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Chaotic vibration of the one-dimensional linear wave equation with a van der Pol nonlinear boundary condition

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Abstract: The one-dimensional linear wave equation with a van der Pol nonlinear boundary condition is one of the simplest models that may cause isotropic or nonisotropic chaotic vibrations. It characterizes the nonisotropic chaotic vibration by means of the total variation theory. Some results are derived on the exponential growth of total variation of the snapshots on the spatial interval in the long-time horizon when the map and the initial condition satisfy some conditions.

Keywords: Chaos; Wave equation; Van der Pol nonlinearity; Total variation; Homoclinic point; Topological transitivity.

1 Introduction

In this paper, we consider the one-dimensional (1D) wave equation

$$\begin{cases} w_{xx}(x,t) - \upsilon w_{xt}(x,t) - w_{u}(x,t) = 0, \ x < x < 1, t > 0, \\ w_t(0,t) = 0, \upsilon > 0, t > 0, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \alpha, \beta > 0, t > 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), 0 < x < 1. \end{cases} \tag{1.1}$$

We will study the dynamical behavior of the above system by using total vibration theory as the two parameters (υ, α) vary in $[0, \infty] \times [0, +\infty]$. Chen et al. [1 ~ 4] firstly studied chaotic vibrations of 1D wave equation on a bounded interval with a van der Pol boundary condition, which is a well-known self-regulating mechanism in automatic control.

As a continuation of those latest work [5] this paper gives larger regimes of the rates of the total variations growing exponentially than in [5] impelled by [6].

Similar to [7], we let

$$\rho_1(\upsilon) \equiv \frac{-\upsilon + \sqrt{4 + \upsilon^2}}{2}, \tag{1.2}$$

$$\rho_2(\upsilon) \equiv \frac{\upsilon + \sqrt{4 + \upsilon^2}}{2}. \tag{1.3}$$

We have

$$\begin{cases} \rho_1(\upsilon)\rho_2(\upsilon) = 1, \\ \rho_2(\upsilon) - \rho_1(\upsilon) = \upsilon > 0, \\ \rho_1(\upsilon) + \rho_2(\upsilon) = \sqrt{4 + \varphi^2}. \end{cases} \tag{1.4}$$

Let

$$\begin{cases} u = \dfrac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_2(\upsilon) w_x + w_t], \\ v = \dfrac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_1(\upsilon) w_x - w_t], \end{cases} \tag{1.5}$$

we can convert (1.1) into the equivalent uncoupled first order hyperbolic system

$$\begin{cases} u_t(x,t) = \rho_1(\upsilon) u_x(x,t), \\ v_t(x,t) = -\rho_2(\upsilon) v_x(x,t), \end{cases} \quad 0 < x < 1, \ t > 0. \tag{1.6}$$

The initial conditions for u and v are

$$\begin{cases} u(x,0) = u_0(x) \\ \quad = \dfrac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_2(\upsilon) w_0'(x) + w_1(x)], \\ v(x,0) = v_0(x) \\ \quad = \dfrac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_2(\upsilon) w_0'(x) - w_1(x)]. \end{cases} \tag{1.7}$$

The boundary conditions at $x = 0$ are

$$v(0,t) = -u(0,t) \equiv G(u(0,t)), \ t > 0, \tag{1.8}$$

and

$$u(1,t) = F_{\upsilon,\alpha}(v(1,t)). \tag{1.9}$$

For given $x \in \mathbb{R}$,

$$F_{\upsilon,\alpha}(x) = \rho_2[\rho_2 x + g_{\upsilon,\alpha}(x)], \tag{1.10}$$

where $y = g_{\upsilon,\alpha}(x)$ is the unique real solution of the cubic equation

$$\beta y^3 + (\rho_2 - \alpha) y + (\rho_2^2 + 1) x = 0, \tag{1.11}$$

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provided that

$$0 < \alpha \leqslant \rho_2 (= \rho_2(x)). \quad (1.12)$$

Since the parameter β in the equation only plays the role of "scaling" (see [7]), it does not affect the properties of the functions $g_{\upsilon,\alpha}(x)$ and $F_{\upsilon,\alpha}(x)$. Thus we can view $g_{\upsilon,\alpha}(x)$ and $F_{\upsilon,\alpha}(x)$ as functions that are dependent only on the two parameters υ and α.

If (1.12) is violated, then $g_{\upsilon,\alpha}(x)$ is multi-valued. From now on, we always assume that $\alpha, \beta, \upsilon > 0$ satisfy condition (1.12). By means of characterization, the solution u and v of (1.6) ~ (1.9) can be expressed explicitly as follows: for $t = k(\rho_1 + \rho_2) + \tau, k = 0, 1, 2, \cdots, 0 \leqslant \tau \leqslant \rho_1 + \rho_2$, and $0 \leqslant x \leqslant 1$,

$$u(x,t) = \begin{cases} (F_{\upsilon,\alpha} \circ G)^k (u_0(x + \rho_1 \tau)), \ \tau \leqslant \rho_2(1-x), \\ F_{\upsilon,\alpha}(G \circ F_{\upsilon,\alpha})_k (v_0(1+\rho_2^2 - \rho_2^2(x+\rho_1\tau))), \\ \quad \rho_2(1-x) < \tau \leqslant \rho_2(1+\rho_1^2 - x), \\ (F_{\upsilon,\alpha} \circ G)^{k+1}(u_0(x + \rho_1\tau - 1 - \rho_1^2)), \\ \quad \rho_2(1+\rho_1^2 - x) < \tau \leqslant \rho_1 + \rho_2, \end{cases} \quad (1.13)$$

$$v(x,t) = \begin{cases} (F_{\upsilon,\alpha} \circ G)^k (v_0(x - \rho_2\tau)), \ \tau \leqslant \rho_1 x, \\ F_{\upsilon,\alpha}(G \circ F_{\upsilon,\alpha})^k (u_0(-\rho_1^2(x - \rho_2\tau))), \\ \quad \rho_1 x < \tau \leqslant \rho_1(x + \rho_2^2), \\ (F_{\upsilon,\alpha} \circ G)^{k+1}(v_0(x - \rho_2\tau + 1 - \rho_2^2)), \\ \quad \rho_1(x+\rho_2^2) < \tau \leqslant \rho_1 + \rho_2. \end{cases} \quad (1.14)$$

For example, $(G \circ F_{\upsilon,\alpha}(x))^k$ denotes the k-folded iterative composition of $G \circ F_{\upsilon,\alpha}(x)$.

From these explicit representations, we can estimate the growth rate of the total variations of $u(\cdot, t)$ and $v(\cdot, t)$ on $[0,1]$ as t goes to infinity by means of the estimation of those of $(G \circ F_{\upsilon,\alpha}(x))^k(\cdot)$ and $(F_{\upsilon,\alpha} \circ G(x))^n(\cdot)$ on some spatial intervals as n goes to infinity.

2 Some property of the map $G \circ F_{\upsilon,\alpha}$ and some basic Lemma

We firstly consider some properties of the map $G \circ F_{\upsilon,\alpha}$. Most of the basic properties of the map have been established by Chen et al. in [7].

Lemma 2.1 Let $\beta > 0$. Then the map $G \circ F_{\upsilon,\alpha}$ has the following properties:

1) $G \circ F_{\upsilon,\alpha}(\cdot)$ is odd;

2) $G \circ F_{\upsilon,\alpha}$ has exactly three fixed points 0, v_0 and $-v_0$, where

$$v_0 = v_0(\upsilon, \alpha) = \frac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)} \sqrt{\frac{\alpha}{\beta}}; \quad (2.1)$$

3) $-G \circ F_{\upsilon,\alpha}(= F_{\upsilon,\alpha})$ has exactly there fixed points $-v_1$, 0 and v_1, where

$$v_1 = v_1(\upsilon,\alpha) = \frac{1}{\rho_2 - \rho_1}\sqrt{\frac{1 + \alpha(\rho_2 - \rho_1)}{\beta(\rho_2 - \rho_1)}} = \frac{1}{\upsilon}\sqrt{\frac{1+\alpha\upsilon}{\beta\upsilon}}, \quad (2.2)$$

in which the last equality in (2.2) follows from (1.4);

4) The equation $G \circ F_{\upsilon,\alpha}(v) = 0$ has exactly three roots $-v_I$, 0 and v_I, where

$$v_I = v_I(\upsilon, \alpha) = \frac{1}{\rho_2}\sqrt{\frac{1+\alpha\rho_2}{\beta\rho_2}}; \quad (2.3)$$

5) $G \circ F_{\upsilon,\alpha}$ has local external values

$$m = G \circ F_{\upsilon,\alpha}(-v_c) = -\frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1 + \rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}}, \quad (2.4)$$

$$M = G \circ F_{\upsilon,\alpha}(v_c) = \frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1+\rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}} = -m, \quad (2.5)$$

where

$$v_c = v_c(\upsilon,\alpha) = \frac{3\rho_2^2 - 2\alpha\rho_2 + 1}{3\rho_2(\rho_2^2+1)}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}}, \quad (2.6)$$

$-v_c$ and v_c are critical points of $G \circ F_{\upsilon,\alpha}$. Here m and M are, respectively, the local minimum and maximum of $G \circ F_{\upsilon,\alpha}$. The function $G \circ F_{\upsilon,\alpha}$ is strictly decreasing on $(-\infty, v_c)$ and $(v_c, +\infty)$, but strictly increasing on $(-v_c, v_c)$;

6) $$(G \circ F_{\upsilon,\alpha})'(v) = -\rho_2^2 + \rho_2\frac{\rho_2^2+1}{D}, \quad (2.7)$$

$$(G \circ F_{\upsilon,\alpha})''(v) = \frac{6\beta\rho_2(\rho_2^2+1)^2 g_{\upsilon,\alpha}(v)}{D^3}, \quad (2.8)$$

where

$$D = 3\beta g_{\upsilon,\alpha}^2(v) + \rho^2 - \alpha,$$

and $g_{\upsilon,\alpha}(\cdot)$ is defined by (1.11).

Proof See Section 2 in [7].

As in [5], we let

$$h_1(\upsilon) \equiv \frac{1}{\rho_2(\upsilon)}\left[\frac{3\sqrt{3}}{2\rho_2(\upsilon)} + \frac{3\sqrt{3}}{2} - 1\right], \quad (2.9)$$

$$h_2(\upsilon) \equiv 3\frac{1+\rho_1^2}{\upsilon}\cos\theta - \rho_1, \quad (2.10)$$

where

$$\theta = \frac{1}{3}\arccos\frac{1}{1+\rho_2^2} \quad \left(\frac{\pi}{9} \leqslant \theta \leqslant \frac{\pi}{6}\right),$$

and let

$$S = \{(\upsilon, \alpha) \in \mathbb{R}^2 \mid 0 < \upsilon < +\infty, 0 < \alpha \leqslant \rho_2(\upsilon)\}, \quad (2.11)$$

$$S_1 = \{(v,\alpha) \mid 0 < v < v_1, 0 < \alpha \leqslant \rho_2(v)\} \cup \{(v,\alpha) \in \mid v_1 \leqslant v < +\infty, 0 < \alpha \leqslant h_1(v)\}, \quad (2.12)$$

$$S_2 = \{(v,\alpha) \mid v_1 < v < v_2, 0 < \alpha \leqslant \rho_2(v)\} \cup \{(v,\alpha) \in \mid h_1(v) \leqslant v < +\infty, h_1(v) < \alpha \leqslant h_2(v)\}, \quad (2.13)$$

$$S_3 = \{(v,\alpha) \in \mid v_2 < v < +\infty, h_2 < \alpha \leqslant \rho_2(v)\}. \quad (2.14)$$

We divided the regime S_1 into two parts

$$S_1^0 = \{(v,\alpha) \mid 0 < v < v_1, 0 < \alpha \leqslant \rho_2(v)\} \cup \{(v,\alpha) \in \mid v_0 \leqslant v < +\infty, 0 < \alpha \leqslant \frac{1}{v}\}, \quad (2.15)$$

$$S_1^1 = \{(v,\alpha) \mid v_0 < v < v_1, \frac{1}{v} < \alpha \leqslant \rho_2(v)\} \cup \{(v,\alpha) \in \mid v_1 \leqslant v < +\infty, \frac{1}{v} < \alpha \leqslant h_1(v)\}, \quad (2.16)$$

We can divide S into four regions. See Fig. 1.

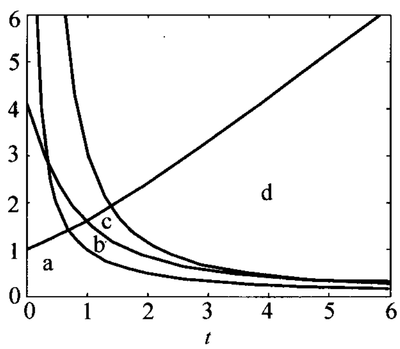

Fig. 1 $S_1^1(b), S_1^0(a)$ $S_2(c)$, and $S_3(d)$.

A routine check shows that

$$v_c(v,\alpha) < v_0(v,\alpha) < M(v,\alpha) < v_I(v,\alpha), \quad \text{if } (v,\alpha) \in S_1^1,$$

and if $(v,\alpha) \in S_1^1$, $[0, v_I]$ and $[-v_I, 0]$ are bounded invariant intervals of $G \circ F_{v,\alpha}$. See Figs. 2 and 3.

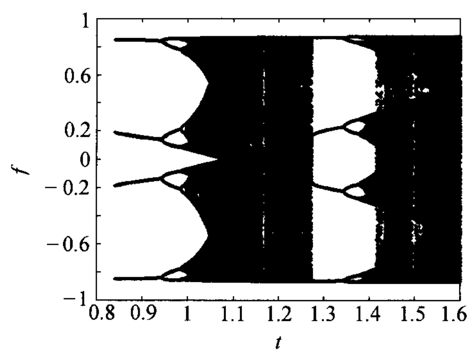

Fig. 2 Orbit diagram of $G \circ F_{v,\alpha} = f$ with $v(\alpha = 1.5)$.

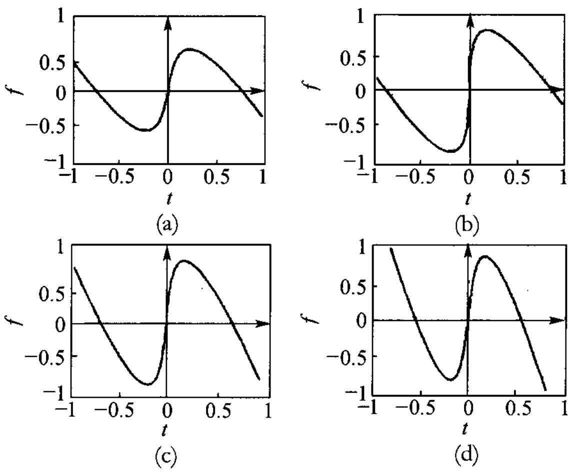

Fig. 3 Graph of $G \circ F_{v,\alpha} = f$ in different domain, eg. $(v,\alpha) \in S_1^1(b)$.

Proposition 2.1 [5, Theorem 3.1] Consider the initial-boundary value problem (1.1). Assume that w_0 and w_1 in (1.1) are sufficiently smooth such that u_0 and v_0 in (1.7) are continuous and piecewise monotone and satisfy the compatibility conditions

$$v_0(0) = -u_0(0), u_0(1) = f_{v,\alpha}(v_0(1)), \quad (2.17)$$

Then we have

1) If $(v,\alpha) \in S_1^0$ and $|u_0(x)| \leqslant v_I$, $|v_0(x)| \leqslant v_I$, $\forall x \in [0,1]$, then, for u and v in (1.6), we have $V_{[0,1]}(u(x,t)) \leqslant C$ and $V_{[0,1]}(u(x,t)) \leqslant C$, $\forall t \geqslant 0$, for some positive constant C independent of t;

2) If $(v,\alpha) \in S_1^1$ and there is a small positive ε_0 such that

$$[0, \varepsilon_0] \subset \text{Range}(u_0) \cap \text{Range}(v_0)$$

or

$$[-\varepsilon_0, 0] \subset \text{Range}(u_0) \cap \text{Range}(v_0), \quad (2.18)$$

then, for u and v in (1.6), we have

$$\lim_{t\to\infty} V_{[0,1]}(u(\cdot,t)) = \infty \text{ and } \lim_{t\to\infty} V_{[0,1]}(v(\cdot,t)) = \infty;$$

3) If $(v,\alpha) \in S_2$ and there is a small positive constant ε_0 such that (2.18) holds, then for u and v in (1.6), the growth rates of $V_{[0,1]}(u(\cdot,t))$ and $V_{[0,1]}(v(\cdot,t))$ are at least exponential as $t \to \infty$.

Lemma 2.2 [4, Corollary 9.1] Let $f \in C^0(I,I)$. Suppose f is piecewise monotone with a finite number of external points. Then the following conditions are equivalent:

1) f has a periodic point whose period is not a power of 2;

2) f has a homoclinic point, that is, $P_h(f) \neq \emptyset$;

3) f has positive topological entropy;

4) The total variation $V_{[0,1]}(f^n)$ grows exponentially as $n \to \infty$.

Lemma 2.3 Let $f \in C^0(I,I)$ and f be piecewise monotone with finitely many extremal points. If f is topologically mixing, then, for any non-degenerate closed subinterval J of I, we have $V_J(f^n)$ grows exponentially as $n \to \infty$.

Proof Since topological mixing implies that f has positive entropy, from Lemma 2.2, we have

$$V_I(f^n) \text{ grows exponentially as } n \to \infty. \quad (2.19)$$

By [8, Proposition 45 and Theorem 46], we have for non-degenerate of closed subinterval J of I, there exists a positive integer K such that $f^K(J) = I$. So the total variation $V_J(f^n)$ grows exponentially as $n \to \infty$ by (2.19).

Now if we let f be topologically transitive, from [8, Proposition 42 and Theorem 46], it follows that either f is topological mixing on I or there exist nonempty closed subintervals J, K with $J \bigcup K = I$ and $J \bigcap K = \{y\}$, where y is a fixed point of f, such that $f^2 \mid_J$ and $f^2 \mid_K$ are topologically mixing. Combining with Lemma 2.3, we have

Lemma 2.4 Let $f \in C^0(I, I)$ be piecewise monotone with a finite number of extremal points. If f is topologically transitive, then $V_J(f^n)$ grows exponentially as $n \to \infty$ for any non-closed subinterval J of I.

Let $f \in C^0(I, I)$, and A_n denote the graph of f^n. That is, $A_n = \{(x, f^n(x)) \in I \times I \mid x \in I\}$. The superior and inferior limits of this set sequence are defined as usual. If we denote

$$B_k = \bigcup_{n \geqslant k}^{\infty} A_n = \bigcup_{n \geqslant k}^{\infty} \text{graph}(f^n).$$

Then we have $\limsup_{n \to \infty} A_n = \bigcap_{k=1}^{\infty} \bar{B}_k$.

The following proposition characterizes the geometrical features of topological mixing, topological transitivity and positive entropy, respectively, for an interval map.

Proposition 2.2[6] Let $f \in C^0(I, I)$. We have

1) f is topological mixing on I if and only if $\liminf_{n \to \infty} A_n = I \times I$.

2) f is topological transitivity on I if and only if $\limsup_{n \to \infty} A_n = I \times I$.

3) If $\limsup_{n \to \infty} A_n$ contains an interior point, then f has positive entropy (equivalently, f has a homoclinic point).

3 Main result

Since $G \circ F_{\upsilon,\alpha}$ and $F_{\upsilon,\alpha} \circ G$ are topologically conjugate, we only need to consider the map $G \circ F_{\upsilon,\alpha}$. Moreover, since the parameter β in the equation only plays the role of "scaling" [7], it does not affect the properties of the map. In the following, we assume that $\beta \equiv 1$.

Theorem 3.1 Assume that

1) The map $G \circ F_{\upsilon,\alpha}$ has a bounded invariant interval $J = [-M_1, M_1]$ and a homoclinic point in J.

2) The initial conditions u_0 and v_0 of the system (1.6) are continuous such that either the Range of u_0 or the Range of v_0 contains a point $p \in P_h(G \circ F_{\upsilon,\alpha})$ in its interior.

Then, for the solution pair (u, v) of (1.13), we have

$$V_{[0,1]}(u(\cdot, t)) + V_{[0,1]}(v(\cdot, t)),$$

which grows exponentially as $t \to \infty$.

Proof Since $G \circ F_{\upsilon,\alpha}$ and $F_{\upsilon,\alpha} \circ G$ are topologically conjugate and

$$(G \circ F_{\upsilon,\alpha}) \circ G = G \circ (F_{\upsilon,\alpha} \circ G), \quad (3.2)$$

we have

$$P_h(G \circ F_{\upsilon,\alpha}) = G^{-1}(P_h(F_{\upsilon,\alpha} \circ G)) = \{-p \mid p \in P_h(F_{\upsilon,\alpha} \circ G)\}. \quad (3.3)$$

Let $t = k(\rho_1 + \rho_2) + \tau$ for some integer k and $0 \leqslant \tau \leqslant \rho_1 + \rho_2$. The theorem suffices to prove that, for any fixed $\tau \in [0, \rho_1 + \rho_2)$, $V_{[0,1]}(u(\cdot, 2k + \tau)) + V_{[0,1]}(v(\cdot, 2k + \tau))$ grows exponentially as $t \to \infty$. Assume $0 \leqslant \tau \leqslant \rho_1$ (the proof is similar if $\rho_1 \leqslant \tau \leqslant \rho_1 + \rho_2$.) By the expressions (1.13) and (1.14) of u and v, we have

$$\begin{aligned} V_{[0,1]}(u(\cdot, t)) &= V_{[0,1-\tau\rho_1]}(u(\cdot, t)) + V_{[1-\tau\rho_1,1]}(u(\cdot, t)) \\ &= V_{[\rho_1\tau,1]}(F_{\upsilon,\alpha} \circ G)^k(u_0(\cdot)) \\ &\quad + V_{[1-\rho_2\tau,1]}(G^{-1} \circ (G \circ F_{\upsilon,\alpha})^{k+1}(v_0(\cdot))), \end{aligned} \quad (3.4)$$

$$\begin{aligned} V_{[0,1]}(u(\cdot, t)) &= V_{[0,\tau\rho_2]}(v(\cdot, t)) + V_{[\tau\rho_2,1]}(v(\cdot, t)) \\ &= V_{[(0,\rho_2\tau]}(G \circ (F_{\upsilon,\alpha} \circ G)^k(u_0(\cdot)) \\ &\quad + V_{[0,1-\rho_2\tau]}(G \circ F_{\upsilon,\alpha})^k(v_0(\cdot)). \end{aligned} \quad (3.5)$$

Firstly we assume that there is a point $p \in P_h(F_{\upsilon,\alpha} \circ G)$ such that the range of u_0 contains a neighborhood $O(p)$ of p. That is, either $u_0([0, \tau\rho_2])$ or $u_0([\tau\rho_1, 1])$ contains the point p as in its interior. Thus it follows (3.4) and (3.5) that there exists a subinterval J which contains p as its interior such that

$$V_{[0,1]}(u(\cdot, t)) + V_{[0,1]}(v(\cdot, t)) \geqslant V_J((F_{\upsilon,\alpha} \circ G)^k). \quad (3.6)$$

It suffices to prove $V_J((F_{\upsilon,\alpha} \circ G)^k)$ grows exponentially as $k \to \infty$.

In the following we will prove $V_J((G \circ F_{\upsilon,\alpha})^k)$ grows exponentially as $k \to \infty$ and subinterval J which contains $p \in P_h(G \circ F_{\upsilon,\alpha})$ for topologically conjugation of the two map.

Let $f = G \circ F_{\upsilon,\alpha}$, since $p \in P_h(f)$, then there is a homoclinic point x such that $x \neq p, x \in W^n(p, f^n)$ where n is the period of p, and $f^{nm} = p$ for some positive integer m. Let s be the smallest positive integer with $f^{ns} = p$, $z = f^{n(s-1)}(x)$ and $g = f^n$. Then p is a fixed point of g and $z \in W^u(p, g)$ with $z \neq p$ and $g(z) = p$.

Since $z \in W^u(p, g)$ and $z \neq p$, for any neighborhood V of p there is a point $z_1 \in V \cap W^u(p, g)$ such that $g^r(z_1) = z$ for some positive integer r ([9, Lemma 6]). Repeating the above procedure infinitely, we obtain a sequence $\{z_k\}_{k \geqslant 0}$ with the following properties:

z_k approaches p as $k \to \infty$; $z_0 = z$; for each positive integer k there is positive integer r such that $g^r(z_k) = z_{k-1}$; $z_k \in W^u(p, g)$.

We assume without loss of generality that infinitively many z_k are in the right of p. For any open interval $O(p)$ containing p, there is a point z_K with $p < z_K$ and $[p, z_K] \in O(p)$ by the property of the sequence $\{z_k\}_{k \geqslant 0}$.

Let y be any point of the sequence $\{z_k\}$ in the interval (p, z_K). Then there exists a positive integer m_1 such that $g^{m_1}(y) = p$. Also, for some positive integer m_2 with $m_2 > m_1$ there is a point y_1 of the sequence $\{z_k\}$ in the interval (p, y) such that $g^{m_2}(y_1) = z_K$. Since $m_2 > m_1$, we have $g^{m_2}(y) = p$, thus

$$g^{m_2}([p, y_1]) \supset [p, z_K],$$
$$g^{m_2}([y_1, y_2]) \supset [p, z_K].$$

It follows from the continuity of g that there exist $y_2 \in (p, y_1)$ and $y_3 \in (y_1, y)$ such that

$$g^{m_2}(y_2) = y,\ g^{m_2}(y_3) = y.$$

Let $J_1 = [p, y_2]$ and $J_2 = [y_3, y]$. Then

$$J_1 \cap J_2 = \emptyset,\ g^{m_2}(J_1) \cap g^{m_2}(J_2) \supset J_1 \cup J_2.$$

It follows that $V_{J_1}(g^{m_2 n_1})$ and $V_{J_2}(g^{m_2 n_1})$ grows exponentially as $n_1 \to \infty$.

Since f is a Lipschitz map, and so is $g^{m_2}(f^{nm_2})$. For any given l with $0 < l < nm_2$, we have

$$V_{J_1}(f^{nm_2(n_1+1)}) = V_{J_1}(f^{nm_2 - l} \circ f^{nm_2 n_1 + l}) \leqslant (\mathrm{Lip}(f))^{nm_2 - l} V_{J_1}(f^{nm_2 n_1 + l}),$$

where Lip (f) is the Lipschitz constant of f. Thus $V_{J_1}(f^{n_1})$ also grows exponentially as $n_1 \to \infty$.

Since $J_1 \subset J$, we have $V_J(f^k)$ grows exponentially as $k \to \infty$.

Theorem 3.2 Assume that

1) Either $(\upsilon, \alpha) \in S_1^1$ such that the map $G \circ F_{\upsilon,\alpha}(F_{\upsilon,\alpha} \circ G)$ is topologically transitive on the invariant interval $[c, M]$, where $c = G \circ F_{\upsilon,\alpha}(M)$, or $(\upsilon, \alpha) \in S_2$ such that the map $G \circ F_{\upsilon,\alpha}(F_{\upsilon,\alpha} \circ G)$ is topologically transitive on the invariant interval $[-M, M]$.

2) The initial conditions u_0 and v_0 are continuous such that the range of one of them contains a non-degenerate subinterval of $[-b, b]$, $b = \max\{v_l, M\}$.

Then, for the solution pair (u, v) of (1.13), we have that the total variation

$$V_{[0,1]}(u(\cdot, t)) + V_{[0,1]}(v(\cdot, t))$$

grows exponentially as $t \to \infty$.

Proof We only consider the following case (other cases can be proved likewise):

$(\upsilon, \alpha) \in S_1^1$ such that the map $G \circ F_{\upsilon,\alpha}(F_{\upsilon,\alpha} \circ G)$ is topologically transitive on the invariant interval $[G \circ F_{\upsilon,\alpha}(M), M]$, and $\forall x, G \circ F_{\upsilon,\alpha}(x) = -F_{\upsilon,\alpha}(x) = F_{\upsilon,\alpha}(-x) = F_{\upsilon,\alpha} \circ G(x)$ by Lemma 2.1. The initial condition u_0 contains two distinct points of $I = [-b, b]$. If $G \circ F_{\upsilon,\alpha}(M) \leqslant M$, we let

$$J = [G \circ F_{\upsilon,\alpha}(M), M] \text{ and } f = G \circ F_{\upsilon,\alpha}.$$

Since f is a unimodel map, for any non-degenerate subinterval $K \subset I - J$, there exists a positive integer k such that $f^k(K)$ contains a non-degenerate subinterval in J. Thus, for any subinterval J_1 of I, we have that $V_{J_1}(f^n)$ grows exponentially as $n \to \infty$ by Lemma 2.4 and the proof Theorem 3.1.

As u_0 is continuous and $u_0([0,1])$ contains two distinct points of $[-b, b]$. For any $\tau \in [0,1]$, either $u_0([0, \tau\rho_2])$ or $u_0([\tau\rho_1, 1])$ contains a subinterval I_1 of $[-b, b]$. By (3.4) and (3.5), as the proof of the theorem 3.1, we have

$$V_{[0,1]}(u(\cdot, t)) + V_{[0,1]}(v(\cdot, t)) \geqslant V_{I_1}((F_{\upsilon,\alpha} \circ G)^k).$$

Thus $V_{[0,1]}(u(\cdot, t)) + V_{[0,1]}(v(\cdot, t))$ grows exponentially as $t \to \infty$.

4 Numerical simulation results

When $(\upsilon, \alpha) \in S_1^1$, there are some pair (υ, α), such that $G \circ F_{\upsilon,\alpha}$ is topologically transitive on some interval $[G \circ F_{\upsilon,\alpha}(M_1), M_1]$, and $(\upsilon, \alpha) \in S_2$ such that $G \circ F_{\upsilon,\alpha}$ is not topologically transitive on the invariant interval $[-M_1, M_1]$. For example the parameter pairs are $(\upsilon, \alpha) = (1.05, 1.5) \in S_1^1$ and $(\upsilon, \alpha) = (1.34, 1.5) \in S_2$. Let $B_{n,k} = \bigcup_{i=n}^{n+k} \mathrm{graph}(G \circ F_{\upsilon,\alpha})$ for n and k are large enough. We will detect whether $B_{n,k}$ always "fill" the whole squares

$$[G \circ F_{\upsilon,\alpha}(M_1), M_1] \times [G \circ F_{\upsilon,\alpha}(M_1), M_1]$$

$$= [0.0288, 0.86] \times [0.0288, 0.86]$$

for the pair $(\upsilon, \alpha) = (1.05, 1.5)$, and the whole squares

$$[-M_1, M_1] \times [-M_1, M_1]$$
$$= [-0.87, 0.87] \times [-0.87, 0.87]$$

for the pair $(\upsilon, \alpha) = (1.34, 1.5)$.

We can see from Fig.4, and Fig.5, that $G \circ F_{\upsilon,\alpha}$ is topologically transitive for the pair $(\upsilon, \alpha) = (1.05, 1.5)$, and is not topologically transitive for the pair $(\upsilon, \alpha) = (1.34, 1.5)$ by Lemma 2.2.

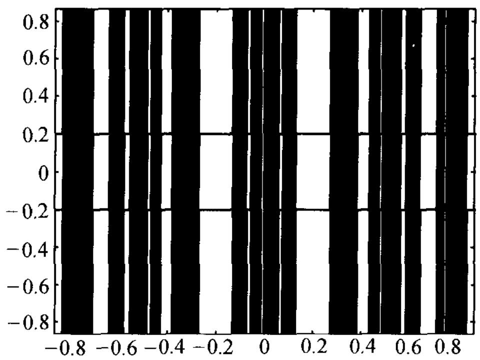

Fig. 4 $G \circ F_{\upsilon,\alpha} = f$ for the pair $(\upsilon, \alpha) = (1.34, 1.5)$.

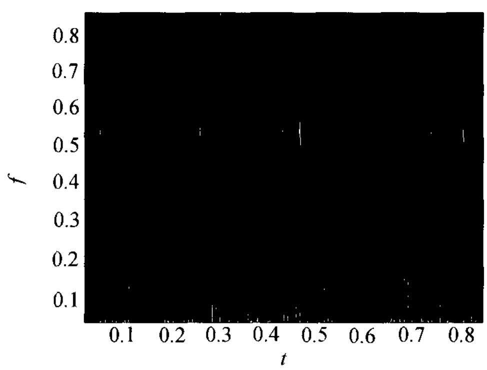

Fig. 5 $G \circ F_{\upsilon,\alpha} = f$ For the pair $(\upsilon, \alpha) = (1.05, 1.5)$.

A few computer simulation results are presented in this section as snapshots of the vibrations of a solution pair $u(\cdot, t)$ and $v(\cdot, t)$ of the (1.6) ~ (1.9).

$$w_0(x) = 0.2\sin(\frac{\pi}{2}x) + 0.6,$$
$$w_1(x) = 0.2\sin(\pi x) + 0.6, \ x \in [0,1], \quad (4.1)$$

Then by (1.7), we have

$$u_0(x) = \frac{0.2}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_2(\upsilon)\frac{\pi}{2}\cos(\frac{\pi}{2}x) + 0.3 + \sin(\pi x)], \ x \in [0,1], \quad (4.2)$$

$$v_0(x) = \frac{0.2}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_1(\upsilon)\frac{\pi}{2}\cos(\frac{\pi}{2}x) - 0.3 - \sin(\pi x)], \ x \in [0,1]. \quad (4.3)$$

Considering the parameter pairs $(\upsilon, \alpha) = (1.05, 1.5) \in S_1^1$ and $(\upsilon, \alpha) = (1.34, 1.5) \in S_2$, and $\rho_1(\upsilon) \approx 0.6044, 0.5337, \rho_2(\upsilon) \approx 0.6544, 1.8737$, then we have the snapshots of $u(\cdot, t)$ and $v(\cdot, t)$ at $t = 50(\rho_1 + \rho_2) \approx 50 \times 2.2589, 50 \times 2.4074$, as shown in Figs.6 and 7.

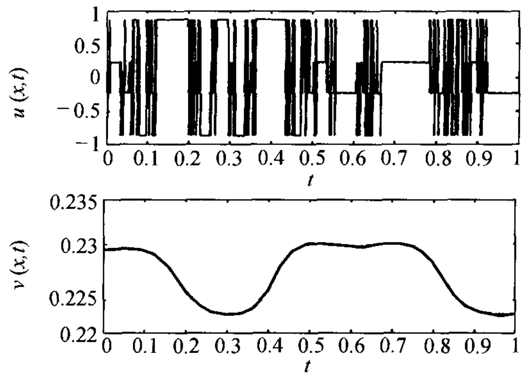

Fig. 6 Profiles of $u(\cdot, t)$ and $v(\cdot, t)$ with $(\upsilon, \alpha) = (1.05, 1.5) \in S_1$.

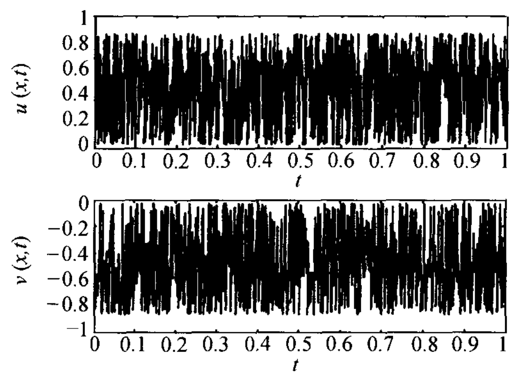

Fig. 7 Profiles of $u(\cdot, t)$ and $v(\cdot, t)$ with $(\upsilon, \alpha) = (1.34, 1.5) \in S_2$.

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