MECHANISM FOR CHAOS IN THE DUFFING EQUATION

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It is shown that period-doubling bifurcations in the Duffing equation arise from parametric instability. Our mechanism requires this to be preceded by generation of zero- and second-harmonic modes by parametric instability. Numerical calculations are presented which confirm the analytic findings.

A recent study of the Duffing equation [1]

$$\ddot{x} + \alpha\dot{x} + x - \beta x^3 = \Gamma \cos ft\,, \qquad (1)$$

has shown that its solution exhibits hysteresis together with a cascade of period-doubling bifurcations on the upper branch resulting ultimately in a chaotic state. It is our object here to propose a mechanism which will explain this behaviour. Unlike discrete systems where similar period-doubling behaviour results as a consequence of tangent bifurcations [2,3], the mechanism responsible for the period-doubling behaviour in systems modelled by a set of differential equations is less well understood. We show that for eq. (1) above, the cascade of period-doubling bifurcations *must* be preceded by a bifurcation from a solution comprising only odd harmonics to one in which both even and odd harmonics are present. The latter are excited by a parametric instability in the system; sub-harmonics then result from a further cascade of sequential parametric instabilities, as shown below.

Our approach is similar to that published recently by Chui and Ma [4]. Here, however, we investigate the mechanism responsible for subharmonic generation whereas they concentrated on the chaotic regime. We assume that in the region of interest the solution for $x(t)$ can be approximated by the truncated mode expansion

$$x(t) = \sum_q C_q \cos(qft + \theta_q)\,, \qquad (2)$$

where q takes the values $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ and 3. Here C_q and θ_q are slowly varying functions of time in the sense that $\dot{C}_q \ll qfC_q$, etc. Substituting (2) into (1) and equating respective Fourier coefficients to zero gives a set of coupled first-order equations for the mode amplitudes and phases. We shall be concerned with the case where steady-state conditions prevail (i.e. where $\dot{C}_q = 0$, etc.), since this is sufficient to derive the required threshold conditions.

Note first that there is no threshold condition for generation of the odd harmonics (q = 3, 5, 7, ...); these are always present irrespective of the amplitude Γ or frequency f of the driving force. By contrast, even harmonics are generated by parametric instability with a concomitant threshold requirement.

The characteristic equation for the fundamental mode in zeroth-order approximation is given by the first of the following:

$$D_1 D_1^* = 3\beta\Gamma^2/4Z_1\,, \qquad (3a)$$

$$D_{3/2} D_{1/2}^* = Z_1^2\,, \qquad (3b)$$

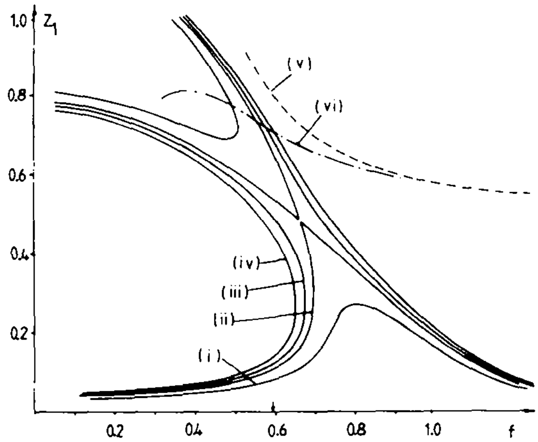

Fig. 1. Plot of characteristic curve eq. (3a) for $\alpha = 0.4$, $\beta = 4$ and the following values of Γ: (i) $\Gamma = 0.1$, (ii) $\Gamma = 0.1108$, (iii) $\Gamma = 0.115$ and (iv) $\Gamma = 0.12$. Curve (v) is the threshold condition eq. (3b), curve (vi) is the threshold condition eq. (3c). The arrow marks the value of f at which curves (iv) and (vi) intersect. See text for details.

$$D_0 = 2Z_1^2 \operatorname{Re}(1/D_2), \quad (3c)$$

$$D_0 = 2Z_1^2 \operatorname{Re}(1/D_2 + \tfrac{2}{3}Z_1/D_2D_3^* + \tfrac{1}{3}Z_1/D_2D_3), \quad (3d)$$

$$D_{1/2}D_{1/2}^* = 9\beta^2 C_1^2 \, |C_0 + \tfrac{1}{2}C_2 \exp[\mathrm{i}(2\theta_1 - \theta_2)]\,|^2. \quad (3e)$$

Here,

$$D_q = 1 - (qf)^2 - \sum_{q'} Z_{q'}(2 - \delta_{qq'}) - \mathrm{i}q\alpha f, \quad q \neq 0,$$
$$(4)$$
$$D_0 = 1 - \sum_{q'} 2Z_{q'} + \tfrac{4}{3}Z_0, \quad q = 0,$$

where δ is the Kronecker δ-function and

$$Z_q = 3\beta C_q^2/4 \quad (q \neq 0), \quad Z_0 = 3\beta C_0^2/2,$$

a suitable measure of the "intensity" of the qth mode. It is sufficient for our purpose to consider only the $q' = 1$ contribution to the sum in (4), which is now done without further comment.

Eq. (3a) is shown in fig. 1 [curves (i)–(iv)] as a plot of Z_1 versus f. Here, we follow the approach adopted in ref. [1], where the frequency is the variable quantity and α, β and Γ are given constant values. Note the different forms for the curves depending on whether

$$\Gamma \gtrless \Gamma_c \equiv (\alpha - \tfrac{1}{4}\alpha^3)/(3\beta)^{1/2}. \quad (5)$$

When $\Gamma < \Gamma_c$ the lower branch of the solution exhibits hysteresis [5] but does not exhibit period-doubling behaviour. When $\Gamma > \Gamma_c$ (corresponding to the case considered in ref. [1]), the two branches reconnect as shown and only then is period doubling observed on the branch obtained by continuation from the vicinity of $f \approx 1$.

Generation of $f/2$ and $3f/2$ by direct parametric instability of these modes requires eqs. (3b) (both real and imaginary parts) to be satisfied simultaneously with (3a). Clearly, these requirements are too stringent to permit sub-harmonic generation by this route. More generally, it is not possible to generate the frequencies ν and $2f-\nu$ by direct parametric instability except for the special case of degenerate frequencies (when the two conditions in (3b) are replaced by a single condition), or when $\nu = 0$, in which case the threshold condition analogous to (3b) is replaced by (3c).

Eq. (3c) is the threshold condition for parametric excitation of the zero-frequency and second-harmonic modes as a result of their interaction with the fundamental mode alone. If proper account is taken of the third-harmonic terms – that is, by incorporating $\frac{3}{2}\beta C_1 C_2 C_3 \cos(\theta_1 + \theta_2 - \theta_3)$ as a driving term in the equation for the zero-frequency mode, and $\frac{3}{2}\beta \times C_0 C_1 C_3 \cos(\theta_3 - \theta_1)$, $\frac{3}{2}\beta C_0 C_1 C_3 \sin(\theta_3 - \theta_1)$ as driving terms in the in-phase and quadrature parts respectively of the second-harmonic mode equation – then the threshold condition (3c) is modified to (3d). Eqs. (3c) and (3d) are also plotted in fig. 1. Note that it is essential to include the effects of the third-harmonic mode in order to get a sensible estimate of the threshold frequency for generation of the even frequency modes. The value predicted by the intersection of curves (iv) and (vi) in fig. 1 ($f = 0.589$) agrees well with that obtained from numerical solution of eq. (1), where even harmonics appear below $f = 0.582$.

With the appearance of the even-harmonic modes, it is now possible to generate the sub-harmonics. The appropriate driving terms are of the form $C_0 C_1 C_{1/2} \times \cos(\theta_1 - 2\theta_{1/2})$, $C_1 C_2 C_{1/2} \cos(\theta_2 - \theta_1 - 2\theta_{1/2})$, etc., and the threshold condition is given by eq. (3e). This is a degenerate form of eq. (3b) discussed previously and, as such, constitutes a single condition which can be satisfied simultaneously with eq. (3a). Clearly, the zero- and second-harmonic modes *must* be present before the mechanism for sub-harmonic

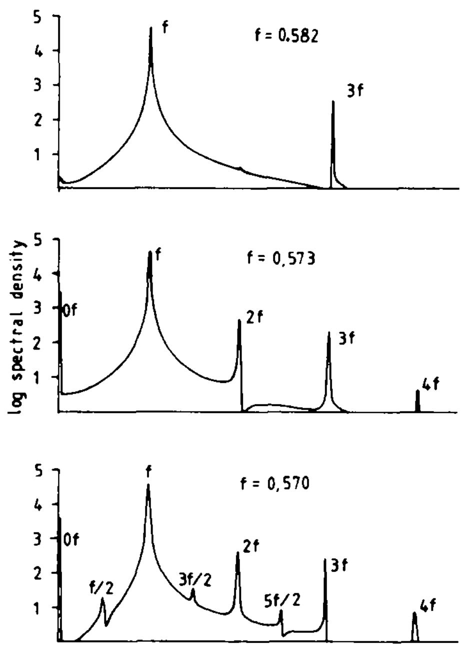

Fig. 2. Spectra obtained by transforming numerical solutions of eq. (1) for the case $\alpha = 0.4$, $\Gamma = 0.12$ and for the f values shown. See text for details.

generation summarised in eq. (3e) can come into effect. At the next bifurcation (to generate $f/4$), a threshold condition similar to (3e) is obtained with the suffixes 2, 1, 0, $\frac{1}{2}$ now replaced by 1, $\frac{1}{2}$, 0, $\frac{1}{4}$, respectively, and so on.

The condition (3e) is obtained by ignoring the effects of coupling the f mode to others such as the $3f/2$ or $5f/2$ modes. This coupling will occur through terms like $C_1^2 C_{1/2} \cos(2\theta_1 + \theta_{1/2} - \theta_{5/2})$, $C_1 C_2 C_{1/2} \times \cos(\theta_1 + \theta_2 - \theta_{1/2} - \theta_{5/2})$, etc. Clearly, such coupling will modify the condition (3e). However, these extra coupling mechanisms differ fundamentally from that embodied in (3e) in that, if considered in isolation, they describe the prohibited processes of two-mode excitation by parametric instability [cf. earlier comments concerning eq. (3b)]. The degenerate mechanism embodied in eq. (3e) is essential for generation of the $f/2$ mode, and hence for subsequent $f/2^n$ modes.

The general sequence of events is summarised in fig. 2, where a spectrum consisting of odd harmonics alone bifurcates to one consisting of both even and odd harmonics, before the sub-harmonic modes appear.

A more detailed study of this problem is in progress and results will be published elsewhere.

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