

ON THE ATTRACTING SET FOR DUFFING'S EQUATION

II. A GEOMETRICAL MODEL FOR MODERATE FORCE AND DAMPING

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After a brief review of some earlier work on Duffing's equation in the small force and damping regions, we use the results of numerical integrations to construct a geometrically defined Poincaré map which captures the qualitative features of the attracting set of larger force and damping levels. This map has a (small) constant Jacobian determinant and can be regarded as a perturbation of a non-invertible one-dimensional map. We give a partial analysis of the map and pose some important open questions regarding perturbations of one-dimensional maps and the creation of "strange attractors" during bifurcation to horseshoes.

1. Introduction

In this paper we develop a geometrical model for the Poincaré map associated with Duffing's equation:

$$\ddot{x} + \delta \dot{x} - x + x^3 = \gamma \cos \omega t, \quad (1.1)$$

for moderate force level γ and dissipation δ . The excitation frequency does not play a prominent role in our analysis and henceforth we will assume $\omega = 1$. For information on applications and related work on the Duffing equation, and particularly for global analytical results in the small γ , δ regime, we refer the reader to the companion review paper by Holmes and Whitley [17], and to Holmes [15], Moon and Holmes [32], Holmes and Marsden [16] and Greenspan and Holmes [8, 9]. We remark that the methods developed in this paper can be applied to many problems in non-linear oscillations. For a rather different example (with non-constant Jacobian) see Levi's [24] work on the van der Pol equation.

The Poincaré map of (1.1) is a crucial tool in our analysis. Letting $u = x$, $v = \dot{x}$ and $\theta = t$, we convert (1.1) into an autonomous system with 2π -periodic phase space $(u, v; \theta) \in \mathbb{R}^2 \times S^1$:

$$\dot{u} = v,$$

$$\dot{v} = u - u^3 - \delta v + \gamma \cos \theta, \quad (1.2)$$

$$\dot{\theta} = 1.$$

We next take a cross section $\Sigma : \{(u, v, \theta) \mid \theta = 0\}$ and define the time 2π first return or *Poincaré map*:

$$P_\gamma : U \rightarrow \Sigma; \quad U \subseteq \Sigma \quad (1.3)$$

induced by the flow $\phi_t : \mathbb{R}^2 \times S^1 \rightarrow \mathbb{R}^2 \times S^1$ of (1.2). Letting $\phi_t(u, v, \theta)$ denote the solution based at (u, v, θ) at $t = 0$, we have

$$P_\gamma(u, v) = \pi \cdot \phi_{2\pi}(u, v, 0), \quad (1.4)$$

where π denotes projection onto the first factor. It is easily proved (Holmes [15], Holmes and Whitley [17]) that all solutions of (1.2) are bounded for all t , provided $\delta > 0$. Hence P_γ is globally defined.

Moreover, it is easy to show that P_γ preserves orientation (or solution curves of (1.2) would cross) and that it contracts areas uniformly at the rate $e^{-2\pi\delta}$, for

$$DP_\gamma = \exp(2\pi Df), \quad (1.5)$$

and

$$\det(DP_\gamma) = \exp(2\pi \operatorname{Tr}(Df)). \quad (1.6)$$

Here $\operatorname{Tr} Df = \partial \dot{u} / \partial u + \partial \dot{v} / \partial v + \partial \dot{\theta} / \partial \theta = -\delta$ (from (1.2)) measures the volume contraction of the flow. Of course, P_γ cannot be computed without solving the equation, and hence is only accessible via perturbation methods, as in the literature cited above, or numerical integrations and inspired guesswork, as in the present paper.

In what follows we shall regard the dissipation δ as a fixed positive constant and vary γ . Thus we denote the explicit parameter dependence of the Poincaré map: P_γ . Since the cross section Σ is isomorphic to the plane, we shall be dealing with families of orientation preserving diffeomorphisms of the plane with constant Jacobian.

In section 2 we outline previous (perturbation) results on the Duffing equation in the small γ, δ regime. In section 3 we give some numerical results for moderate γ, δ due to Ueda [40]. Then, in section 4 we develop a geometrical model for the Poincaré map and partially analyze it in section 5. The desire to complete this analysis leads us, in sections 6–7, to consider families of two-dimensional diffeomorphisms with small Jacobian which are close to one dimensional maps of the form

$$y \mapsto f_\mu(y), \quad (1.7)$$

where $f_\mu: I \rightarrow R$ is defined on the unit interval and has a single critical point and negative Schwarzian derivative. Such one-dimensional maps are now quite well understood and, under suitable restrictions it is known that the set of parameter (μ) values for which (1.7) has a strange attractor is of positive measure (Jakobsen [20]). The one-dimensional theory carries over in a limited way to high iterates of two-dimensional maps such as the Duffing Poincaré map, and we discuss some positive results in this area and point out some important differences and problems.

The space available here prevents us from giving a full background to the differentiable dynamical

and global analytical methods used in this paper. For general background we recommend the books by Chillingworth [3], the notes of Newhouse [35] and Bowen [2] and the forthcoming text by Guckenheimer and Holmes [12].

In particular, some familiarity with the horseshoe construction due to Smale [37, 38], cf. Moser [33], Newhouse [35], will be assumed in this paper. A nice elementary discussion of the horseshoe can be found in Chillingworth [3]. We feel that a proper understanding of this example is an essential prerequisite to attempts to understand chaos and strange attractors in dynamical systems.

The reader may also find it helpful to consult a number of earlier papers on the Duffing equation and variants of it, especially Holmes [15], Moon and Holmes [32], and Greenspan and Holmes [8, 9]. Andronov, Vitt and Khaiken [1] continues to provide the best background in nonlinear oscillations and planar systems.

2. The attracting set for small γ and δ

In Holmes and Whitley [17] (cf. Holmes [15]) it is shown that an open disc $D \subset \mathbb{R}^2$ can be chosen such that $\overline{P_\gamma(D)} \subset D$ (overbar denotes closure). We can therefore define the attracting set A_γ as

$$A_\gamma = \bigcap_{n \geq 0} \overline{P_\gamma^n(D)}. \quad (2.1)$$

Since $\det(DP_\gamma) = e^{-2\pi\delta} < 1$, and the boundary of D is a simple closed curve, it follows that A_γ is a compact, invariant connected set with empty interior.

We suppose that δ is fixed positive, and sufficiently small, for validity of perturbations. Then, the work of Holmes [15], Greenspan and Holmes [8] and Holmes and Whitley [17], using Melnikov's [28] method; establishes the following:

Theorem 2.1. If γ and δ are sufficiently small and $\gamma > (4/3\sqrt{2\pi}) \cosh(\pi/2)\delta \approx 0.753\delta$, then

- (i) $A_\gamma = W^u(p_{2m+1})$, where $W^u(p_{2m+1})$ is the un-

stable manifold of a subharmonic ($2m + 1$ -periodic cycle) of P_γ .

(ii) A_γ contains invariant hyperbolic sets A^N_γ (Smale horseshoes) on which some iterate P^N_γ of P_γ is conjugate to a shift on two symbols.

(iii) A_γ contains finitely many stable periodic orbits, and hence is not indecomposable. The number of stable periodic orbits grows without bound as $\gamma, \delta \rightarrow 0$.

For pictures of this attracting set, see Greenspan and Holmes [8] and Holmes and Whitley [17].

The geometrical structure of A_γ is such that it contains orbits which circulate arbitrarily many times to the left or right in any sequence, but all such chaotic orbits are, alas, unstable and one expects to see stable subharmonic behavior after a period of transient chaos. Numerical integrations of (1.1) for small γ and δ support this conclusion. Thus A_γ is not a strange attractor, although the high period stable subharmonics may well be unobservable even on the most expensive computers.

There is no generally accepted definition of a strange attractor, except in the case of one-dimensional maps (cf. section 6). In this paper we therefore adopt the following:

Definition. A strange attractor is a closed, invariant attracting set which contains a dense orbit and hence is indecomposable, and in which solutions exhibit sensitive dependence on initial conditions: that is, almost all orbits in the attractor separate locally exponentially fast.

We will see that finding a dense orbit, or equivalently, getting rid of stable periodic orbits, is the hard part. We have seen that such stable orbits *always* exist for small γ and $\delta > 0$. We now will see what happens to them for larger γ and δ .

3. Numerical work

In this section we review some numerical results. We first point out that, for zero and small force γ ,

with $\delta > 0$ fixed and moderate, the Poincaré map P_γ is particularly simple. In fig. 1 we show results due to Ueda [40], who plotted segments of the stable and unstable manifolds of the saddle point $p_1 \approx (0, 0)$ which represents a continuation for $\gamma \neq 0$ of the original saddle at $(u, v) = (0, 0)$ of the unforced Duffing equation. For low γ (fig. 1a), the manifolds do not intersect and almost all solutions approach either one or other of the sinks near $(u, v) = (\pm 1, 0)$. Here it is clear that $A_\gamma = \overline{W^u(p_1)}$ and A_γ contains two sinks. As γ is increased, however, the stable and unstable manifolds approach, touch, and thereafter intersect transversely

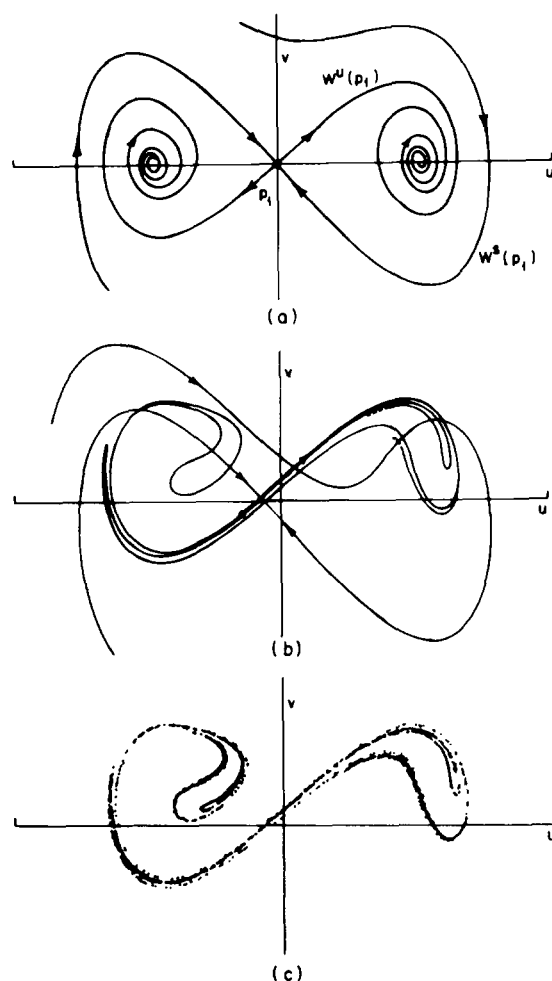


Fig. 1. Poincaré maps for Duffing's equation, $\delta = 0.25$, $\omega = 1.0$. (a), (b) invariant manifolds of p_1 . (c) A single orbit.

in a (pair of) homoclinic orbits (fig. 1b). This leads to the horseshoes and 'unstable chaos' of theorem 2.1. In fig. 1c we show a typical single orbit $\{P^n(x)\}_{n=50}^{1000}$ for the same parameter values as 1b. The starting transient has been deleted. We note that this orbit appears to lie on a set equal to the closure of the unstable manifold $W^u(p_1)$. This set is locally the product of a curve and a Cantor set (cf. Hénon [14])*. The set of parameter values for which such apparently chaotic motions are observed is relatively large, but careful work has revealed thin bands within it, in which stable motions of periods 3, 4, 5 and even 7 exist. However these orbits have different structures from the stable subharmonics found analytically for small γ and δ (Greenspan and Holmes [7]). The map developed below will share this type of behavior.

4. A geometrically defined attracting set

We now construct a geometrical attractor on the basis of the numerical work outlined above. A careful examination of the structure of manifolds of fig. 1b reveals the topological structure 'straightened out' and sketched in fig. 2a. Symmetry implies that the same structure exists in the left-hand half of the plane also. The transverse homoclinic points labelled q, r, s, t are mapped under P_γ to q^+, r^+, s^+, t^+ and under P_γ^{-1} to q^-, r^-, s^-, t^- . The point p_1 is the (fixed) saddle point. In fig. 2b we show three regions bounded by pieces of these invariant manifolds and their images under P_γ .

Those familiar with Smale's construction (Smale [37, 38], Moser [33], Chillingworth [3]) will clearly recognize a horseshoe in fig. 2b in the rectangle $p_1 q r q^+$ and its preimage $p_1 q^- r^- q$, but we still have not constructed an attracting set, since the image of the vertically shaded region falls outside the

* With some reservations: there must be points where the unstable manifold 'turns back on itself', almost everywhere in the attracting set, cf. Zeeman [42].

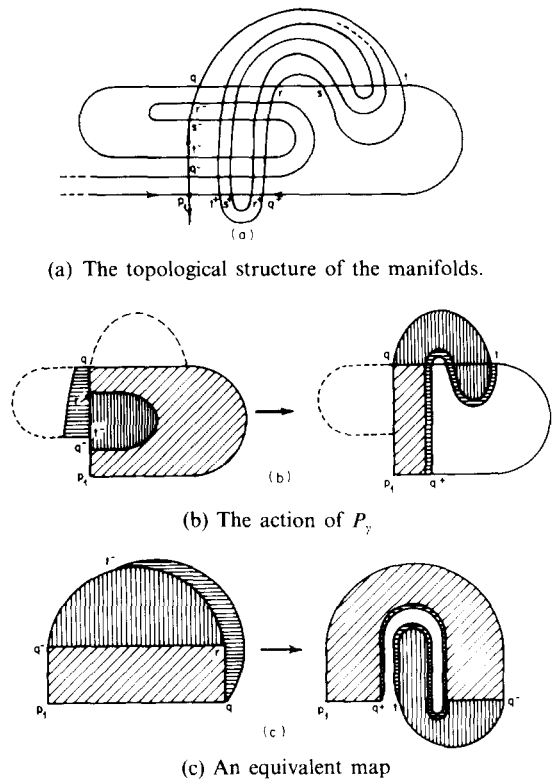


Fig. 2. Towards a geometric model of the attracting set (not to scale).

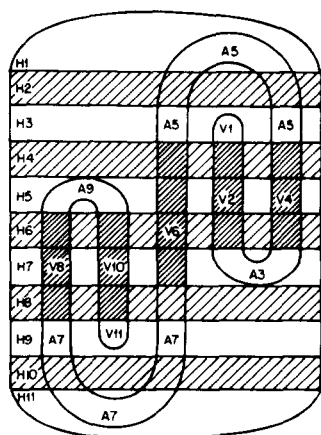
original three regions chosen. To construct a trapping region analogous to D of section 2, we enlarge our basic domain and once more 'straighten out' (cf. fig. 2c), so that the set $p_1 q^- r^- q$ becomes a 'horizontal' rectangle. Taking a symmetric structure in the negative half-plane, we finally obtain the diffeomorphism of a (closed) disc D into its interior shown in fig. 3. To distinguish this from the true (incalculable) Poincaré map, we denote this geometrically defined map G_γ .

We now state various properties which we shall assume for the map G_γ :

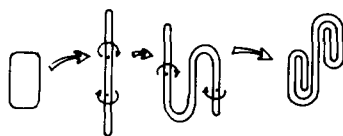
(A1) The map has constant Jacobian determinant $\alpha (= e^{-2\pi\delta}) < 1$.

(A2) The eleven horizontal strips H_1-H_{11} are mapped into the corresponding vertical strips $V_2, V_4, V_6, V_8, V_{10}$, the 'buttons' V_1, V_{11} , and the arches A_3, A_5, A_7, A_9 .

(A3) The map is linear hyperbolic (or at least its



(a)

(a) The map G_γ 

(b)

(b) Cartoon of the action of G_γ

Fig. 3. The geometrical attracting set.

expansion and contractions are nicely bounded, cf. Moser [33]) on the strips $H_2, H_4, H_6, H_8, H_{10}$.

We believe that careful numerical estimates on the Duffing equation would reveal that, for P_γ , regions and their images could be chosen essentially consistent with our model G_γ . Although P_γ would not, of course, be piecewise linear, it still seems likely that hyperbolicity estimates would be obtainable. We also note that, from Ueda's computations, we can deduce that an increase in the force level γ has the effect of increasing the hyperbolic expansion and contraction rates, so that the arches A_3 and A_7 are pulled down and A_5 and A_9 pulled up. Such parameter dependence will be important later.

We now turn to a partial analysis of the invariant attracting set for this map, which we denote

$$A_\gamma = \overline{\bigcap_{n=0}^{\infty} G_\gamma^n(D)}.$$

5. Shoes and sinks for G_γ

We first show explicitly that the invariant set A_γ of G_γ contains horseshoes. In particular, the three horizontal strips H_4, H_6 and H_8 form the basis of a Markov Partition (Bowen [2], Newhouse [35], Guckenheimer and Holmes [12]) and, since

$$G_\gamma(H_4) \cap H_4, H_6 \neq \emptyset,$$

$$G_\gamma(H_6) \cap H_4, H_6, H_8 \neq \emptyset,$$

$$G_\gamma(H_8) \cap H_6, H_8 \neq \emptyset,$$

we have the associated *transition matrix*

$$A = (a_{ij}) = \begin{matrix} & \begin{matrix} (4) & (6) & (8) \end{matrix} \\ \begin{matrix} (4) \\ (6) \\ (8) \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad (5.1)$$

where $a_{ij} = 1$ if an orbit passing from i to j exists and $a_{ij} = 0$ if $G_\gamma(H_i) \cap H_j = \emptyset$. (In fact we need more than just non-empty intersections: the images of the strips H_j must "overlap" the H_j suitably, cf. Bowen [2].) Thus the invariant set A_γ of G_γ contains a subshift on the three symbols 4, 6, 8; the combinations 48 and 84 being disallowed. The general theory of symbolic dynamics permits us to conclude that, to any bi-infinite sequence containing the symbols 4, 6, 8, but not the pairs 48 or 84, there corresponds precisely one point of A_γ . Moreover, the orbit of this point visits the strips H_4, H_6 and H_8 in the order prescribed by the occurrence of symbols in the sequence. That is, if $a(x) = \{a_j\}_{j=-\infty}^{\infty}$ is the symbol sequence for the point x , that $a_j = 4, 6$ or 8 depending on whether $G_\gamma^j(x) \in H_4, H_6$ or H_8 , respectively.

Thus the symbolic dynamics shows that there are three fixed points $\dots 444.444\dots$; $\dots 666.666\dots$ and $\dots 888.888\dots$; two orbits of period two ($\dots 46.46\dots$; $\dots 68.68\dots$) and so forth. In general G_γ^k has $N = \text{Tr}(A^k)$ fixed points. All the periodic orbits are of saddle type, since on $H_4 \cup H_6 \cup H_8$, G_γ expands vertically and contracts horizontally. We remark that the 'central' fixed

point $\dots 666.666\dots$ is just the original saddle of Duffing's equation near $(u, v) = (0, 0)$. In addition to this countable set of periodic orbits, there is an uncountable set of orbits which are not asymptotically periodic and which correspond to symbol sequences with positive or negative (infinite) non-periodic tails. Thus, chaotic orbits visiting the left (V_8) and the right (V_4) halves of the disc in *any* prespecified order exist. This is in essence part of the result we had in theorem 2.1. except that the numerical evidence has permitted us to conclude that such orbits exist for the map G_γ itself- and not merely for some (perhaps high) iterate.

Examination of fig. 3 reveals that regions H_4 , H_6 and H_8 are the only ones intersected by their images ($G(H_i)$), and so to construct more of the invariant set A_γ we must go on to look at G_γ^2 , G_γ^3 , etc. We will now sketch a little of this analysis. First consider $G_\gamma^2(H_2)$ (or, equivalently, by the symmetry, $G_\gamma^2(H_{10})$). In fig. 4 we show the two successive images, indicating that there are points of H_2 which return to H_2 after two iterates. Once more a Markov partition can be chosen, based on the horizontal substrips H_{21} , $H_{22} \subset H_2$ with images $V_{21} = G_\gamma^2(H_{21})$, $V_{22} = G_\gamma^2(H_{22}) \subset H_{22}$, and we deduce that $G^2|_{H_2}$ has a shift on two symbols. However, the nonlinearity of the map of H_5 needs to be specified more precisely before we can conclude that the invariant set is hyperbolic.

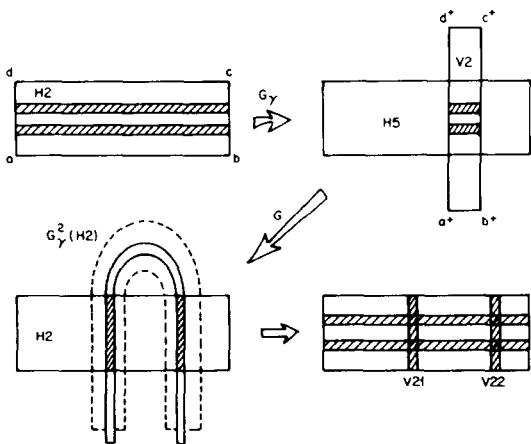


Fig. 4. $G_\gamma^2|_{H_2}$, showing an orbit with sequence $\dots 2525\dots$

In a similar fashion we can find horseshoe-like (hyperbolic) subshifts of finite type for higher iterates of G_γ , by restricting our attention to orbits which do not contain points near the peaks of the arches A_3 , A_5 , A_7 or A_9 . For the specific example G_γ of fig. 3, G_γ^5 turns out to be the lowest iterate for which orbits can pass near these peaks and return to their starting points. We now discuss the implications of this.

We consider an orbit passing through the following regions: $\dots H_9 \rightarrow H_5 \rightarrow H_3 \rightarrow H_7 \rightarrow H_{11} \rightarrow H_9 \rightarrow \dots$ (It has a symmetric partner $H_3 \rightarrow H_7 \rightarrow H_9 \rightarrow H_5 \rightarrow H_1 \rightarrow H_3 \rightarrow \dots$). A happy morning's drawing should convince the reader that a thin horizontal substrip $H \subset H_{11}$ can be chosen such that $G_\gamma^5(H)$ lies in H_{11} as shown in fig. 5 for three force levels γ . (Recall that, as γ is increased, the images $A_9 = G_\gamma(H_9)$ and $A_3 = G_\gamma(H_3)$ move up and down respectively). Thus, as γ varies, we create a full shift on two symbols – a horseshoe – for G_γ^5 . As Newhouse noted [34, 35], in this situation we necessarily create stable sinks (of period 5), cf. fig. 5b.

This construction, and analogues of it, show how stable periodic orbits can “suddenly spring out of chaos” as a parameter is varied in the Duffing equation. However, we note that motions such as these are not directly related to the weakly dissipative “regular” subharmonics found in earlier analyses (section 2 and Greenspan and Holmes [8], Holmes and Whitley [17]). In that case we note that we had to take γ and δ smaller the higher the order of the subharmonic, so that averaging remained valid. In fact a careful examination of the order of the terms involved shows that in studying

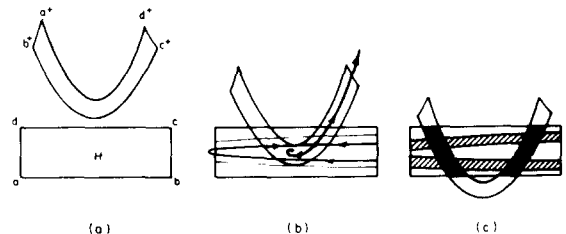


Fig. 5. $G_\gamma^5|_{H_{11}}$. The creation of a 5-shoe $a, b, c, d \rightarrow a^+, b^+, c^+, d^+$: (a) γ small (b) γ medium (c) γ high.

P_γ^N by perturbation methods we must take $\det(P_\gamma^N) \approx \exp(-2\pi N^6 \exp(-4\pi N)) \rightarrow 1$ as $N \rightarrow \infty$. In studying the geometrically defined map G_γ , in contrast, we fix $\det(G_\gamma) = \alpha$ and thus $\det(G_\gamma^N) = \alpha^N \rightarrow 0$ as $N \rightarrow \infty$. For the particular values of Ueda's computations, on which our geometrical construction is based we have

$$\det DP_\gamma = \exp\left(-\frac{2\pi \times 0.25}{1}\right) \approx 0.208, \quad (5.2)$$

and

$$\det(DP_\gamma^5) \approx 0.00039. \quad (5.3)$$

The two analyses can therefore be expected to yield different (and complementary) results.

Examination of higher iterates G_γ^N show that the qualitative picture of fig. 5 occurs over and over again in various regions of the disc D as γ is varied. For example, the reader might like to imagine γ decreased so that the peaks of the arches A_5 and A_7 of fig. 3 lie in regions H_2 and H_{10} respectively. In this case, stable orbits of period three can be constructed which pass through regions $H_2 \rightarrow H_4 \rightarrow H_5$ and $H_{10} \rightarrow H_8 \rightarrow H_7$, respectively. Such orbits correspond to numerically obtained period three subharmonics (Ueda [40]).

We end this section by noting that it follows from our geometrical construction that the attracting set A_γ is the closure of the unstable manifold of the stable point $\dots 666.666\dots$ in $H_6 \cap V_6$. This follows in much the same way as the usual horseshoe construction, where it is shown that the invariant set lies in the closure of any of the unstable manifolds of saddle points within it (cf. Holmes and Whitley [18]).

Thus we have a similar structure to that proven to exist for the Duffing equation in theorem 2.1, with the added advantage that the features of iterates of G_γ relevant to the creation of periodic orbits and horseshoes can be understood by the analysis of small perturbations of one-dimensional maps. Before going on to sketch such analyses, we outline how an iterate G_γ^N such as that of fig. 5 can

be regarded as a small perturbation of such a noninvertible map.

Consider the family of maps

$$F_{(\epsilon, \gamma)}(x, y) = (y, -\epsilon x + f_\gamma(y)), \quad (5.4)$$

where $f_\gamma: I \rightarrow \mathbb{R}$ is a one-dimensional map with a single critical point. A more precise formulation is provided in section 6. For $\epsilon > 0$ (5.4) is a diffeomorphism while for $\epsilon = 0$ it is non-invertible, the image of all points in $\mathbb{R}^2$ lying on the curve $y = f_\gamma(x)$. If we pick the family f_γ appropriately, then $F_{\epsilon, \gamma}$ behaves as shown in fig. 5, and as $\epsilon \rightarrow 0$ the image of the rectangle $abcd$ shrinks in area to a (parabolic) arch, the points a^+ , b^+ and c^+ , d^+ coalescing. Thus we expect the features of $F_{\epsilon, \gamma}$, $\epsilon > 0$ small, to be well approximated in some sense by those of $F_{0, \gamma}$ and hence of the one dimensional map f_γ . In the final sections we will review some of the many results on maps of the interval and describe recent work on perturbations of such maps in the form of diffeomorphisms with small Jacobian determinant, like (5.4). This will enable us to provide a partial description of the bifurcation sequence in which sinks, saddles and ultimately horseshoes are created for iterates G_γ^N as γ increases.

6. Maps of the interval

In this section we describe and illustrate some of the many results on one-parameter families of maps of the interval. We will work with the class $\mathcal{C}$ of C^3 maps $f: I \rightarrow I$ of a closed interval $I = [a, b] \subset \mathbb{R}$, containing the origin, which satisfy the following properties:

- (1) $f(a) = f(b) = a$;
- (2) f has a single critical point, which we assume to lie at the origin $x = 0$, and $f''(0) < 0$ so that this critical point is a maximum; and
- (3) the Schwarzian derivative,

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2, \quad (6.1)$$

is negative on $I \setminus \{0\}$.

Condition (3) is included since for maps f with negative Schwarzian derivative a result due to Singer [36] (see also Misiurewicz [30]) tells us that if p is a stable periodic point of f , then there is either a critical point of f or an endpoint of I whose ω -limit set is the orbit of p . If we further assume within the class $\mathcal{C}$ that, either the fixed point $x = a$ at the left-hand end of the interval is unstable, or that $\omega(0) = \{a\}$ then our maps will have at most one stable periodic orbit. In this case Singer's theorem implies that, if the orbit of the critical point does not tend to a stable periodic orbit, then f has no stable periodic orbit.

Since the appearance of Li and Yorke's [25] 'period 3 implies chaos' result, itself a corollary of an earlier theorem of Sarkovskii (see Stefan [39]), there has been much interest in the dynamics of one-dimensional maps. For maps in $\mathcal{C}$ we now have a topological classification (Guckenheimer [11]) and a decomposition of the nonwandering set (Jonker and Rand [21], van Strien [41]), both largely based on the unpublished 'kneading theory' of Milnor and Thurston [29]. Apart from these topological results, Feigenbaum [6] has discovered universal metric properties, and Jakobsen [19, 20] has important theorems on the existence of invariant measures and resulting chaotic dynamics in families of maps. For more general introductory material we refer the reader to the monograph by Collet and Eckmann [4].

Here our main concern is to outline some of the properties of one-parameter families of maps in $\mathcal{C}$. We consider families f_γ , depending continuously on a real parameter γ , which are *full* in the sense that there are parameter values $\gamma_1^s < \gamma_1^h$ so that:

- (a) $f_\gamma \in \mathcal{C}$ for $\gamma \in [\gamma_1^s, \gamma_1^h]$;
- (b) the nonwandering set $\Omega_{\gamma_1^s}$ of $f_{\gamma_1^s}$ consists of a single fixed point;
- (c) for $\gamma > \gamma_1^h$, f_γ satisfies conditions (1)–(3) above and maps I onto (but not into) itself; in which case one may show (cf. van Strien [41]) that $f_\gamma|_{\Omega_\gamma}$ is conjugate to a full (one-sided) shift on two symbols. Ω_γ is a one-dimensional version of the standard Smale horseshoe.

In general the interval I (i.e. its endpoints) will

vary with the parameter as in the archetypal example

$$x \rightarrow \gamma - x^2; \quad \gamma_1^s = -\frac{1}{4}, \quad \gamma_1^h = 2. \quad (6.2)$$

Graphs of a typical family f_γ for various values of γ are shown below.

We will now describe the major features of the bifurcation set in the parameter range $\gamma_1^s \leq \gamma \leq \gamma_1^h$. We assume that, as in the quadratic family (6.2), f_γ has a fold or saddle-node bifurcation at $\gamma = \gamma_1^s$ which creates two fixed points, one stable and one unstable, which exist for $\gamma > \gamma_1^s$, and that γ_1^h is the least γ -value such that $f_\gamma(0) = b$. Let $\gamma_1^{ss} = \sup\{\gamma \mid f_\gamma(0) \leq 0\}$ and $\gamma_2^s = \sup\{\gamma \mid f_\gamma(p) = p \text{ and } f'_\gamma(p) = -1\}$ where $p = p_\gamma$ is the unique positive fixed point of f_γ . Clearly $\gamma_1^{ss} \in (\gamma_1^s, \gamma_1^h)$ and $\gamma_2^s \in (\gamma_1^{ss}, \gamma_1^h)$.

Now one can find parameters $\gamma > \gamma_2^s$ so that $f_\gamma^2(0) > 0$ and, since $f_{\gamma_1^h}^2(0) = a$, it follows by continuity that there is at least one γ with $f_\gamma^2(0) = p' \in (a, 0)$, where $f_\gamma(p') = f_\gamma(p)$. Let γ_2^h be the infimum of such γ 's. Then $\gamma_1^s < \gamma_2^s < \gamma_2^h < \gamma_1^h$ and for $\gamma \in [\gamma_2^s, \gamma_2^h]$ f_γ^2 maps the interval $[p', p]$ into itself. In fact f_γ maps $[p', p]$ onto $[p, f_\gamma(0)]$ and $[p, f_\gamma(0)]$ into $[p', p]$ so that, since all points in $I \setminus \{a, b\}$ eventually map into $[p', p]$, all periodic points of f_γ , $\gamma \in [\gamma_2^s, \gamma_2^h]$, except the fixed points, have even periods. Moreover, for this parameter range, $f_\gamma^2|_{[p', p]}$ is a full family and behaves just like $f_\gamma|_I$ 'in miniature'. Repeating this argument inductively we have (cf. Collet, Eckmann and Lanford [5]):

Proposition 6.1. There are two sequences of parameters $\gamma_{2^n}^s, \gamma_{2^n}^h$ with $\gamma_1^s < \gamma_2^s < \gamma_4^s < \dots < \gamma_{2^n}^s < \dots < \gamma_{2^n}^h < \dots < \gamma_4^h < \gamma_2^h < \gamma_1^h$ so that, for $n > 1$:

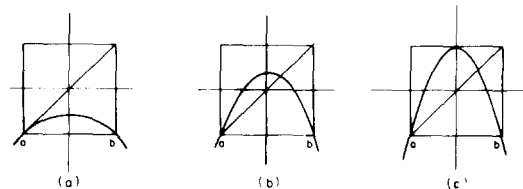


Fig. 6. Three graphs of f_γ .

- (1) $f_{\gamma_{2^n}}^s$ has a point of period 2^{n-1} with eigenvalue $\lambda = -1$;
- (2) $f_{\gamma_{2^n}}^{2^n+1}(0)$ is an unstable point of period 2^{n-1} ;
- (3) If $\gamma \in [\gamma_{2^n}^s, \gamma_{2^n}^h]$ there are 2^n closed subintervals $J_1, \dots, J_{2^n}$ with $0 \in J, f_\gamma^{2^n}|_{J_i} \in \mathcal{C}$, and the nonwandering set $\Omega(f_\gamma)$ is $\Omega(f_\gamma|_{\bigcup_{i=1}^{2^n} J_i})$ together with an unstable orbit of period 2^k , one for each $k = 0, 1, \dots, 2^{n-2}$, plus the fixed point a .

Thus we see that the bifurcation set is as indicated in fig. 7. There is a fold bifurcation at γ_1^s creating one stable and one unstable fixed point. The stable fixed point loses stability at γ_2^s where a stable orbit of period 2 is created through a flip bifurcation, and this process repeats as γ increases: at $\gamma_{2^n}^s$ an orbit of period 2^{n-1} flips to period 2^n . At $\gamma = \gamma_{2^n}^h$, $f_\gamma^{2^n}$ maps the subinterval J_1 exactly onto itself with a single fold at the origin, and $f_{\gamma_{2^n}^h}$ is conjugate to the piecewise linear map

$$x \rightarrow s - 1 - s|x|; \quad s = 2^{1/2^n}. \quad (6.3)$$

For a class of maps including the quadratic family (6.2) Jakobsen [19] has shown that the $f_{\gamma_{2^n}^h}$'s have absolutely continuous, ergodic invariant measures supported on the subintervals J_i .

In the interval $[\gamma_{2^n}^s, \gamma_{2^n}^h]$ all periodic orbits except those involved in the initial period-doubling sequence have periods which are multiples of 2^n . For families with a quadratic maximum satisfying a certain transversality condition, of which (6.2) is an example, (Lanford [22]), the sequences $\gamma_{2^n}^s$ and $\gamma_{2^n}^h$ converge, from below and above respectively, on a parameter γ_1^F . Furthermore, if $l_n = \gamma_{2^n}^h - \gamma_{2^n}^s$ denotes the length of the n th nested 'box' in the parameter space, then these lengths converge to zero at Feigenbaum's [6] universal rate:

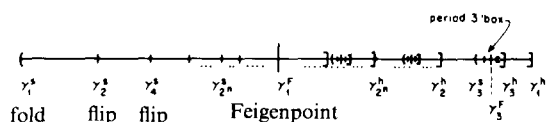


Fig. 7. The bifurcation set of f_γ .

$$\lim_{n \rightarrow \infty} \frac{l_n - l_{n+1}}{l_{n+1} - l_{n+2}} = 4.669 \dots, \quad (6.4)$$

independently of the particular family.

This view of the bifurcation set shows clearly the central role of the parameter value γ_1^F . For $\gamma < \gamma_1^F$ the nonwandering set consists of finitely many periodic orbits each with period a power of 2, while when γ crosses γ_1^F there is an explosion into a region of complicated, *although not necessarily always chaotic*, dynamics. The map $f_{\gamma_1^F}$ itself, which could be thought of as an 'organising centre' for the whole family, falls into the class of maps with zero topological entropy studied by Misiurewicz [30]. Its nonwandering set consists of the fixed point a in the boundary of I , one (unstable) orbit of period 2^k for each $k \geq 0$ and an invariant minimal set homeomorphic to a Cantor set. Almost all points in I are attracted to the Cantor set, but do not display sensitive dependence upon initial conditions.

Embedded in the large box $[\gamma_1^s, \gamma_1^h]$ between each pair $\gamma_{2^n+1}^h, \gamma_{2^n}^h$ are other 'boxes' corresponding to periods of the form $k \cdot 2^n$, in which $f_\gamma^{k \cdot 2^n}$ is a full family on some subinterval of I . The internal structure of these boxes is exactly the same as the outer one. (The embedded box idea was first stressed by Mira, cf. Gumowski and Mira [13].) For example, in (γ_2^h, γ_1^h) there is a 3-box $[\gamma_3^s, \gamma_3^h]$. At γ_3^s , f_γ^3 has a saddle-node of period 3 and at γ_3^h f_γ^3 maps a subinterval exactly onto itself with a single fold. There are sequences $\gamma_{3 \cdot 2^n}^s$ and $\gamma_{3 \cdot 2^n}^h$ for which at $\gamma_{3 \cdot 2^n}^s$, f_γ has a flip bifurcation from period $3 \cdot 2^{n-1}$ to period $3 \cdot 2^n$ and $f_{\gamma_{3 \cdot 2^n}^h}^{3 \cdot (2^n+1)}(0)$ is a point of period $3 \cdot 2^{n-1}$, exactly analogous to the behavior of f_γ for $\gamma \in [\gamma_1^s, \gamma_1^h]$.

The position of the interior boxes is restricted by the form of the periods allowed in each box $[\gamma_{2^n}^s, \gamma_{2^n}^h]$, and by Šarkovskii's theorem on the ordering of orbits (Stefan [39], cf. Guckenheimer [10]). Also, we know that boxes corresponding to periods $k \cdot 2^n$, k odd, accumulate on $\gamma_{2^n+1}^h$ from above. This is because, for $\gamma > \gamma_{2^n+1}^h$ the periodic orbit which at $\gamma_{2^n+1}^h$ contains the point $f_{\gamma_{2^n+1}^h}^{2^n+1}(0)$ is a snap-back repeller for f^{2^n} (Marotto [26], Marotto

and Rogers [27]). This means that, for $\gamma > \gamma_{2^n+1}^h$, f^{2^n} has periodic points of all periods greater than some period N . In particular f^{2^n} has periodic points with odd periods, i.e. f has periodic points with periods of the form $\text{odd} \times 2^n$. These periodic orbits do not exist when $\gamma < \gamma_{2^n+1}^h$, since then all periods have the form 2^k or $k \cdot 2^{n+1}$. Thus $\gamma_{2^n+1}^h$ is an accumulation point of boxes with associated periods $\text{odd} \times 2^n$.

The parameter γ_1^F is then seen to be an accumulation point of accumulation points and it is γ_1^F which divides the parameter space into those maps with simple dynamics ($\gamma < \gamma_1^F$) from those which possibly have chaotic behavior ($\gamma > \gamma_1^F$). We emphasize however that not all f_γ with $\gamma > \gamma_1^F$ are chaotic. For many parameter values (most likely for an open, dense set, though this is not proven) f_γ has a stable periodic orbit and in this case an open, dense set of points in I , with full Lebesgue measure, are attracted to the stable orbit (van Strien [41]). There may be complicated transient behavior but we do not consider this to be 'chaos'.

Even if the map has no stable periodic orbit it may not display the sensitive dependence on initial conditions (Guckenheimer [11]) that we feel a 'strange' attractor should possess. (This is the case, for example, when $\gamma = \gamma_1^F$). However, Jakobsen [20] has shown that in the quadratic family (6.2) there is a set Γ of parameter values of positive Lebesgue measure for which f_γ has an absolutely continuous, ergodic invariant measure. These maps exhibit sensitive dependence, have indecomposable attractors, and may be considered truly chaotic. This set Γ contains the countable set $\{\gamma_{k \cdot 2^n}^h\}_{k,n=0}^\infty$ for which $f^n(0)$ falls on an unstable periodic point, and also an uncountable set (Misiurewicz [30, 31]) for which $f^n(0)$ falls on an unstable Cantor set whose closure does not contain 0. Both these sets are of zero measure, and Jakobsen's set involves maps for which the orbit of critical point returns arbitrarily closely to the critical point.

7. Diffeomorphisms of $\mathbb{R}^2$ near maps of the interval

In this penultimate section we consider

diffeomorphisms with small Jacobian determinant of the form

$$F_{(\epsilon, \gamma)}(x, y) = (y, -\epsilon x + f_\gamma(y)), \quad \epsilon \ll 1, \quad (7.1)$$

where f_γ is a one-dimensional map of the type discussed in the previous section. When $\epsilon = 0$, $F_{(0, \gamma)}$ maps $\mathbb{R}^2$ onto the parabola $y = f_\gamma(x)$ which is the graph of f_γ , and the resulting dynamics on the invariant parabola are identical to those of f_γ . We describe two theorems of van Strien [41] which indicate both the similarities and important differences in the case when $\epsilon \neq 0$. Our proof of the second theorem differs from van Strien's and is, we feel, more direct. We note that the diffeomorphism studied by Hénon [14] can be put in the form (7.1).

First we show that for some open sets of γ -values the behavior of $F_{(0, \gamma)}$ persists for $F_{(\epsilon, \gamma)}$, ϵ small. Let $\tilde{P}$ be the interior of the set of parameter values for which f_γ has an attracting periodic orbit, and let P be a component of $\tilde{P}$. For $\delta > 0$, define $P(\delta) = \{\gamma \mid (\gamma - \delta, \gamma + \delta) \in P\}$ so that $P(\delta)$ is a slightly smaller set than P .

Two one-parameter families of maps ϕ_γ, ψ_γ of $\mathbb{R}^2$ are Ω -semiconjugate on Q if there is a homeomorphism $\rho: Q \rightarrow Q$ and a family of continuous surjections $h_\gamma: \Omega(\phi_\gamma) \rightarrow \Omega(\psi_\gamma)$ so that $\psi_{\rho(\gamma)} \circ h_\gamma = h_\gamma \circ \phi_\gamma$ for $\gamma \in Q$, and so that h_γ is a bijection from $\text{Per}(\phi_\gamma)$, the set of periodic points of ϕ_γ , to $\text{Per}(\psi_\gamma)$. We say that ϕ_γ and ψ_γ are Ω -conjugate on Q if h_γ is a homeomorphism.

We will assume that the family f_γ has generic flip bifurcations, i.e. if f_{γ^*} has a periodic point p with period k and eigenvalue -1 then $d(df_\gamma^k(p)/dx)/d\mu|_{\gamma=\gamma^*} \neq 0$. We remark that the second condition required for a generic flip, $d^3(f_{\gamma^*}^k(p))/dx^3 \neq 0$, is automatically satisfied since our maps have negative Schwarzian derivative.

Van Strien proves the following:

Theorem 7.1. For $\epsilon \geq 0$ there is a $\delta(\epsilon)$ depending continuously on ϵ , with $\delta(0) = 0$, so that

(1) $\forall \epsilon > 0$, $F_{(\epsilon, \gamma)}$ is Ω -semi conjugate to $F_{(0, \gamma)}$ on $P(\delta(\epsilon))$;

(2) If $\epsilon_1, \epsilon_2 \neq 0$ and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ then $F_{(\epsilon_1, \gamma)}$ and $F_{(\epsilon_2, \gamma)}$ are Ω -conjugate on $P(\delta(\epsilon))$.

Essentially this says that for γ values such that f_γ is in a period doubling sequence, the behavior of $F_{(0,\gamma)}$ persists for $F_{(\epsilon,\gamma)}$, ϵ small, on a possibly smaller γ -range. It follows from the implicit function theorem that the bifurcation points in the period doubling sequences of f_γ extend to sets of 'parallel' curves in the ϵ - γ plane which are bifurcation curves for period doubling sequences of $F_{(\epsilon,\gamma)}$. Lanford [23] has remarked that this result can be made uniform in ϵ by using the universality ideas carefully as one approaches γ_1^F . In fact for fixed $\epsilon < 1$ the flip bifurcations accumulate at the same asymptotic rate $\delta = 4.669 \dots$ as in the one-dimensional case.

At other points, however, the dynamics of $F_{(\epsilon,\gamma)}$ are radically different from those of $F_{(0,\gamma)}$:

Theorem 7.2. Suppose there is a parameter γ_* so that

- (1) For γ near γ_* , f has a periodic point $p(\gamma)$ of period k with $|f_\gamma^k(p(\gamma))| > 1$;
- (2) there is an integer n with $f_{\gamma_*}^n(0) = p(\gamma_*)$,
- (3) $\frac{\partial}{\partial \mu} (f_\mu^n(c) - p(\gamma))|_{\gamma=\gamma_*} \neq 0$.

Then there is $\epsilon_0 > 0$ so that for each ϵ with $0 < |\epsilon| < \epsilon_0$

- (a) For γ near γ_* , $F_{(\epsilon,\gamma)}$ has a periodic saddle $p(\epsilon, \gamma)$ of period k with $p(0, \gamma) = p(\gamma)$;
- (b) there is a continuous curve $\gamma(\epsilon)$, with $\gamma(0) = \gamma_*$, so that the stable and unstable manifolds of $p(\epsilon, \gamma(\epsilon))$ have a tangency.

We outline the proof of this result (cf. van Strien [41]) which, together with the results of Newhouse [34, 35], implies that close to maps $F_{(\epsilon,\gamma)}$ satisfying the conditions of the theorem there are diffeomorphisms of $\mathbb{R}^2$ which have infinitely many sinks. For example, we can take $\gamma_* = \gamma_{k \cdot 2^n}^h$; $k, n \in \mathbb{Z}$. In contrast, the maps $F_{(0,\gamma)}$ can, as we remarked in section 6, have at most one sink. More details will appear in Holmes and Whitley [18].

Conclusion (a) of the theorem is an immediate consequence of the implicit function theorem. For (b), we define $g_\epsilon: I \rightarrow \mathbb{R}^2$ by $g_\epsilon(\gamma) = F_{(\epsilon,\gamma)}^n(0, f_\gamma(0))$ and let $M_\epsilon = \{(\gamma, W^s(p(\epsilon, \gamma)))\} \subset \mathbb{R}^3$. Hypothesis (3) says that g_0 is transverse to M at some point

$(\gamma_*(\epsilon), p(\epsilon, \gamma_*))$ near $(\gamma_*, p(0, \gamma_*))$. We then find small open balls $B_\epsilon(g_\epsilon(\gamma_1))$ and $B_\epsilon(g_\epsilon(\gamma_2))$, $\gamma_1 < \gamma_* < \gamma_2$, whose radii are independent of ϵ , and which lie on opposite sides of $W^s(p(\epsilon, \gamma))$ for all sufficiently small ϵ .

Since $F_{(0,\gamma_*)}^n(0, f_{\gamma_*}(0)) = p(0, \gamma_*)$, for any $\delta > 0$ we can choose ϵ small enough so that $B_\delta(0, f_\gamma(0))$ contains part of the unstable manifold $W^u(p(\epsilon, \gamma))$ for $\gamma \in (\gamma_1, \gamma_2)$. The continuity of F^{-1} ensures that δ may be chosen so that

$$F^n(B_\delta(0, f_\gamma(0))) \subset B_\epsilon(g_\epsilon(\gamma)).$$

Then for $\gamma \in (\gamma_1, \gamma_2)$, $B_\epsilon(g_\epsilon(\gamma))$ contains a piece of $W^u(p(\epsilon, \gamma))$. Varying γ from γ_1 to γ_2 this ball crosses $W^s(p(\epsilon, \gamma))$ and so must part of the unstable manifold.

As an illustration of the theorem consider the case $\gamma = \gamma_2^h$ where $f_\gamma^h(0)$ is the unstable fixed point in the interior of I . Conditions (1) and (2) of (9.2) are satisfied and (3) is easily checked in any given family (e.g. (8.1)). At a point of tangency (the 'first') of the stable and unstable manifolds of $p(\epsilon, \gamma(\epsilon))$ these manifolds are arranged as shown in fig. 8.

Note that there will be more than one tangency for $F_{(\epsilon,\gamma)}$ as γ passes through γ_2^h with ϵ small, in fact there will be infinitely many. This follows because when $\epsilon = 0$ the unstable manifold of p collapses onto the parabola $y = f_\gamma(x)$ and the points at the ends of the loops in $W^u(p)$ (such as s, t in fig. 8) will coincide. Thus at $\epsilon = 0$ there are in effect countably many (coincident) tangencies occurring between $W^u(p)$ and $W^s(p)$. When $\epsilon \neq 0$ these tangencies separate into fans of curves through $\epsilon = 0$, $\gamma = \gamma_*$, each curve representing a tangency. More-

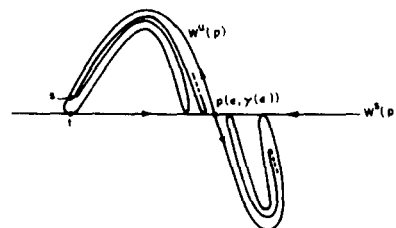


Fig. 8. Homoclinic Tangency.

over, since higher order invariant hyperbolic sets Λ^j created in preceding fans already exist, each fan contains a further *uncountable* set of curves radiating from γ_* on which we have tangencies between invariant manifolds $W^u(x)$, $W^s(y)$ of arbitrary pairs of points $x, y \in \Lambda^j$. This leads to the picture of the ϵ - γ plane of fig. 9. There are further "parallel" bands of curves representing period doubling sequences along with fans of homoclinic tangencies. Each tangency is preceded and followed by sequences of saddle-node and period doubling bifurcations (Gavrilov and Silnikov [7]). However, it appears that all these periodic points have their counterparts in the one-dimensional map and this lends weight to the conjecture of van Strien [41] that there are no more periodic orbits for $F_{(\epsilon, \gamma)}$, $\epsilon \neq 0$ than there are for $F_{(0, \gamma)}$; they are simply created in a different order.

These results show that, while the bifurcation sequence during the creation of the horseshoe for a strongly dissipative two-dimensional map is grossly the same as that for the one-dimensional family sketched in fig. 7, the details differ dramatically. Specifically, for an orbit of period $\leq N$ (fixed), we can choose ϵ sufficiently small to guarantee the 'correct' (i.e. Sarkovskii's) order of bifurcation, but for any $\epsilon > 0$ we can find orbits of sufficiently high period M which come 'out of order' and such that more than one stable orbit can coexist.

8. Conclusions and some open problems

We leave the reader to draw his own conclusions on precisely how the results of sections 6–7 should be applied to G_γ , and end with a conjecture based

on the following ideas. Recall that, for f_γ , the points $\gamma_{k \cdot 2^n}^h$ correspond to strange attractors. In contrast, in the fans above these points for $\epsilon \neq 0$ we expect to find residual subsets of open sets of parameter values for which $F_{(\epsilon, \gamma)}$ has countably many sinks (Newhouse [34, 35]). However, the 'windows' in which these sinks are stable are extremely short and it is still possible and even likely, that there is a set of parameter values of positive measure for which $F_{(\epsilon, \gamma)}$ has no stable sinks. We conjecture that this is the case, but note that, as in Jakobsen's work, a proof (or disproof) of this conjecture will require careful estimates of the derivatives along orbits which continually reenter a small neighborhood of the 'bends' in the manifolds of fig. 8.

Conjecture 8.1. For ϵ sufficiently small there is a set $\Gamma_\epsilon \in (\gamma_1^F, \gamma_1^h)$ of positive Lebesgue measure such that for $\gamma \in \Gamma_\epsilon$, $F_{(\epsilon, \gamma)}$ has a strange attractor.

To cut our possible losses, we remark that, even if this conjecture is untrue, and all the Jakobsen points vanish when $\epsilon \neq 0$, the stable periodic orbits remaining are of such high periods that they are effectively unobservable and one would see a pseudo-strange attractor.

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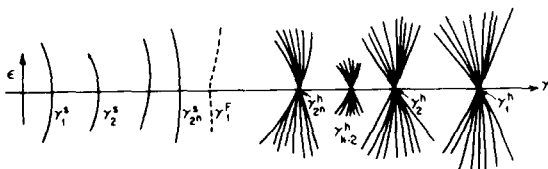


Fig. 9. The bifurcation set of $F_{(\epsilon, \gamma)}$.

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