{b_i}$$

where

$$\widehat{\mathcal{L}}^*(g_i(H)) \stackrel{\text{def}}{=} -\frac{1}{T_i(H)} \left(B_i(H) g_i(H)\right)' + \frac{1}{2} \frac{1}{T_i(H)} \left(\sigma_i^2(H) g_i'(H)\right)' \tag{21}$$

and once again the '+' sign is taken if the coordinate H on the edge I_i is greater than $H(\mathcal{O})$ and the '$-$' sign is taken otherwise. First, at the exterior vertices indicated by b_i, the first boundary condition is easily satisfied since $f_i' = 0$ at an entrance or natural boundary. The second boundary condition implies that the probability current at the boundaries, b_i is zero. However, at the interior vertex $\mathcal{O}$ we have to be very careful. Imposing the condition that the probability density is continuous at the vertex $\mathcal{O}$ (i.e., $g_1(\mathcal{O}) = g_2(\mathcal{O}) = g_3(\mathcal{O}) = g(\mathcal{O})$),



(a) (b)

FIGURE 3. Variation of the stationary density with the modulus, m

and taking into consideration the gluing condition (11), the first boundary condition in the above expression is identically zero. Now, making use of the fact that f is continuous at the vertex $\mathcal{O}$ (i.e., $f_1(\mathcal{O}) = f_2(\mathcal{O}) = f_3(\mathcal{O}) = f(\mathcal{O})$, the second boundary condition reduces to

$$\sum_{i=1}^{3}(\pm)\left\{B_i(\mathcal{O})g_i(\mathcal{O}) - \frac{1}{2}\sigma_i^2(\mathcal{O})g_i'(\mathcal{O})\right\} \stackrel{\text{def}}{=} \sum_{i=1}^{3}(\pm)\mathcal{J}_i \tag{22}$$

where $\mathcal{J}_i$ denotes the probability flux in each edge I_i of the vertex $\mathcal{O}$. Eq. (22) implies that the *probability flux* is conserved. Now, taking the appropriate signs and making use of Eq.(17) and Eq.(**??**) the flux condition (22) for the Duffing-van der Pol equation is obtained as

$$g_1'(\mathcal{O}) + g_2'(\mathcal{O}) - 2g_3'(\mathcal{O}) = 0 \tag{23}$$

It is clear that

$$\left(\sigma_i^2(H)g_i'(H)\right) - 2\left(B_i(H)g_i(H)\right) = -2G_i^{st} \tag{24}$$

Integrating Eq. 24, we obtain

$$g_i(H) = \exp\left\{2\int_{H_0}^{H}\frac{B_i(\eta)}{{\sigma_i}^2(\eta)}d\eta\right\}\left[-2G_i^{st}\int_{H_0}^{H}\frac{\exp\left\{-2\int_{H_0}^{\eta}\frac{B_i(\xi)}{\sigma_i^2(\xi)}d\xi\right\}}{\sigma_i^2(\eta)}d\eta + D_i\right] \tag{25}$$

where G_i^{st} and D_i are to be determined from the appropriate boundary conditions and the flux condition at the vertex $\mathcal{O}$. Since the flux is zero at exterior vertices indicated by b_i, it can be shown that G_i^{st} are identically zero. Making use of the continuity conditions, the flux condition and the normalization condition,

$\int_H g(H)dH = 1$, we can determine the constants D_i. Figure 3 shows the variation of the stationary density with the modulus m for $\alpha = 1, \beta = 1, \nu_1 = 1, \nu_2 = 1$. (the relation between the modulus, m, and the Hamiltonian, H, is given in Eq. (13) and Eq. (14)).

CONCLUSIONS

In this paper we extend the work by Arnold *et al.* [1] and Liang and Sri Namachchivaya [7] to obtain analytical results, close to the homoclinic orbits, for the behavior of the stochastic version of the *Duffing-van der Pol* equation, away from the trivial solution (global).

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