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LETTERS TO THE EDITOR

THE STOCHASTIC SENSITIVITY OF THE VAN DER POL EQUATION

The van der Pol equation has been considered in references [1, 2], for example. In a general form, the equation can be written as

$$\ddot{x} - \varepsilon(1-\gamma x^2)\dot{x} + c_1 x + c_3 x^3 = B\cos\xi t, \tag{1}$$

where $\varepsilon \neq 0$, γ, c_1, c_3, B, and ξ are constant with respect to the time t, and $x(0) = x_0$ and $\dot{x}(0) = x_1$ are initial conditions. ($\dot{}$) denotes differentiation with respect to time. The excitation is assumed to be a random process of the form

$$B\cos\xi t = B(\omega)\cos[\xi(\omega)t], \tag{2}$$

where $B(\omega)$ and $\xi(\omega)$ are random variables with uniform distribution in the intervals $[B_1, B_2]$ and $[N_1, N_2]$ respectively. The stochastic sensitivity, in respect to ε, of the van der Pol equation (1) under the excitation given by equation (2) can be investigated as follows.

Taking the derivative with respect to ε (for $\varepsilon = \varepsilon_0$) of equation (1) gives

$$\ddot{\sigma} - \varepsilon(1-\gamma x^2)\dot{\sigma} + (2\varepsilon\gamma x\dot{x} + c_1 + 3c_3 x^2)\sigma = (1-\gamma x^2)\dot{x}, \tag{3}$$

where $\sigma = \partial x/\partial\varepsilon|_{\varepsilon=\varepsilon_0}$ is the stochastic sensitivity function (SSF) of first order [3, 4]. One can obtain also the second order SSF. Differentiating the system (3) with respect to ε (for $\varepsilon = \varepsilon_0$), and using the second order SSF $\nu = \partial\sigma/\partial\varepsilon = \partial^2 x/\partial\varepsilon^2|_{\varepsilon=\varepsilon_0}$ gives

$$\ddot{\nu} - \varepsilon(1-\gamma x^2)\dot{\nu} + (2\varepsilon\gamma x\dot{x} + c_1 + 3c_3 x^2)\nu = (1-\gamma x^2)\dot{\sigma} - 2\gamma x\dot{x}\sigma. \tag{4}$$

This procedure can be carried on for the third and higher order SSFs. Because the initial conditions for system (1) are independent of ε, the initial conditions for the stochastic sensitivity equations of arbitrary order (greater than or equal to unity) are zeros.

For numerical analysis it has been assumed that $\varepsilon_0 = 0{\cdot}05$, $\varepsilon = 0{\cdot}06$, $\gamma = 10$, $c_1 = c_3 = 1$, $B_1 = 0$, $B_2 = 0{\cdot}05$, $N_1 = 0$, $N_2 = 2$, $x_0 = 1$, and $x_1 = 0$. The simulation of a uniform distribution was made by using a generator given in reference [5] and the simulation of solutions by using Adams' routine [6].

Figure 1 presents graphs of the mean values $E\sigma(t, \varepsilon_0, \omega)$ and $E\nu(t, \varepsilon_0\,\omega)$. The values of $E\nu$ are greater than those of $E\sigma$. Table 1 shows the ranges of the fluctuations of the mean values. The ratio between the respective maxima of the values of $E\sigma$ and $E\nu$ ($\max|E\sigma|/max|E\nu|$) is 5·32%.

Figure 2 presents graphs of the variances $\mathrm{Var}\,\sigma(t, \varepsilon_0, \omega)$ and $\mathrm{Var}\,\nu(t, \varepsilon_0, \omega)$. The values of $\mathrm{Var}\,\nu$ are greater than those of $\mathrm{var}\,\sigma$. Table 2 shows the maximum values of the variances. The ratio between the respective maxima of $\mathrm{Var}\,\sigma$ and $\mathrm{Var}\,\nu$ ($\max \mathrm{Var}\,\sigma/\max$-$\mathrm{Var}\,\nu$) is 0·3%.

If the solution and the SSF are known for $\varepsilon = \varepsilon_0$, then the solution for arbitrary ε can be presented in the Taylor series form to a

$$x(t, \varepsilon, \omega) = x(t, \varepsilon_0, \omega) + \sigma(t, \varepsilon_0, \omega)\Delta\varepsilon + (1/2!)\nu(t, \varepsilon_0, \omega)(\Delta\varepsilon)^2 + \cdots, \tag{5}$$

where $\Delta\varepsilon = \varepsilon - \varepsilon_0$.

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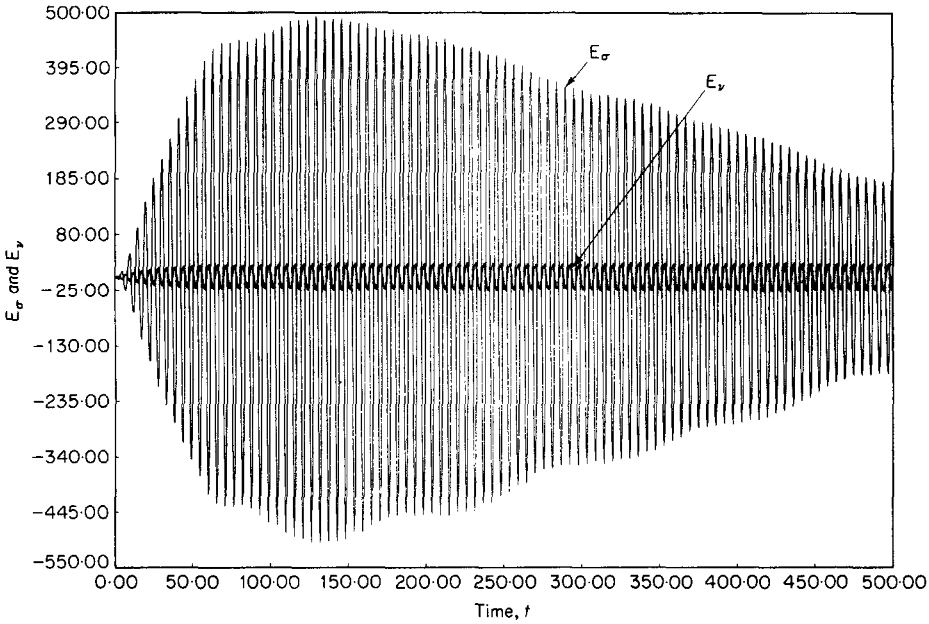

Figure 1. Mean values Eσ and Eν.

TABLE 1

Ranges of fluctuations of mean values

Mean value	Range
Eσ	(−26·858, 26·516)
Eν	(−504·60, 493·07)

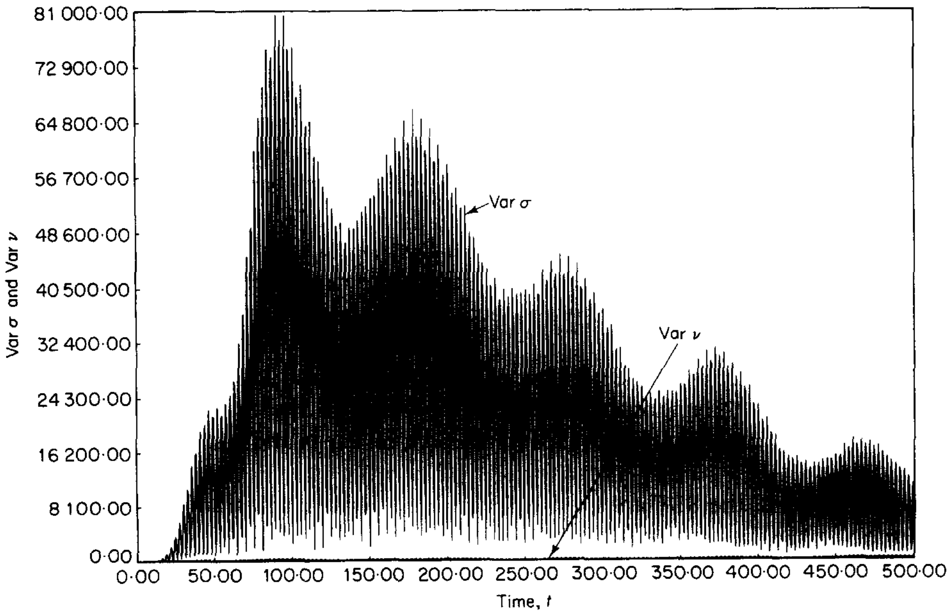

Figure 2. Variances Var σ and Var ω.

TABLE 2
Maximum values of variances

Variances	Maximum values
Var σ	244·86
Var ν	80410

One can investigate the difference between the solution (for arbitrary ε) and the finite sum, to a certain order, of the terms on the right hand side expression of formula (5):

$$R_0(t, \varepsilon_0, \varepsilon, \omega) = x(t, \varepsilon, \omega) - x(t, \varepsilon_0, \omega)$$

$$R_1(t, \varepsilon_0, \varepsilon, \omega) = x(t, \varepsilon, \omega) - [x(t, \varepsilon_0, \omega) + \sigma(t, \varepsilon_0, \omega)\Delta\varepsilon],$$

$$R_2(t, \varepsilon_0, \varepsilon, \omega) = x(t, \varepsilon, \omega) - [x(t, \varepsilon_0, \omega) + \sigma(t, \varepsilon_0, \omega)\Delta\varepsilon + (1/2!)\nu(t, \varepsilon_0, \omega)(\Delta\varepsilon)^2],$$

$$\vdots \qquad\qquad \vdots$$

$$R_p(t, \varepsilon_0, \varepsilon, \omega) = x(t, \varepsilon, \omega) - [x(t, \varepsilon_0, \omega) + \sigma(t, \varepsilon_0, \omega)\Delta\varepsilon + \cdots + (1/p!)\eta_p(t, \varepsilon_0, \omega)(\Delta\varepsilon)^p], \tag{6}$$

where η_p is the SSF of order p.

The sequence R_p is convergent to zero, since the series expression (5) is convergent. So, without solving the system (1) for arbitrary ε, one can find good approximations (the expressions in the brackets in equation (6)) when the solution for $\varepsilon = \varepsilon_0$ and also the SSF are known. If one wants to obtain many solutions or moments for system (1), for arbitrary ε, then by using this method the computer time may be considerably shortened. The differences given by formula (6) are stochastic processes. Their realizations can be calculated from equations (1), (2), (3), etc., by using simulation methods for each random parameter ω. From these realizations, one can find the moments such as the mean value, variance etc. The rate of convergence to zero of the sequence R_p can be investigated by

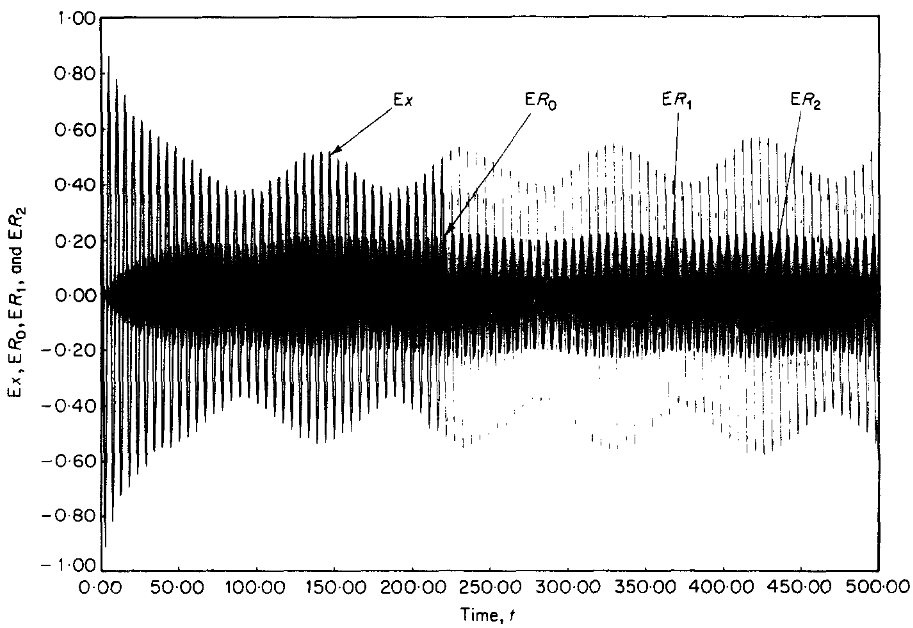

Figure 3. Mean values $Ex(t, \varepsilon_0, \omega)$, $ER_0(t, \omega)$, $ER_1(t, \omega)$ and $ER_2(t, w)$.

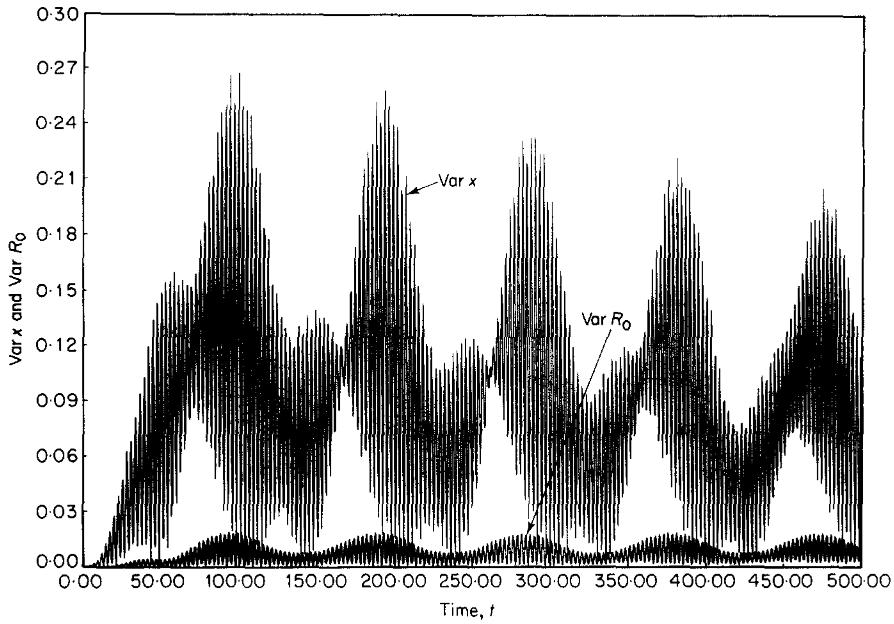

Figure 4. Variances Var $x(t, \varepsilon_0, \omega)$ and Var $R_0(t, \omega)$.

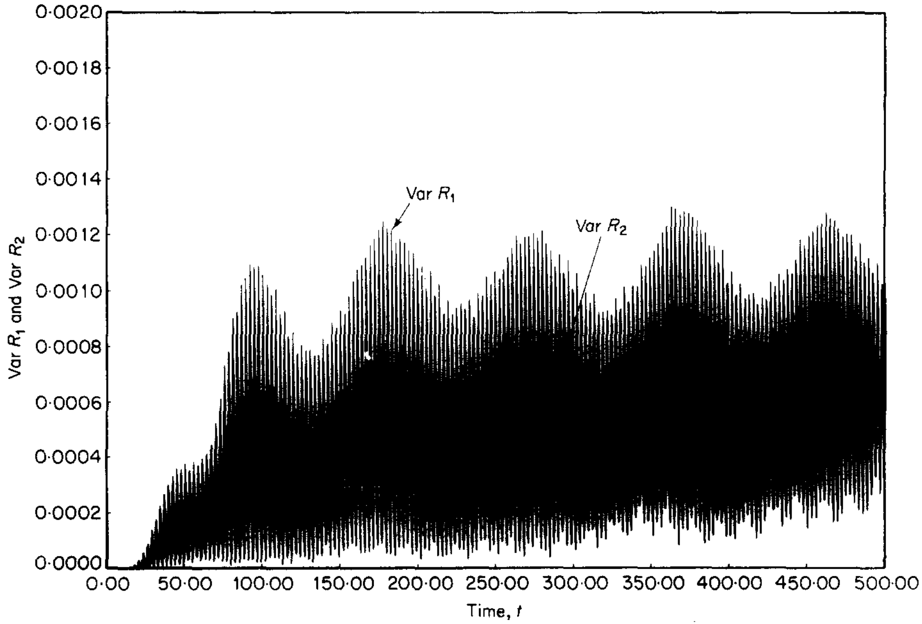

Figure 5. Variances Var $R_1(t, \omega)$ and Var $R_2(t, \omega)$.

determining the rates of convergence of the respective moments (the moments are deterministic functions). When realizations of the stochastic process have been made, then the moments can be calculated, and the computer time required for this calculation is short.

For van der Pol equations calculations of the sensitivity of the mean value and variance to a change in the parameter ε and also the convergence of the mean values and variances of the differences R_0, R_1 and R_2 have been carried out.

Figure 3 presents graphs of the mean values $Ex(t, \varepsilon_0, \omega)$, $ER_0(t, \omega)$, $ER_1(t, \omega)$ and $ER_2(t, \omega)$. It can be noted that $ER_p(t, \omega)$ is smaller for greater $p = 0, 1, 2$ nd also smaller than $Ex(t, \varepsilon_0, \omega)$.

Figure 4 presents graphs of the variances $\mathrm{Var}\, x(t, \varepsilon_0, \omega)$ and $\mathrm{Var}\, R_0(t, \omega)$. In Figure 5 graphs of $\mathrm{Var}\, R_1(t, \omega)$ and $\mathrm{Var}\, R_2(t, \omega)$ are shown. It can be seen that $\mathrm{Var}\, R_p(t, \omega)$ is smaller for greater p and also smaller than $\mathrm{Var}\, x(t, \varepsilon_0, \omega)$.

Table 3 shows the ranges of the fluctuations of the mean values. The ratio between the respective maxima of the absolute mean values is shown in Table 4. The ratio between $\max|ER_2|$ and $\max|E_x|$ is 6·91%. The results show a good convergence of the mean values of the approximations (the expressions in brackets in equations (6)) to zero when the SSF, of second order is used, for the numerical data assumed in these calculations. Table 5 shows the maximum values of the variances. The ratios between the respective maxima of the variances is shown in Table 6. The ratio $\max \mathrm{Var}\, R_2/\max \mathrm{Var}\, x$ is 0·41%.

TABLE 3

Ranges of fluctuations of mean values

Mean value	Range
Ex	(−0·913 28, 0·860 94)
ER_0	(−0·228 05, 0·226 80)
ER_1	(−0·068 072 3, 0·068 478 3)
ER_2	(−0·063 136 9, 0·061 418 4)

TABLE 4

Ratios of maxima of absolute mean values

$\max\|ER_0\|/\max\|Ex\|$	24·97%
$\max\|ER_1\|/\max\|ER_0\|$	30·03%
$\max\|ER_2\|/\max\|ER_1\|$	92·20%

TABLE 5

Maximum values of variances

$\mathrm{Var}\, x(t, \varepsilon_0, \omega)$	0·266 82
$\mathrm{Var}\, R_0(t, \omega)$	0·018 577 2
$\mathrm{Var}\, R_1(t, \omega)$	0·001 300 72
$\mathrm{Var}\, R_2(t, \omega)$	0·001 105 27

TABLE 6

Ratios between respective maxima of the variances

$\max \mathrm{Var}\, R_0/\max \mathrm{Var}\, x$	6·96%
$\max \mathrm{Var}\, R_1/\max \mathrm{Var}\, R_0$	7·00%
$\max \mathrm{Var}\, R_2/\max \mathrm{Var}\, R_1$	84·97%

The results show good convergence to zero of the variances of the successive approximations given in the brackets in equation (6) when the SSF of second order is used, again for the data assumed in these calculations. Generally, by using the first and second order SSFs good approximations of the solution moments of the van der Pol equation have been obtained for changes in ε up to 20%.

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