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ON THE RESPONSE OF THE VAN DER POL OSCILLATOR TO WHITE NOISE EXCITATION†

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The response of the Van der Pol oscillator to white noise excitation has been investigated by Stratonovich, Caughey, Piszczek and others using different approximate methods. There occurred a controversy over this problem in the IUTAM Symposium on Stochastic Problems in Dynamics held at Southampton in 1976. In the present paper it is shown that in certain ranges of non-linearity and noise intensity Stratonovich's stochastic averaging method yields reasonable results while Caughey's and Piszczek's techniques give incorrect results. It is also shown that the Gaussian closure technique is inapplicable to this problem.

1. INTRODUCTION

The Van der Pol oscillator is the model of many electrical and mechanical systems and is representative of many self-excited systems. It is of both theoretical and engineering importance to study the behaviour of the Van der Pol oscillator under random perturbation.

In the past three decades, the stationary response of the Van der Pol oscillator to white noise excitation has been investigated by Stratonovich [1], Caughey [2], Piszczek [3] and others [4, 5] using different approximate methods. The solutions obtained by different approaches are quite different or even contradictory. Stratonovich showed that the mean square amplitude of the response $\langle A^2\rangle$ increases with increasing noise intensity while Caughey and Piszczek showed that $\langle A^2\rangle$ decreases. There occurred a controversy over this problem in the IUTAM Symposium on Stochastic Problems in Dynamics held at Southampton in 1976. In addition, as an example in his book [6], Bolotin applied the Gaussian closure technique to a Rayleigh oscillator subjected to white noise excitation. If one applies the technique to the Van der Pol oscillator, another different solution will be obtained.

The equation of the motion for the problem can be written in the form

$$\ddot{x}+\varepsilon(-1+x^2)\dot{x}+x=\xi(t), \tag{1}$$

where ε is a positive parameter characterizing the severity of non-linearity and $\xi(t)$ is a Gaussian white noise with correlation function $R(\tau)=s\,\delta(\tau)$, where s is the intensity of the noise.

In the present paper, Stratonovich's stochastic averaging method (SAM), and Caughey's, Piszczek's and the Gaussian closure techniques are applied to equation (1) sequentially, and the statistics of the response are obtained for each approach. Then digital simulation results are used to decide the correctness of each approximate solution. In order to verify the effectiveness of the simulation results, the simulation method is first applied to a system whose exact solution is available, then the comparison is made between the simulation results and the exact solutions and finally the accuracy of the

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method is determined. Following that, a detailed comparison is made between the simulation results and approximate solutions for the Van der Pol equation(1), and analyses are made to explain why SAM can, in certain ranges of values ε and s, yield good approximate solutions for the problem and other techniques can not.

2. STOCHASTIC AVERAGING METHOD

Introducing the following transformation to equation (1),

$$x = A(t)\cos\Phi(t) = A(t)\cos[t+\varphi(t)],$$
$$\dot{x} = -A(t)\sin\Phi(t) = -A(t)\sin[t+\varphi(t)], \tag{2}$$

one obtains

$$\dot{A} = \tfrac{1}{2}\varepsilon A(1-\tfrac{1}{4}A^2-\cos 2\Phi+\tfrac{1}{4}A^2\cos 4\Phi)-\xi(t)\sin\Phi,$$
$$\dot{\varphi} = \tfrac{1}{2}\varepsilon[(1-A^2/2)\sin 2\Phi-\tfrac{1}{4}A^2\sin 4\Phi]-(1/A)\xi(t)\cos\Phi. \tag{3}$$

Then applying the deterministic and stochastic averaging principles to equations (3), one obtains the following averaged Itô equations:

$$\mathrm{d}A = [\tfrac{1}{2}\varepsilon A(1-\tfrac{1}{4}A^2)+s/4A]\,\mathrm{d}t+\sqrt{s/2}\,\mathrm{d}w_1(t),$$
$$\mathrm{d}\varphi = (1/A)\sqrt{s/2}\,\mathrm{d}w_2(t). \tag{4}$$

Here $w_1(t)$ and $w_2(t)$ are independent unit Wiener processes. It is seen from equations (4) that $A(t)$ is approximately a Markov processes. The transition probability density function (p.d.f.) for the amplitude, $p(A,t)$, is governed by the following one-dimensional FPK equation:

$$\frac{\partial p(A,t)}{\partial t} = -\frac{\partial}{\partial A}\left\{\left[\frac{1}{2}\varepsilon A\left(1-\frac{1}{4}A^2\right)+\frac{s}{4A}\right]p\right\}+\frac{1}{2}\frac{\partial^2}{\partial A^2}\left(\frac{s}{2}p\right). \tag{5}$$

The solution of equation (5) is not available but the stationary p.d.f. $p_s(A)$ can be easily obtained as

$$p_s(A) = N^{-1}A\exp[-(\varepsilon/8s)(A^2-4)^2], \tag{6}$$

where N^{-1} is the normalization constant with

$$N = \int_0^\infty A\exp[-(\varepsilon/8s)(A^2-4)^2]\,\mathrm{d}A = \sqrt{\pi s/2\varepsilon}\,\operatorname{erf}(-\sqrt{2\varepsilon/s}). \tag{7}$$

Equation (6) shows that $p_s(A)$ depends only on the ratio $c = s/\varepsilon$. The curves of p.d.f. for different c values are shown in Figure 1.

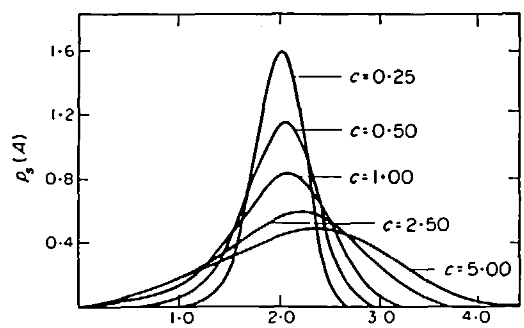

Figure 1. Steady-state p.d.f. of the response amplitude A.

The most probable amplitude A_{mp} is obtained from equation (6) by letting $\mathrm{d}P_s(A)/\mathrm{d}A = 0$:

$$A_{mp} = (2+2\sqrt{1+s/2\varepsilon})^{1/2}. \tag{8}$$

According to the form of equation (4), it is reasonable to assume that the joint p.d.f. of the response $p_s(A, \varphi)$ is separable. Based on this assumption it can be shown [7] that φ is uniformly distributed in $(0, 2\pi)$ and $p_s(A, \varphi)$ is of the following form:

$$p_s(A, \varphi) = p_s(A)p_s(\varphi) = (1/2\pi)p_s(A). \tag{9}$$

The stationary joint p.d.f. of the displacement and velocity $p_s(x, \dot{x})$ can then be obtained from equation (9) by using transformation (2):

$$p_s(x, \dot{x}) = M^{-1} \exp[-(\varepsilon/8s)(x^2+\dot{x}^2-4)^2]. \tag{10}$$

Here M^{-1} is the normalization constant with

$$M = \int\!\!\int_{-\infty}^{\infty} \exp[-(\varepsilon/8s)(x^2+\dot{x}^2-4)^2]\,\mathrm{d}x\,\mathrm{d}\dot{x}. \tag{11}$$

The marginal p.d.f. $p_s(x)$ and $p_s(\dot{x})$ can be obtained from equation (10) numerically. $p_s(x)$ and $p_s(\dot{x})$ are identical due to the symmetry of equation (10) with respect to x and $\dot{x}$.

The mean square amplitude of the response $\langle A^2\rangle$ can be obtained from equation (6):

$$\langle A^2\rangle = 4+(2s/N\varepsilon)\exp(-2\varepsilon/s). \tag{12}$$

Equation (12) shows that $\langle A^2\rangle$ increases with increasing noise intensity s.

The mean amplitude of the response $\langle A\rangle$ can be obtained from equation (6) by numerical integration. The results show that with increasing noise intensity s, $\langle A\rangle$ decreases slightly at first and then increases.

It is interesting to note that the stochastic averaging method may be regarded as a kind of equivalent non-linearization method. To show this, one can consider the energy-dependent-damping system

$$\ddot{x}+\varepsilon[-1+k(x^2+\dot{x}^2)]\dot{x}+x = \xi(t), \tag{13}$$

where ε and $\xi(t)$ are the same as in equation (1), and k is a positive constant. The exact p.d.f. of the amplitude of stationary response is

$$p_s(A) = N_1^{-1}A\exp[-(k\varepsilon/2s)(A^2-1/k)^2], \tag{14}$$

where N_1^{-1} is the normalization constant with

$$N_1 = \int_0^{\infty} A\exp[-(k\varepsilon/2s)(A-1/k)^2]\,\mathrm{d}A. \tag{15}$$

With $k = \frac{1}{4}$, equation (14) is equal to equation (6). In this sense, equation (13) with $k = \frac{1}{4}$ can be considered as the equivalent non-linear system of equation (1).

If one applies SAM to equation (13), the same $p_s(A)$ as in equation (14) can be obtained. On the other hand, Caughey [8] showed that equation (9), holds exactly for system (13). It seems that for a system having an exact solution of the form (2) with joint p.d.f. of the form (9), SAM yields exact statistics of the response. Therefore, the extent to which equation (9) is satisfied may be regarded as an indication of the effectiveness of SAM for the system.

It is also worthy noting that if the averaging method for the coefficients in the FPK equation is applied to equation (1), the same result as in equation (6) can be obtained. The result obtained by Piszczek [3] using the averaging method for the coefficients in FPK equation is incorrect.

3. CAUGHEY'S AND PISZCZEK'S TECHNIQUES

3.1. CAUGHEY'S TECHNIQUE

In the treatment of equation (1), Caughey [2] assumed that the response is of the form

$$x(t)=x_p(t)+x_n(t)=F\sin t+x_n(t), \tag{16}$$

where $x_p(t)$ represents the periodic part of the response while $x_n(t)$ is the random part, and F is an unknown constant. He believed that $x_n(t)$ could be treated as a narrow band Gaussian process in the case of small non-linearity. Using Rice's method [9] he showed that

$$(x_p+x_n)^3\approx[\tfrac{3}{4}F^2+3R_n(0)]x_p+[\tfrac{3}{2}F^2+3R_n(0)]x_n, \tag{17}$$

where $R_n(\tau)$ is the correlation function of $x_n(t)$. Then equation (1) can be approximately replaced by two linear equation for x_p and x_n, respectively:

$$\begin{aligned}&\ddot{x}_p+\varepsilon\{-1+\tfrac{1}{3}[\tfrac{3}{4}F^2+3R_n(0)]\}\dot{x}+x_p=0,\\&\ddot{x}_n+\varepsilon\{-1+\tfrac{1}{3}[\tfrac{3}{2}F^2+3R_n(0)]\}\dot{x}_n+x_n=0.\end{aligned} \tag{18}$$

If there exists a non-zero stationary x_p, then the damping ratio in the first of equations (18) must be zero, and from this

$$F^2=4-4R_n(0). \tag{19}$$

By substituting expression (19) into the second of equations (18), one obtains

$$R_n(0)=\langle x_n^2\rangle=\langle \dot{x}_n^2\rangle=\tfrac{1}{2}(1-\sqrt{1-2s/\varepsilon}). \tag{20}$$

Thus

$$F^2=2+2\sqrt{1-2s/\varepsilon}. \tag{21}$$

The mean square amplitude of the response of equation (1) is then

$$\langle A^2\rangle=\langle x^2+\dot{x}^2\rangle=\langle x_p^2\rangle+\langle x_n^2\rangle+\langle \dot{x}_p^2\rangle+\langle \dot{x}_n^2\rangle=4-2R_n(0)=3+\sqrt{1-2s/\varepsilon}. \tag{22}$$

Equation (22) shows that $\langle A^2\rangle$ decreases with increasing noise intensity s, which contradicts the result obtained by SAM.

The p.d.f. of $x(t)$ can be obtained by using Rice's formula [9]:

$$p_s(x)=[1/\sqrt{2\pi^3 Rn(0)}]\int_0^\pi \exp[-(x-F\cos t)^2/2R_n(0)]\,\mathrm{d}t. \tag{23}$$

Expression (23) can be evaluated numerically. Note that expressions (20)-(23) hold only for $s<\varepsilon/2$.

3.2. PISZCZEK'S TECHNIQUE

In the treatment of equation (1), Piszczek [3] assumed that the response is of the form

$$x(t)=x_p(t)+x_n(t)=F\sin\omega t+x_n(t), \tag{24}$$

where F and ω are both unknown constants.

Substituting equation (24) into equation (1) and separating sine, cosine and random terms with higher "harmonics" discarded, he obtained

$$\begin{aligned}&-\omega^2+2\varepsilon x_n\dot{x}_n+1=0,\qquad \tfrac{1}{4}F^2+x_n^2-1=0\\&\ddot{x}_n+\varepsilon(-1+x_n^2+\tfrac{1}{2}F^2)\dot{x}_n+x_n=\xi(t).\end{aligned} \tag{25}$$

ω and F^2 are obtained by taking the average of the first two of equations (25):

$$\omega^2 = \langle 1 - 2x_n\dot{x}_n \rangle = 1, \qquad F^2 = \langle 4 - 4x_n^2 \rangle = 4 - 4R_n(0). \tag{26}$$

Applying the equivalent linearization method to the third of equations (25), one can obtain

$$R_n(0) = \langle x_n^2 \rangle = \langle \dot{x}_n^2 \rangle = \tfrac{1}{2}(1 - \sqrt{1 - 2s/\varepsilon}). \tag{27}$$

It is seen that the results of Piszczek's technique are the same as those obtained by Caughey's technique.

It should be noted that in equations (25) F and ω are random and this contradicts the presumption.

4. GAUSSIAN CLOSURE TECHNIQUE

If the Gaussian closure technique, which was used for a Rayleigh oscillator by Bolotin ([6] see p. 307), is applied to equation (1), the following results can be obtained:

$$\langle x^2 \rangle = \langle \dot{x}^2 \rangle = \tfrac{1}{2}(1 + \sqrt{1 + 2s/\varepsilon}), \qquad \langle A^2 \rangle = \langle x^2 + \dot{x}^2 \rangle = 1 + \sqrt{1 + 2s/\varepsilon}. \tag{28}$$

These results are completely wrong because they do not approach the deterministic limit cycle solution when the noise intensity s approaches zero.

5. DIGITIAL SIMULATION METHOD

Sample functions of Gaussian white noise can be obtained by linear variation of the sequence (see Figure 2)

$$\xi_i = \sqrt{s/\Delta t}\,\zeta_i, \qquad i = 1, 2, 3, \ldots, \tag{29}$$

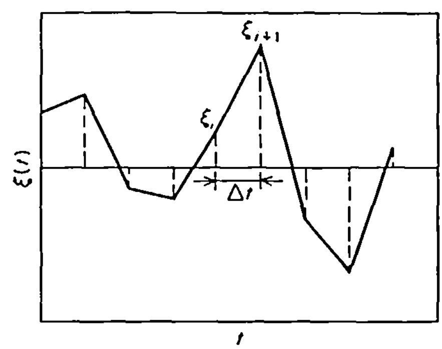

Figure 2. A typical sample of the simulated white noise.

where the ζ_i are generated by a standard Gaussian random number generator. In fact, the sequence (29) may be regarded as being "sampled" from the white noise, and Δt as the "sampling interval". When Δt is sufficiently small, the power spectral density function of the artificial process will be nearly equal to s over a sufficiently wide frequency range, and may be regarded as "white" for the system considered.

In the simulation of equation (1), we selected $\Delta t \doteq 0{\cdot}025$, which is about 1/250 of the natural period of the system. The power spectral density obtained from the samples by using F.F.T. technique is shown in Figure 3. It is very close to the corresponding theoretical value. It is seen that the spectrum remains approximately constant up to 6 Hz, and this is nearly 40 times of the natural frequency of the system.

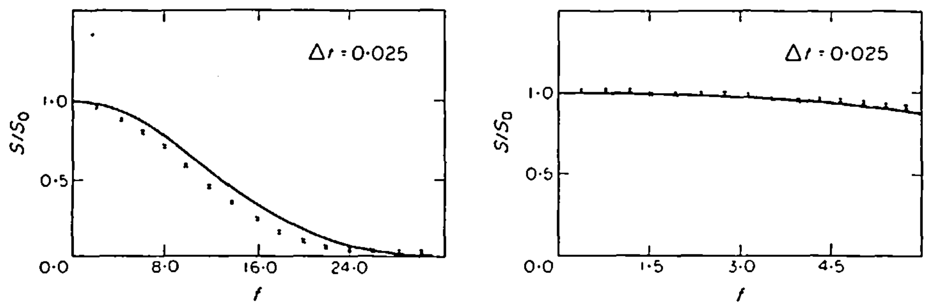

Figure 3. The power spectral density of the simulated white noise. ——, Theoretical spectrum; x, spectrum obtained from samples.

The equation was discretized and solved numerically by using the Runge–Kutta method of order 4. For each pair of ε and s, 300 samples of responses were obtained. Each sample contains 2600 points, and the last 800 points are regarded as the steady state response. Thus the total points used for evaluating the statistics were $300 \times 800 = 240\,000$.

In general, the error of a statistic obtained from simulation, due to statistical uncertainty, may be estimated by using the formula

$$p\{a(\sigma/\sqrt{N}) < M_s - M_e < b(\sigma/\sqrt{N})\} \approx (1/\sqrt{2\pi}) \int_a^b \exp(-u^2/2)\, du, \tag{30}$$

where $M_e = \langle M \rangle$ is the unknown exact value, $M_s = (1/N) \sum_{i=1}^{N} M_i$ is the simulation result, σ is the standard deviation of M_i and N is the total number of points used in statistics. From equation (30) one has, with probability 0·99, that the error of M_s is

$$e_{0\cdot 99} = |M_s - M_e| < 3\, \sigma/\sqrt{N}. \tag{31}$$

For the Van der Pol equation (1), the standard deviation of A and A^2 are not larger than $O(1)$ when the noise is not very large. So, if $N = 240\,000$, $e_{0\cdot 99}$ of the obtained moment values will be of the order of $O(10^{-3})$ or $O(10^{-2})$.

The round-off error of the computer and the error of the Runge–Kutta method are much smaller and will not be considered here.

To check the accuracy of our digital simulation results, we made a simulation of equation (13) with $k = 1$. Table 1 shows the exact and simulation results for the mean value and mean square value of the response amplitude. The exact results were obtained through numerical integration.

TABLE 1

The comparison between the exact and simulation results for $\langle A \rangle$ and $\langle A^2 \rangle$ ($\varepsilon = 0\cdot 20$, $k = 1$)

	Noise level s	0·05	0·20	0·50	1·50
	exact	0·981622	1·070433	1·218364	1·493568
⟨A⟩	simulation	0·995833	1·070927	1·218766	1·496941
	error	$+1\cdot 4 \times 10^{-2}$	$+4\cdot 9 \times 10^{-4}$	$+4\cdot 0 \times 10^{-4}$	$+3\cdot 4 \times 10^{-3}$
	exact	1·027624	1·287600	1·701254	2·590799
$\langle A^2 \rangle$	simulation	1·054732	1·287945	1·698011	2·596560
	error	$+2\cdot 7 \times 10^{-2}$	$+3\cdot 4 \times 10^{-4}$	$-3\cdot 2 \times 10^{-3}$	$+5\cdot 8 \times 10^{-3}$

6. COMPARISON AND DISCUSSION

6.1. COMPARISONS

Comparisons between the SAM solutions and digital simulation results for $p_s(x)$, $p_s(\dot{x})$, $p_s(A)$, $\langle A\rangle$ and $\langle A^2\rangle$ are shown in Figures 4–14, where the solid curves represent the SAM solutions while the crosses are the simulation results. It is shown that for $\varepsilon = 0{\cdot}05$, the SAM solutions are in full agreement with the simulation results; for $\varepsilon = 0{\cdot}20$, the SAM solutions are still quite close to the simulation results; but in the case of $\varepsilon = 0{\cdot}50$,

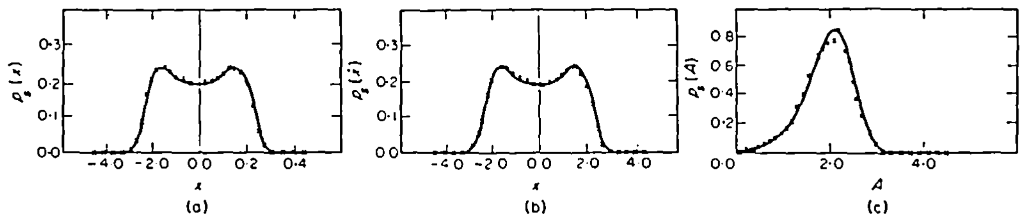

Figure 4. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}05$; $s = 0{\cdot}05$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

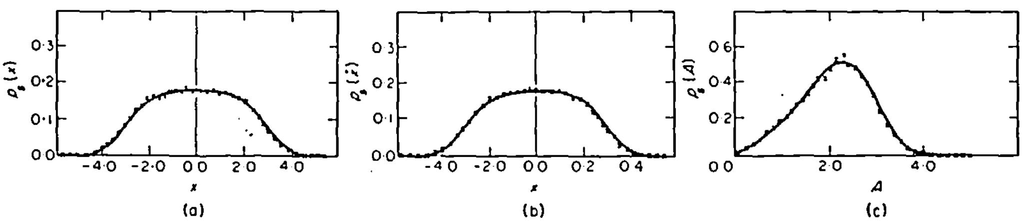

Figure 5. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}05$; $s = 0{\cdot}20$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

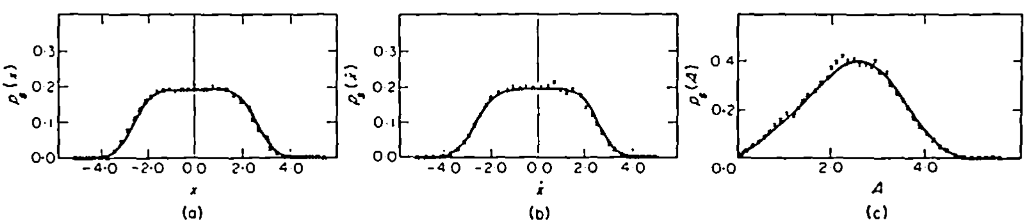

Figure 6. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}05$; $s = 0{\cdot}50$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

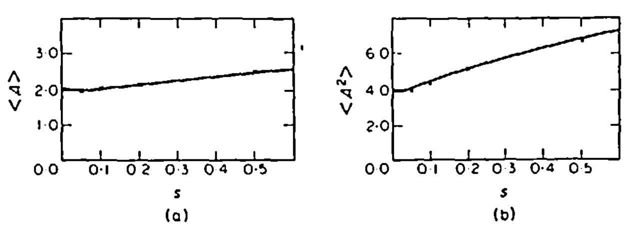

Figure 7. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}05$. (a) Mean value of A; (b) mean square value of A.

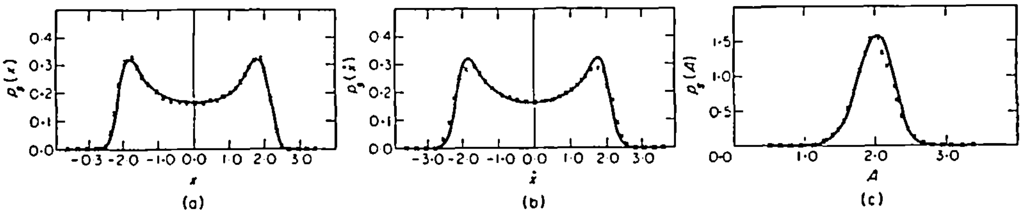

Figure 8. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}20$; $s = 0{\cdot}05$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

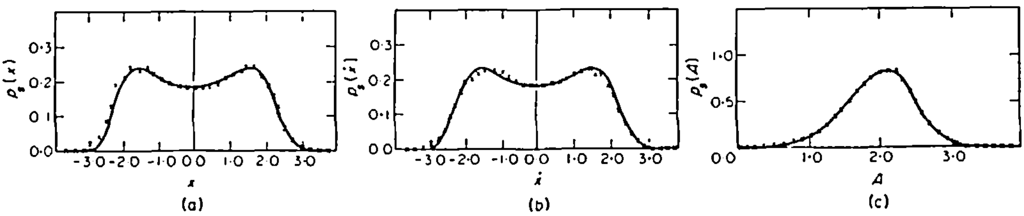

Figure 9. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}20$; $s = 0{\cdot}20$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

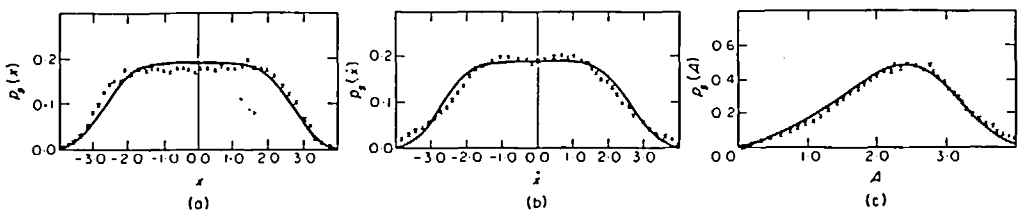

Figure 10. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}20$; $s = 1{\cdot}00$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

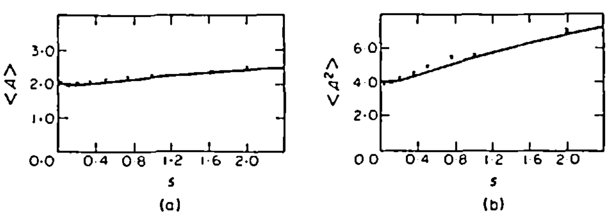

Figure 11. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}20$. (a) Mean value of A; (b) mean square value of A.

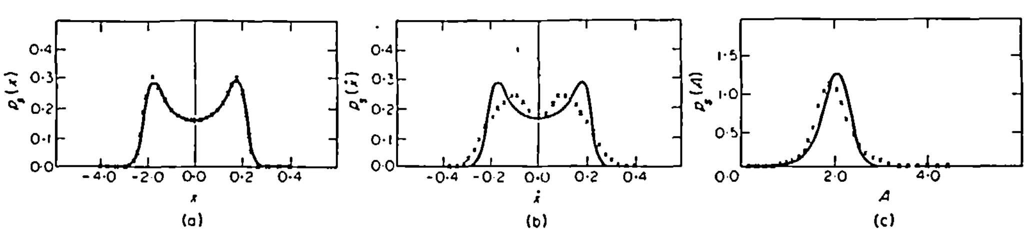

Figure 12. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}50$; $s = 0{\cdot}20$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

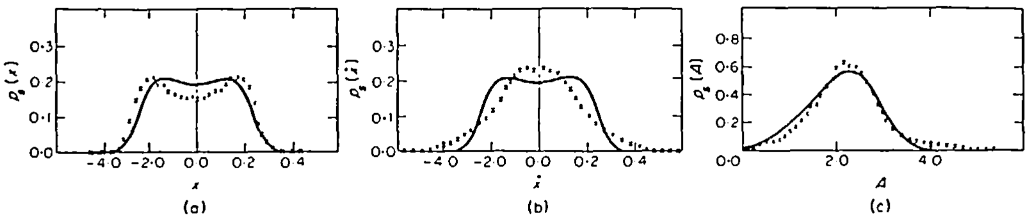

Figure 13. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}50$; $s = 1{\cdot}50$. (a) p.d.f. of x; (b) p.d.f. of $\dot{x}$; (c) p.d.f. of A.

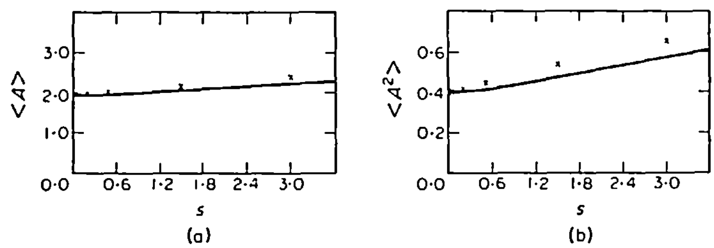

Figure 14. SAM solution and digital simulation results. $\varepsilon = 0{\cdot}50$. (a) Mean value of A; (b) mean square value of A.

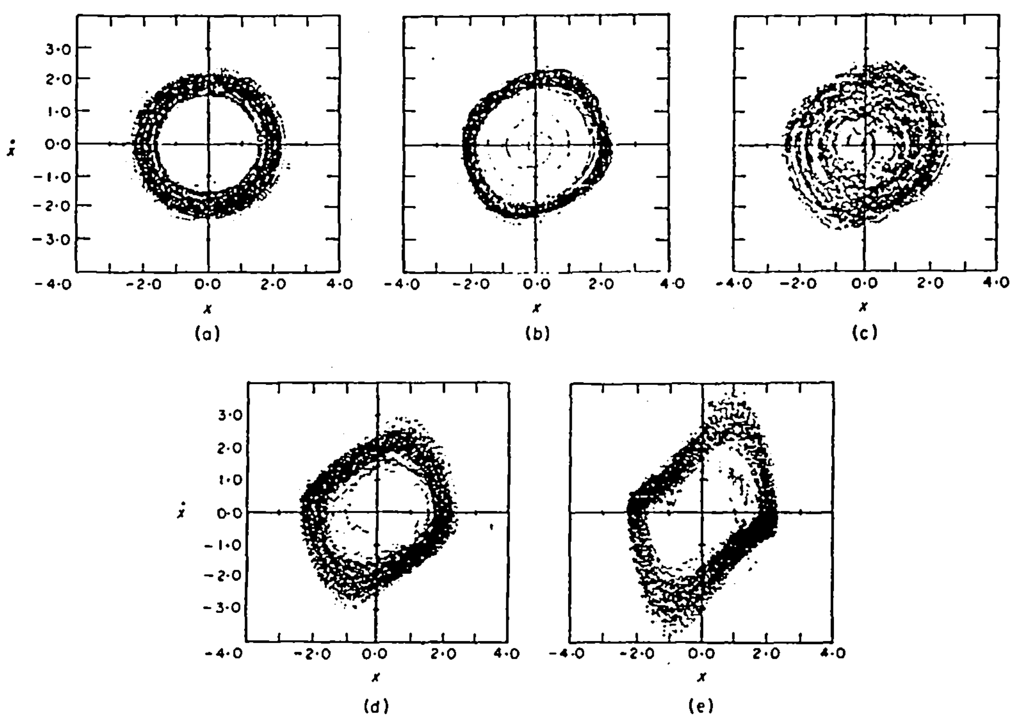

Figure 15. Discrete phase trajectories of the response. (a) $\varepsilon = 0{\cdot}05$, $s = 0{\cdot}01$; (b) $\varepsilon = 0{\cdot}20$, $s = 0{\cdot}02$; (c) $\varepsilon = 0{\cdot}20$, $s = 0{\cdot}10$; (d) $\varepsilon = 0{\cdot}50$, $s = 0{\cdot}10$; (e) $\varepsilon = 1{\cdot}20$, $s = 0{\cdot}20$.

remarkable differences between the SAM solutions and the simulation results are observed, especially for $p_s(\dot{x})$. In the last case SAM yields conservative mean and mean square amplitudes, and the error may be up to 12%.

The behaviour of the stationary response of the Van der Pol oscillator to white noise excitation can be seen from the discrete phase trajectories (phase points) shown in Figure 15, which are typical samples of the response sequences obtained from digital simulation. Each phase point represents the system state in a time interval Δt. The phase points are usually distributed in a ring; the shape of the ring is similar to that of a corresponding deterministic limit cycle and depends on the value ε, while the width of the ring depends on the ratio s/ε. The density of the phase points is proportional to the value of the p.d.f. $p_s(A, \varphi)$ or $p_s(x, \dot{x})$.

It was pointed out previously that the accuracy of SAM depends on the extent to which $p_s(A, \varphi)$ follows the form (9). When ε is small, for instance, $\varepsilon = 0{\cdot}05$, the phase points are distributed almost axi-symmetrically and the stationary p.d.f. $p_s(\varphi)$ is nearly a constant in $[0, 2\pi)$ (see Figure 16). In this case $p_s(A, \varphi)$ is almost of the form (9) and SAM yields

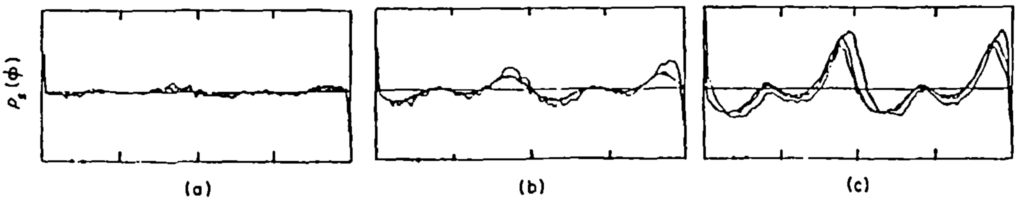

Figure 16. The p.d.f. of φ. (simulation results). (a) $\varepsilon = 0{\cdot}05$; $s = 0{\cdot}05$, $0{\cdot}20$, $0{\cdot}50$; (b) $\varepsilon = 0{\cdot}20$, $s = 0{\cdot}01$, $0{\cdot}10$, $1{\cdot}00$; (c) $\varepsilon = 0{\cdot}50$, $s = 0{\cdot}05$, $0{\cdot}50$, $2{\cdot}00$.

good approximation for the response. When ε is large, the distribution of the phase points becomes axially non-symmetric, apparently, and $p_s(\varphi)$ is obviously not uniform in $(0, 2\pi)$. In this case, $p_s(A, \varphi)$ is apparently not of the form (9) and SAM no longer yields a good approximation for the response. It is shown that the accuracy of SAM for this problem depends mainly on the non-linear factor ε and the errors of SAM increases with increasing ε. When $\varepsilon = 0{\cdot}20$, the relative errors of $\langle A\rangle$ and $\langle A^2\rangle$ of SAM when compared to the simulation results are about 3% and 6%, respectively.

6.2. DISCUSSION

As shown in equation (22), the mean square amplitude of the response obtained by Caughey's or Piszczek's techniques decreases with increasing noise intensity. This trend contradicts the simulation results shown in Figures 7, 11 and 14. On the other hand, as shown in Figure 17, there exists a discrepancy between $p_s(x)$ obtained by digital simulation and those obtained by Caughey's or Piszczek's techniques. The discrepancy increases

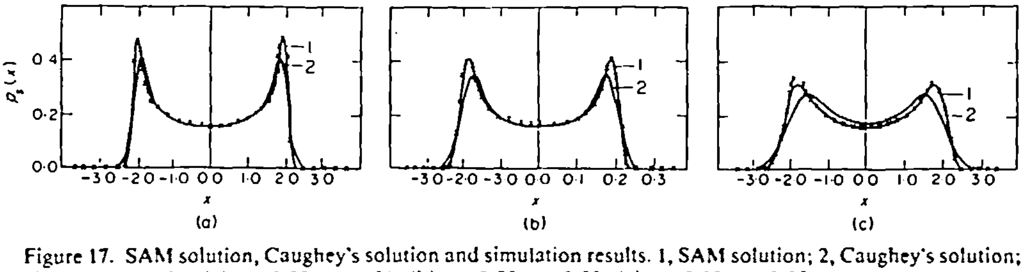

Figure 17. SAM solution, Caughey's solution and simulation results. 1, SAM solution; 2, Caughey's solution; x, simulation results. (a) $\varepsilon = 0{\cdot}20$, $s = 0{\cdot}01$; (b) $\varepsilon = 0{\cdot}20$, $s = 0{\cdot}02$; (c) $\varepsilon = 0{\cdot}20$, $s = 0{\cdot}05$.

with increasing noise intensity s. In addition, if Piszczek's technique is applied to equation (13), results inconsistent with the exact solutions will also be obtained. All these show that there must exist some mistakes in these two techniques.

In Caughey's technique, the solution form (16) is incompatible with the actual response. The actual response of the system to white noise excitation is a narrow band random process when ε is small and the bandwidth increases with increasing s. Hence the correlation time of the response is finite and there is no periodic component in the response. On the other hand, the linearization of the cubic term is questionable. In Caughey's original article [2], the term h_{21} may also have a contribution at ω_0.

In Piszczek's technique, there is a similar problem about the solution form. In addition, the harmonic balance method is not appropriate in the random case.

It is seen from Figures 4–14 that the stationary response of the Van der Pol oscillator to white noise excitation is obviously non-Gaussian. When s/ε is small, the distribution of the response is close to that of the deterministic limit cycle and is completely non-Gaussian; when s/ε is large, $p_s(x)$ will also be non-Gaussian, as can be seen from the joint p.d.f. $p_s(x, \dot{x})$ shown in equation (10). This explains why the Gaussian closure technique yields bad results for the problem. This also suggests that the Gaussian closure technique may also be invalid for other self-excited systems.

7. CONCLUSIONS

Based on the comparison and discussion in the last section, the following conclusions can be drawn.

1. The stochastic averaging method yields good approximate statistics of the stationary response of the Van der Pol oscillator to white noise excitation when the non-linearity of the system is small (for example, $\varepsilon < 0{\cdot}2$). The errors of the method increase with increasing nonlinearity.
2. Caughey's and Piszczek's techniques yield incorrect solutions for the problem because of the inadequate solution forms and unreasonable analyzing procedures.
3. The Gaussian closure technique is inapplicable to the problem because of the completely non-Gaussian property of the response.

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