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ON THE OCCURRENCE OF CHAOS IN VAN DER POL–DUFFING'S OSCILLATOR

1. INTRODUCTION

Some simple deterministic non-linear vibratory systems behave strangely: i.e., their response to harmonic excitation for certain parameter sets of the system is chaotic. The term "chaos" is generally used to distinguish such behaviour from a true random process, and these chaotic motions have been called strange attractors. The Duffing equation and the Duffing–Van der Pol equation belong to the simplest non-linear differential equations for which strange attractors have been detected. The strange attractor solutions for these equations were first studied by Ueda [1] and Ueda and Akamatsu [2], who employed experimental studies using analog and digital computers. These works have explained the difference between the almost periodic oscillations and the chaotically transitional processes. Using Poincaré maps and the Fast Fourier Transform algorithm, Ueda has investigated the effect of the damping and forcing amplitude on the transition into and out of chaos as well as the effects of these parameters on the spectral characteristics of the chaotic vibration.

Other examples of chaotic motions in simple physical systems have been presented by Holmes and Moon [3].

Chaotic motions are often related to the classic bifurcation theory in dynamical systems. In some systems a succession of bifurcations to higher and higher subharmonics, period doublying bifurcation, can lead to the occurrence of chaotic motion. Ruelle and Takens [4] were the first to suggest that strange attractors could arise after a finite sequence of bifurcations and might provide models for turbulent motions.

In what follows the criterion for the occurrence of chaos in the system governed by the Van der Pol–Duffing's equation, which is not related to the bifurcation theory, is presented. This work is a continuation of reference [5], where a similar criterion for the occurrence of chaos in a system governed by Duffing's equation was presented.

2. ANALYTICAL CRITERION OF CHAOS

Consider the particular Van der Pol–Duffing's equation examined by Dmitriev *et al.* [6] in the form of an equivalent two-equation system:

$$\mathrm{d}x/\mathrm{d}t = y - \varepsilon\left(\frac{x^3}{3} - x\right),$$

$$\mathrm{d}y/\mathrm{d}t = -x - \xi x^3 + B\cos\omega t. \tag{1}$$

For $\{\omega, \varepsilon, \xi, B\} = \{2{\cdot}46; 0{\cdot}33; 0{\cdot}7; 5{\cdot}0\}$ the authors presented a chaotic motion (see Figure 1).

Equations (1) can be presented in the form of one equation,

$$\mathrm{d}^2x/\mathrm{d}\tau^2 + x = \mu[n\varepsilon'(1-x^2)\,\mathrm{d}x/\mathrm{d}\tau + a'x - n^2\xi' x^3] + n^2\lambda\cos n\tau, \tag{2}$$

where

$$\omega t = n\tau, \qquad 1/\omega^2 = (1/n^2) - (a/n^2), \qquad a = \mu a', \qquad \xi/\omega^2 = \mu\xi' = \xi_1,$$

$$\lambda = B/\omega^2, \qquad \mu\varepsilon' = \varepsilon/\omega = \varepsilon_1. \tag{3}$$

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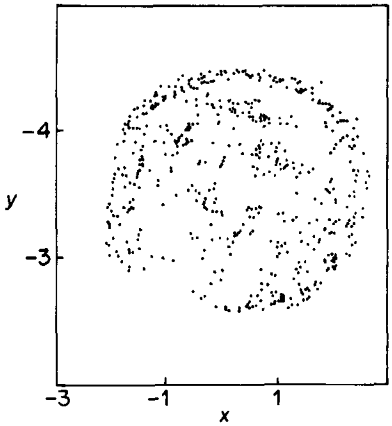

Figure 1. Strange attractor in the system governed by Van der Pol–Duffing equation (from Dmitriev *et al.* [6]).

The second and third of the equalities (3) are valid for small differences between the frequency ω and the natural number n, while μ is the perturbation parameter.

The analysis of the resonance of the nth order will be reduced in this case to determining the periodic solutions of equation (2), of period 2π. These solutions are sought in the form

$$x(t)=x_0(t)+\mu x_1(t)+\mu^2 x_2(t)+\cdots, \tag{4}$$

where

$$x_0(t)=[n^2\lambda/(1-n^2)]\cos n\tau+M_0\cos\tau+N_0\sin\tau \tag{5}$$

and M_0, N_0 are constants to be determined from the periodic condition of the function $x_1(t)$.

In view of the fact that chaos has been obtained for the frequency $\omega=2{\cdot}46$, it is of interest to perform the analysis for $n=2$ and $n=3$. In the first case ($n=2$), the condition of periodicity $x_1(t)$ gives:

$$\begin{gathered}\varepsilon_1 M_0(2+\tfrac{8}{9}\lambda^2-\tfrac{1}{2}A^2)+N_0(a+\tfrac{16}{3}\xi_1\lambda^2-3\xi_1 A^2)=0,\\ M_0(a+\tfrac{16}{3}\xi_1\lambda^2-3\xi_1 A^2)+\varepsilon_1 N_0(2+\tfrac{8}{9}\lambda^2-\tfrac{1}{2}A^2)=0,\end{gathered} \tag{6}$$

where

$$A^2=M_0^2+N_0^2 \tag{7}$$

and A is the vibration amplitude. Equation (6) gives

$$\begin{gathered}A^4(9\xi_1^2+\tfrac{1}{4}\varepsilon_1^2)-A^2[6\xi_1(a+\tfrac{16}{3}\xi_1\lambda^2)+\varepsilon_1^2(2+\tfrac{8}{9}\lambda^2)]\\ +(2+\tfrac{8}{9}\lambda^2)^2\varepsilon_1^2+(a+\tfrac{16}{3}\xi_1\lambda^2)^2=0.\end{gathered} \tag{8}$$

In the second case ($n=3$), the condition of periodicity $x_1(t)$ gives

$$\begin{gathered}M_0(a-\tfrac{243}{128}\xi_1'\lambda^2-\tfrac{3}{4}\xi_1'(M_0^2+N_0^2)]+\tfrac{27}{32}\xi_1'\lambda(M_0^2-N_0^2)\\ +N_0\varepsilon_1[3-\tfrac{3}{4}(M_0^2+N_0^2)-\tfrac{243}{128}\lambda^2]-\tfrac{27}{16}\lambda\varepsilon_1 M_0 N_0=0,\\ -M_0\varepsilon_1[3-\tfrac{3}{4}(M_0^2+N_0^2)-\tfrac{243}{128}\lambda^2]-\tfrac{27}{32}\varepsilon_1\lambda(M_0^2-N_0^2)\\ +N_0[a-\tfrac{243}{128}\xi_1'\lambda^2-\tfrac{3}{4}\xi_1'(M_0^2+N_0^2)]+\tfrac{27}{16}\xi_1'\lambda M_0 N_0=0,\end{gathered} \tag{9}$$

where $\xi_1'=9\xi_1$.

In order to avoid real solutions for the amplitude A, the following inequality is obtained from equation (8):

$$[6\xi_1(a+\tfrac{16}{3}\xi_1\lambda^2)+\varepsilon_1^2(2+\tfrac{8}{9}\lambda^2)]^2 \\ -4(9\xi_1^2+\tfrac{1}{4}\varepsilon_1^2)[(2+\tfrac{8}{9}\lambda^2)^2\varepsilon_1^2+(a+\tfrac{16}{3}\xi_1^2\lambda^2)^2]<0. \qquad (10)$$

The parameters for which Dimitriev *et al.* obtained chaos satisfy condition (10).

Next one can proceed to the analysis of the equation system (9). Let

$$M_0=A\cos\varphi, \qquad N_0=A\sin\varphi,$$
$$\Phi=a-\tfrac{243}{128}\xi_1'\lambda^2-\tfrac{3}{4}\xi_1'A^2, \qquad \Theta=(3-\tfrac{243}{128}\lambda^2-\tfrac{3}{4}A^2)\varepsilon_1. \qquad (11)$$

With expressions (11) taken into account in equations (9), one obtains

$$\phi\cos\varphi+\tfrac{27}{32}\xi_1\lambda A\cos2\varphi+\Theta\sin\varphi-\tfrac{27}{32}\lambda\varepsilon_1 A\sin2\varphi=0,$$
$$\Theta\cos\varphi+\tfrac{27}{32}\varepsilon_1\lambda A\cos2\varphi-\phi\sin\varphi-\tfrac{27}{32}\xi_1\lambda A\sin2\varphi=0. \qquad (12)$$

Substituting the values $\varphi\in[0,2\pi]$ into the equation system (12) gives quadratic equations for the A's. The dependencies of the discriminants Δ_1 and Δ_2 of these equations on the angle φ are shown in Figure 2. It is interesting to note that regular trigonometric curves

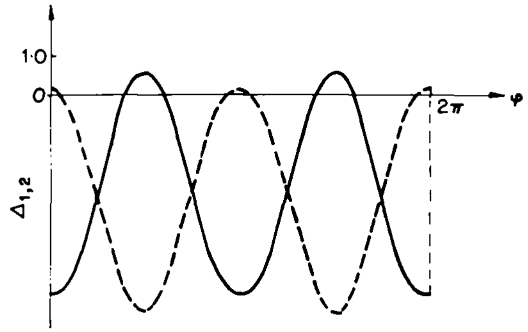

Figure 2. Graphic representation of the solution of the equation system (12). ——, Δ_1; - - -, Δ_2.

have been obtained. The product of the sets of the values of φ for which $\Delta_1>0$ and $\Delta_2>0$ is empty; thence the equation system (12) for the parameters for which chaos occurs has no real solutions for the amplitude A.

Therefore, in equation (1), chaos occurs for such sets of parameters for which no real roots of A of equation (8) and of the equation system (12) exist.

Institute of Applied Mechanics,
Technical University of Łódź,
ul. Stefanowskiego 1/15, 90-924 *Łódź,*
Poland

J. AWREJCEWICZ

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