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# PERTURBATION PROCEDURE FOR THE VAN DER POL OSCILLATOR BASED ON THE HOPF BIFURCATION THEOREM

Systems whose dynamic behavior exhibit limit-cycles arise in many areas of science and engineering [1, 2]. Often, these systems can be modeled by a non-linear differential equation having the form

$$d^2x/dt^2 + x = \varepsilon f(x, \mu)dx/dt, \quad (1)$$

where $\varepsilon$ is a positive parameter which gives a measure of the non-linearity; $\mu$ is another parameter or set of parameters which characterize the system; and the function $f(x, \mu)$ has the property

$$f(-x, \mu) = +f(x, \mu). \quad (2)$$

When $\varepsilon$ satisfies the condition

$$0 < \varepsilon \ll 1, \quad (3)$$

approximations to the solution of equation (1) can be constructed in terms of a perturbation series in $\varepsilon$ [2], i.e.,

$$x(\theta, \varepsilon) = x_0(\theta) + \varepsilon x_1(\theta) + \varepsilon^2 x_2(\theta) + \cdots \quad (4a)$$

$$\theta = [1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots]t, \quad (4b)$$

where the initial conditions are

$$x(0, \varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \qquad dx(0, \varepsilon)/d\theta = 0. \quad (5a, b)$$

The *a priori* unknown constants $(\omega_1, \omega_2, \ldots)$ and $(A_0, A_1, \ldots)$ are used to eliminate secular terms and thus, to obtain the values of the amplitudes and frequency to the periodic limit-cycle solution. For example, the substitution of equations (4) into equation (1) leads to the following hierarchy of linear, inhomogeneous differential equations that can be successively solved for the $x_i(\theta)$ for $i = 0, 1, 2, \ldots$:

$$d^2x_0/d\theta^2 + x_0 = 0, \qquad x_0(0) = A_0, \qquad dx_0(0)/d\theta = 0; \quad (6a)$$

$$d^2x_1/d\theta^2 + x_1 = G_1(x_0, A_0, \omega_1, \mu), \qquad x_1(0) = A_1, \qquad dx_1(0)/d\theta = 0; \quad (6b)$$

$$\vdots \qquad \vdots$$

$$\begin{aligned} d^2x_i/d\theta^2 + x_i &= G_i(x_{i-1}, A_{i-1}, \omega_i, \mu) \\ &= H_i^{(1)}(A_{i-1}, \omega_i, \mu)\cos\theta + H_i^{(2)}(A_{i-1}, \omega_i, \mu)\sin\theta \\ &\quad + (\text{higher-order harmonics}), \\ x_i(0) &= A_i, \qquad dx_i(0)/d\theta = 0. \end{aligned} \quad (6c)$$

The functions $G_i(x_{i-1}, A_{i-1}, \omega_i, \mu)$ are known functions. At each stage, the elimination of secular terms requires

$$H_i^{(1)}(A_{i-1}, \omega_i, \mu) = 0 = H_i^{(2)}(A_{i-1}, \omega_i, \mu), \quad (7)$$

which gives two equations that can be solved for the "amplitude" and "frequency" corrections $A_{i-1}$ and $\omega_i$. In this way, one can calculate a perturbation solution to equation



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(1) to any desired order in the parameter $\varepsilon$. (Complete details are given in the book of Mickens [2].) Thus, for instance, the van der Pol equation

$$d^2x/dt^2 + x = \varepsilon(1 - x^2)(dx/dt), \tag{8}$$

under the condition given by equation (3), has the following perturbation solution for its limit cycle [2]:

$$x(\theta, \varepsilon) = 2\cos\theta + (\varepsilon/4)(3\sin\theta - \sin 3\theta)$$
$$+ (\varepsilon^2/96)(-13\cos\theta + 18\cos 3\theta - 5\cos 5\theta) + O(\varepsilon^3) \tag{9a}$$
$$\theta = [1 - (\varepsilon^2/16) + O(\varepsilon^3)]t. \tag{9b}$$

Note that in the general procedure above, the "other" parameter $\mu$ is held fixed, while the expansion is done in $\varepsilon$. There are a number of important cases where $\varepsilon$ may not be small, but, where a variation in $\mu$ is required. Under certain conditions the Hopf Bifurcation Theorem [1, 3, 4] can be applied. The main purpose of this letter is to construct a perturbation procedure based on the Hopf Bifurcation Theorem for equation (1) and illustrate its use by applying it to the van der Pol equation. A future, more important application is discussed at the end.

To begin, consider the system of equations

$$dx_1/dt = f_1(x_1, x_2, \mu), \qquad dx_2/dt = f_2(x_1, x_2, \mu), \tag{10a, b}$$

where in a domain $D$, the functions $f_1(x_1, x_2, \mu)$ and $f_2(x_2, x_2, \mu)$ are continuous with continuous first derivatives, and they are also assumed to depend on a parameter $\mu$. A critical point of equations (10) is a solution to the equation

$$f_1(\bar{x}_1, \bar{x}_2, \mu) = 0, \qquad f_2(\bar{x}_1, \bar{x}_2, \mu) = 0. \tag{11}$$

Let for each $\mu$ in some interval, say, $-\mu_0 < \mu < \mu_0$, where $\mu_0$ is real and positive, there exists an isolated critical point of equations (10) that depends on the parameter $\mu$, i.e.,

$$(\bar{x}_1, \bar{x}_2) = (\bar{x}_1(\mu), \bar{x}_2(\mu)). \tag{12}$$

Let the linearized system corresponding to equations (10) have the Jacobian matrix $M(\mu)$, where the $k$th row, $l$th column is given by [1]

$$[M(\mu)]_{kl} = \partial f_k(\bar{x}_1, \bar{x}_2)/\partial x_l; \qquad k, l = 1, 2. \tag{13}$$

Denote the eigenvalues of $M(\mu)$ by $\lambda_1(\mu)$ and $\lambda_2(\mu)$. Assume for some interval of $\mu$ values, $-\delta < \mu < +\delta$, where $\delta$ is real and positive, that $\lambda_1(\lambda)$ and $\lambda_2(\mu)$ are differentiable in $\mu$ and complex, i.e.,

$$\lambda_1(\mu) = [\lambda_2(\mu)]^* = \alpha(\mu) + i\gamma(\mu), \tag{14}$$

for some functions $\alpha(\mu)$ and $\gamma(\mu)$.

One can now state the Hopf Bifurcation Theorem [1] as follows. Assume that the critical point $(\bar{x}_1(\mu), \bar{x}_2(\mu))$ is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$, and that

$$\alpha(0) = 0. \tag{15}$$

If

$$d\alpha(0)/d\mu > 0 \quad \text{and} \quad \gamma(0) \neq 0, \tag{16, 17}$$

then for sufficiently small $|\mu|$, i.e., $|\mu| < \mu_c$ where $\mu_c \leq \mu_0$ and $\delta$, there exists a periodic solution to equations (10). If the critical point $(\bar{x}_1(0), \bar{x}_2(0))$ is locally asymptotically stable, then there is a stable limit-cycle about $(\bar{x}_1(\mu), \bar{x}_2(\mu))$ for $|\mu| < \mu_c$.

At this point, it should be noted that equation (1) can always be rewritten to the form given by equations (10).

The above theorem does not lead to a direct procedure for calculating the properties, i.e., amplitude and frequency, of a possible stable limit-cycle. However, this can be done by forming expansions in the parameter $\mu$. Actually, it can be shown that the relevant expansions are given by the following expressions [4]:

$$x(\psi, \mu) = \sqrt{\mu}\, x_0(\psi) + \mu^{3/2} x_1(\psi) + \mu^{5/2} x_2(\psi) + \cdots, \qquad \psi = \omega t = (1 + \omega_1 \mu + \omega_2 \mu^2 + \cdots) t. \tag{18a, b}$$

As indicated, the expansion parameter is actually $\sqrt{\mu}$ rather than $\mu$; furthermore, the new independent variable $\psi$ is an even function of $\sqrt{\mu}$.

The initial conditions can be selected as follows:

$$x(0, \mu) = \sqrt{\mu}\, B_0 + \mu^{3/2} B_1 + \mu^{5/2} B_2 + \cdots, \qquad dx(0, \mu)/d\psi = 0. \tag{19a, b}$$

The *a priori* unknown constants $(\omega_1, \omega_2, \ldots)$ and $(B_0, B_1, B_2, \ldots)$ will be chosen such that no secular terms appear in the solution $x(\psi, \mu)$.

Substitution of equations (18) into equation (1) collecting terms of the same power of $\sqrt{\mu}$ together and setting them equal to zero, gives the following perturbation equations, where $x' = dx/d\psi$ and $x'' = d^2x/d\psi^2$:

$$x_0'' + x_0 = 0, \qquad x_0(0) = B_0, \qquad x_0'(0) = 0; \tag{20a}$$

$$x_1'' + x_1 = K_1(x_0, B_0, \omega_1, \varepsilon), \qquad x_1(0) = B_1, \qquad x_1'(0) = 0; \tag{20b}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_i'' + x_i = K_i(x_{i-1}, B_{i-1}, \omega_i, \varepsilon) = L_i^{(1)}(B_{i-1}, \omega_i, \varepsilon) \cos \psi + L_i^{(2)}(B_{i-1}, \omega_i, \varepsilon) \sin \psi + (\text{higher-order harmonics}),$$

$$x_i(0) = B_i, \qquad x_i'(0) = 0. \tag{20c}$$

The absence of secular terms at the $i$th stage gives the two equations

$$L_i^{(1)}(B_{i-1}, \omega_i, \varepsilon) = 0 = L_i^{(2)}(B_{i-1}, \omega_i, \varepsilon), \tag{21}$$

which can be solved for $B_{i-1}$ and $\omega_i$. In this way, the solution, equations (18), can be determined for equation (1).

To illustrate the use of this perturbation procedure based on the Hopf Bifurcation Theorem, application will be made to the van der Pol equation. However, an easy and direct calculation of the usual form of the van der Pol equation, as given by equation (8), shows that it does not satisfy the requirements for the application of the Hopf Bifurcation Theorem [1, 3]. This problem can be resolved by making the transformations

$$x = \bar{x}/\sqrt{\mu}, \qquad \varepsilon = \mu \bar{\varepsilon}, \tag{22}$$

where $\mu$ is a positive constant. Doing this gives

$$(d^2\bar{x}/dt^2) + \bar{x} = \bar{\varepsilon}(\mu - \bar{x}^2)\, d\bar{x}/dt. \tag{23}$$

(Note that for $\mu = 1$ one obtains the usual van der Pol equation. For the remainder of this letter, equation (23) will be used without the "bars" over $x$ and $\varepsilon$.) Direct application of the Hopf Bifurcation Theorem can now be used on equation (23). The conclusion is that equation (23) has a stable limit-cycle for $\mu$ small positive, provided that $\varepsilon > 0$.

Substitution of equations (18) into equation (23) gives

$$x_0''+x_0=0, \qquad x_0=B_0, \qquad x_0^1(0)=0; \tag{24}$$

$$x_0''+x_1=-2\omega_1 x_0''+\varepsilon(1-x_0^2)x_0', \qquad x_1(0)=B_1, \qquad x_1'(0)=0; \tag{25}$$

$$x_2''+x_2=-2\omega_1 x_1''-(\omega_1^2+2\omega_2)x_0''+\varepsilon(1-x_0^2)(x_1'+\omega_1 x_0')-2\varepsilon x_0 x_1 x_0',$$

$$x_2(0)=B_2, \qquad x_2'(0)=0. \tag{26}$$

Now equation (24) has the solution

$$x_0(\psi)=B_0\cos\psi. \tag{27}$$

Using this in equation (25) gives

$$x_1''+x_1=(2\omega_1 B_0)\cos\psi-\varepsilon B_0(1-B_0^2/4)\sin\psi+\varepsilon(B_0^3/4)\sin 3\psi. \tag{28}$$

Elimination of secular terms in the solution for $x_1(\psi)$ requires

$$\omega_1=0, \qquad B_0=2. \tag{29}$$

Therefore, $x_1(\psi)$ satisfies the equation and initial conditions

$$x_1''+x_1=2\varepsilon\sin 3\psi, \qquad x_1(0)=B_1, \qquad x_1'(0)=0. \tag{30}$$

This problem has the solution

$$x_1(\psi)=B_1\cos\psi+(\varepsilon/4)(3\sin\psi-\sin 3\psi). \tag{31}$$

Substitution of equations (27), (29) and (31) into equation (26) gives

$$x_2''+x_2=(4\omega_2+\varepsilon^2/4)\cos\psi+(2\varepsilon B_1)\sin\psi-(3\varepsilon^2/2)\cos 3\psi$$

$$+(3\varepsilon B_1)\sin 3\psi+(5\varepsilon^2/4)\cos 5\psi. \tag{32}$$

Again, no secular terms in the solution for $x_2(\psi)$ gives the requirement

$$\omega_2=-\varepsilon^2/16, \qquad B_1=0. \tag{33}$$

Thus, to terms of order $\mu^{5/2}$, the solution to equation (23) is

$$x(\psi,\mu)=2\sqrt{\mu}\cos\psi+\mu^{3/2}(\varepsilon/4)(3\sin\psi-\sin 3\psi)+O(\mu^{5/2}), \tag{34a}$$

$$\psi=[1-(\varepsilon^2/16)\mu^2+O(\mu^3)]t. \tag{34b}$$

Comparison of equations (34) and (9) shows that when $\mu=1$, equations (34) reduce to equations (9).

The following summarizes what has been obtained. In equation (1), let $\varepsilon$ be a measure of the nonlinearity and let $\mu$ be a bifurcation parameter such that the conditions of the Hopf Bifurcation Theorem are satisfied. Solutions to equation (1) can then be constructed by either of the following procedures.

(a) *Case I*: $0<\varepsilon\ll 1$ and $\mu$ fixed. In this situation the usual perturbation techniques can be applied to obtain the limit-cycle solution. (See reference [2].)

(b) *Case II*: $0<\mu<\mu_c$ and $\varepsilon$ fixed. For this case the above discussed expansion in $\sqrt{\mu}$ can be applied.

It should be indicated that for any particular nonlinear differential equation, having the form of equation (1), the determination of the bifurcation parameter may require some thought as to the proper transformation of the dependent variable. (See, for example, equation (22) in the case of the van der Pol equation.)

Finally, it should be indicated that the above perturbation procedure, based on the Hopf bifurcation parameter, $\mu$, should be of value in investigating the properties of limit-cycle behavior for non-linear, singular differential equations [5, 6]. For these equations the function $f(x)$ in equation (1) takes the form

$$f(x, \mu) = g(x^2)/(1 - x^2),$$

and the usual perturbation expansion techniques (in $\varepsilon$) cannot be applied. The author is currently considering this class of equations.

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