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On the existence and uniqueness of limit cycles for generalized Liénard systems

Dongmei Xiao [a,*,1], Zhifen Zhang [b,2]

[a] Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China
[b] Department of Mathematics, Peking University, Beijing 100871, China

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Abstract

In this paper, we consider a generalized Liénard system

$$\frac{dx}{dt} = \phi(y) - F(x),$$
$$\frac{dy}{dt} = -g(x), \tag{0.1}$$

where F is continuous and differentiable on an open interval (b_1, a_1) with $-\infty \leqslant b_1 < 0 < a_1 \leqslant +\infty$. Assume that there exist a and b with $b_1 < b < 0 < a < a_1$ such that $xF(x) < 0$ as $b < x < a$, and $xF(x) > 0$ as $a < x < a_1$ or $b_1 < x < b$. A new uniqueness theorem of limit cycles for the Liénard system (0.1) is obtained. An example is given to show the application of the theorem.

Keywords: Liénard system; Limit cycles; Uniqueness

1. Introduction

The existence and number of limit cycles for planar systems are related to Hilbert 16th problem and self-sustaining oscillatory problems in mathematical models. It is a challenging problem to find out conditions so as to guarantee the uniqueness of limit cycles for planar systems. As one knows, there are some systems for which the uniqueness of limit cycles has been extensively studied in the last century, such as Liénard system with the following form:

$$\frac{dx}{dt} = y - F(x),$$

* Corresponding author.
E-mail addresses: xiaodm@sjtu.edu.cn (D. Xiao), zhangzf@pku.edu.cn (Z. Zhang).
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$$\frac{dy}{dt} = -x, \tag{1.1}$$

as well as the more general form

$$\begin{aligned} \frac{dx}{dt} &= \phi(y) - F(x), \\ \frac{dy}{dt} &= -g(x), \end{aligned} \tag{1.2}$$

where the function $g(x)$ is continuous and $F(x)$ is continuously differentiable on an open interval (b_1, a_1), $-\infty \leqslant b_1 < 0 < a_1 \leqslant +\infty$. The function $\phi(y)$ is continuously differentiable on $(-\infty, +\infty)$. Moreover, it is always assumed that

(A1) $xg(x) > 0$ for $x \neq 0$. Let $G(x) = \int_0^x g(s)\,ds$;
(A2) $\phi(0) = 0$, $\phi'(y) > 0$ for $-\infty < y < +\infty$. Let $\Phi(y) = \int_0^y \phi(s)\,ds$.

There have been various techniques for establishing the uniqueness of limit cycles for (1.1) and (1.2) (see [1,2, 8–17] and references therein). In these sufficient conditions, the following conditions are usually assumed.

(H1) There exist a and b, $b_1 < b < 0 < a < a_1$ such that $F(b) = F(0) = F(a) = 0$.

(H2) $$\begin{cases} xF(x) > 0 & \text{if } a < x < a_1 \text{ and } b_1 < x < b, \\ xF(x) < 0 & \text{if } b < x < a. \end{cases}$$

Sansone and Conti in [10] obtained that if the hypotheses (H1) and (H2) hold, and if the derivative of $F(x)$ with respect to x satisfies $F'(x) > 0$ as $a < x < a_1$ and $b_1 < x < b$, then for system (1.1) the number of limit cycles, which enclose two points $(a, 0)$ and $(b, 0)$ inside is unique. Recently, Carletti and Villari in [1] further gave the sufficient conditions for uniqueness of limit cycle of (1.1). For system (1.2), in [11] we gave the conditions to guarantee that a limit cycle encloses two points $(a, 0)$ and $(b, 0)$ inside, which generalized the conclusion of Sansone and Conti in [10]. It is an interesting problem what conditions should add to guarantee the uniqueness of limit cycles for (1.2) if the limit cycle encloses only one point of $(a, 0)$ and $(b, 0)$. Zeng et al. in [12] and [13] studied this problem and used the Filippov transformation $z = G(x)$ to reduce system (1.2) to two equations

$$\frac{dz}{dy} = F_i(z) - \phi(y), \tag{E$_i$}$$

respectively, where $F_i(z) = F(x_i(z)) \in C^1[0, z_i)$, $x_i(z)$ is the inverse function of $z = G(x)$ for $(-1)^{i+1}x \geqslant 0$, $i = 1, 2$, $z_1 = G(a_1)$ and $z_2 = G(b_1)$. In [13] (cf. [13, Theorem 1]), it has been proved that system (1.2) has at most one limit cycle if besides (H1) and (H2), the following hypotheses hold

(H3) if $G(b) > G(a)$ $(G(b) < G(a))$, then $F_1(z)F_1'(z)$ $(F_2(z)F_2'(z)$, respectively) is nondecreasing (nonincreasing, respectively) as $z > G(a)$ $(z > G(b)$, respectively);
(H4) if $G(b) > G(a)$ $(G(b) < G(a))$, then $F_1'(z) \geqslant F_2'(u)$ for any pair (z, u) satisfying $G(a) < z < u$ $(G(b) < u < z$, respectively) and $F_1(z) = F_2(u)$ $(F_2(u) = F_1(z)$, respectively).

From the above sufficient conditions in [1,10,12] and [13], we can see that if limit cycles enclose both points $(a, 0)$ and $(b, 0)$, then the monotonicity of $F(x)$ on $a_1 > x > a$ and $b > x > b_1$ is sufficient to guarantee the uniqueness of the limit cycles. However, if limit cycles enclose only one of the two points $(a, 0)$ and $(b, 0)$, then to guarantee the uniqueness of the limit cycles, it needs more restrictions on the functions $F(x)$ and $G(x)$, such as assumptions (H3) and (H4).

In this paper, we discuss further the uniqueness of limit cycles of (1.2) when the limit cycles enclose only one of two points $(a, 0)$ and $(b, 0)$. We obtain a new sufficient criterion for the uniqueness of limit cycles of (1.2), which is different from those sufficient conditions appeared in [2,9–15] and [16]. For simplicity of statement, we make the following assumption on the derivative of $F(x)$.

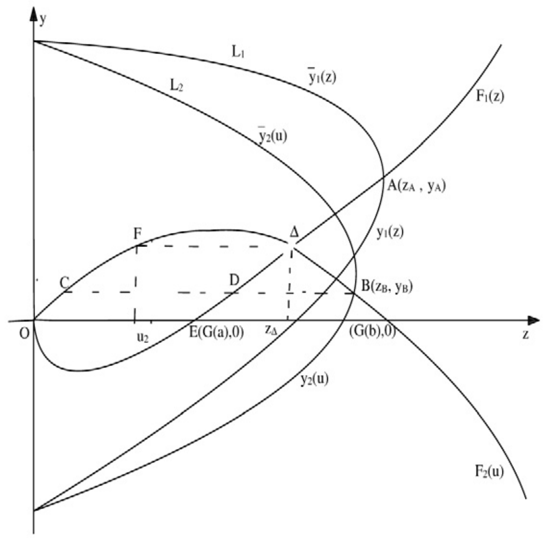

Fig. 1. When $G(b) > G(a)$, the vertical isocline and a closed orbit of system (E_i).

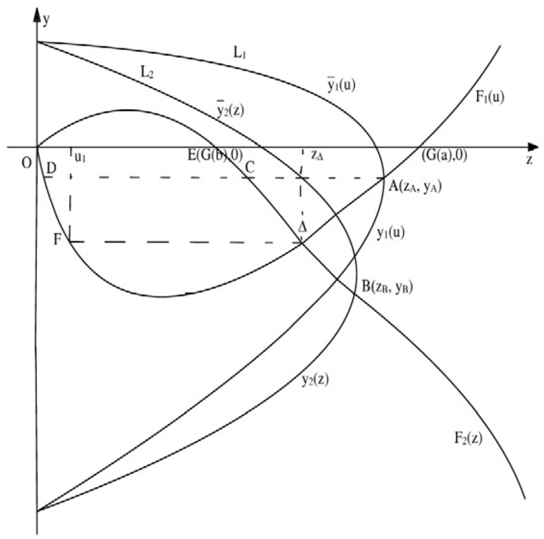

Fig. 2. When $G(b) < G(a)$, the vertical isocline and a closed orbit of system (E_i).

(H5) When $b_1 < x < a_1$, $f(x) \overset{\text{def}}{=} F'(x) = 0$ has only two roots x_1 and x_2, $x_1 > 0$ and $x_2 < 0$. Moreover, $z_\triangle$ is a unique positive root of $F_1(z) - F_2(z) = 0$, which implies that $G(x_i) \leqslant z_\triangle$.

Now we state a new uniqueness theorem as follows.

Theorem 1.1. *Suppose that system* (1.2) *with* (A1) *and* (A2) *satisfies the hypotheses* (H1), (H2), (H5)*, and its equivalent Eqs.* (E_i) *satisfy one of the following conditions.*

(i) *If $G(b) > G(a)$ (which implies $G(a) < z_\triangle < G(b)$), then $F_2'(u) > F_1'(z)$ for any pair (z, u) satisfying $G(a) < z < z_\triangle$, $0 < u < z$ and $F_1(z) = F_2(u)$ (see Fig.* 1*).*
(ii) *If $G(a) > G(b)$ (which implies $G(b) < z_\triangle < G(a)$), then $F_2'(z) > F_1'(u)$ for any pair (z, u) satisfying $G(b) < z < z_\triangle$, $0 < u < z$ and $F_1(u) = F_2(z)$ (see Fig.* 2*).*

Then

(R1) *system* (1.2) *has at most one limit cycle, and it is stable if it exists*;
(R2) *system* (1.2) *has a unique stable limit cycle if* $G(b_1) = G(a_1) = +\infty$ *and* $\Phi(\pm\infty) = \pm\infty$.

2. Main results

In this section, we first give the proof of Theorem 1.1, then compare it with Theorem 1 in [13] and provide an alternative statement of Theorem 1.1.

We now prove the conclusion (R1) of Theorem 1.1. Our main task is to prove the uniqueness of limit cycles of (1.2) if it exists, which can be derived as follows: if any one closed orbit of system (1.2) is stable, then system (1.2) has at most one limit cycle since adjacent closed orbits can not have the same stability. Hence, we will only discuss the stability of closed orbits of (1.2).

Assume that L is a closed orbit of (1.2), then $L = L_1 \cup L_2$, L_i is an orbit of Eqs. (E_i), $i = 1, 2$. And L_1 (L_2) intersects the vertical isocline $\phi(y) = F_1(z)$ ($\phi(y) = F_2(z)$, respectively) at $A(z_A, y_A)$ ($B(z_B, y_B)$, respectively). It is easy to see that $y_A > y_B$. Let $\bar{y}_i(z)$ and $y_i(z)$ represent a part of L_i, which is located above and below of the vertical isocline $\phi(y) = F_i(z)$, respectively (see Fig. 1 or Fig. 2). To determine the stability of L, we have to calculate the integral of divergent of (1.2) along L,

$$\oint_L \operatorname{div}(1.2)\, dt = \oint_L -F'(x)\, dt.$$

Utilizing the notations in [13], we denote

$$V\big(F_i(z), y_i(z), \bar{y}_i(z)\big) \overset{\text{def}}{=} \frac{F_i'(z)}{F_i(z) - \phi(y_i(z))} + \frac{F_i'(z)}{\phi(\bar{y}_i(z)) - F_i(z)}.$$

Hence, we have

$$\begin{aligned}\oint_L -F'(x)\, dt &= -\Big(\int_{L_1} F_1'(z)\, dy - \int_{L_2} F_2'(z)\, dy\Big) \\ &= -\bigg(\int_0^{z_A} V\big(F_1(z), y_1(z), \bar{y}_1(z)\big)\, dz - \int_0^{z_B} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\, dz\bigg),\end{aligned} \tag{2.1}$$

where $dt > 0$ along the integration path and $dy > 0$ along the curves L_1 and L_2.

To estimate the sign of the integral (2.1), we first introduce two lemmas. They can be found in [12] and [13]. For convenience to read, we give the proof of Lemma 2.2

Lemma 2.1. *Let* $0 \leqslant c < d \leqslant z_A$ (*or* z_B). *If* $F_i(d) - F_i(z) \geqslant 0$ (*or* $\leqslant 0$) *for* $c < z < d$, *then* $\int_c^d V(F_i(z), y_i(z), \bar{y}_i(z))\, dz \geqslant 0$ (*or* $\leqslant 0$, *respectively*).

Lemma 2.2. *Consider systems* (E_1) *and* (E_2), *if there exist* $0 \leqslant p_1 < u < z_1$, $0 \leqslant p_2 < z < z_2$ *such that*

(1) $F_1(p_1) = F_2(p_2)$, $F_1(z_1) = F_2(z_2)$;
(2) $F_1'(u) < 0$ *if* $p_1 < u < z_1$, *and* $F_2'(z) < 0$ *if* $p_2 < z < z_2$;
(3) $F_1'(u) < F_2'(z)$ *if* $F_1(u) = F_2(z)$.

Let $\bar{y}_i(\cdot)$ *and* $y_i(\cdot)$ *be solutions of* (E_i), $i = 1, 2$, *which satisfy the following inequalities*:

$$\begin{aligned}&\phi\big(y_1(u)\big) < F_1(u) < \phi\big(\bar{y}_1(u)\big), \quad \text{as } p_1 < u < z_1;\\ &\phi\big(y_2(z)\big) < F_2(z) < \phi\big(\bar{y}_2(z)\big), \quad \text{as } p_2 < z < z_2;\\ &\bar{y}_1(p_1) \geqslant \bar{y}_2(p_2) \quad \text{and} \quad y_1(p_1) \leqslant y_2(p_2).\end{aligned}$$

Then there exist two functions $\bar{W}_2(z)$ and $W_2(z)$ in the interval (p_2, z_2) such that

$$\bar{W}_2(z) = \bar{y}_1\big(F_1^{-1}\big(F_2(z)\big)\big), \qquad W_2(z) = y_1\big(F_1^{-1}\big(F_2(z)\big)\big),$$

and

$$\int_{p_1}^{z_1} V\big(F_1(u), y_1(u), \bar{y}_1(u)\big)\, du > \int_{p_2}^{z_2} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\, dz.$$

Proof. When $p_1 < u < z_1$ and $p_2 < z < z_2$, we let $F_1(u) = F_2(z)$, which define a transformation from variables u into z. Consider Eq. (E_1) with the following form:

$$\frac{dy}{du} = \frac{1}{F_1(u) - \phi(y)}. \tag{2.2}$$

Then (2.2) can be transformed to

$$\frac{dy}{dz} = \frac{1}{F_2(z) - \phi(y)} \frac{F_2'(z)}{F_1'(F_1^{-1}(F_2(z)))}, \tag{E_3}$$

and solutions $\bar{y}_1(u)$ and $y_1(u)$ of (2.2) are transformed to

$$\bar{W}_2(z) = \bar{y}_1\big(F_1^{-1}\big(F_2(z)\big)\big) \quad \text{and} \quad W_2(z) = y_1\big(F_1^{-1}\big(F_2(z)\big)\big),$$

respectively, where $\bar{W}_2(z)$ and $W_2(z)$ are solutions of Eq. (E_3).

It is clear that $\bar{W}_2(p_2) = \bar{y}_1(p_1)$ and $W_2(p_2) = y_1(p_1)$. From $\bar{y}_1(p_1) \geqslant \bar{y}_2(p_2)$ and $y_1(p_1) \leqslant y_2(p_2)$, we have

$$W_2(p_2) \leqslant y_2(p_2), \qquad \bar{W}_2(p_2) \geqslant \bar{y}_2(p_2). \tag{2.3}$$

Next we prove the inequality of integral. Since $\frac{F_2'(z)}{F_1'(F_1^{-1}(F_2(z)))} < 1$,

$$\frac{1}{F_2(z) - \phi(\bar{y}_2(z))} < \frac{1}{F_2(z) - \phi(\bar{y}_2(z))} \frac{F_2'(z)}{F_1'(F_1^{-1}(F_2(z)))},$$
$$\frac{1}{F_2(z) - \phi(y_2(z))} > \frac{1}{F_2(z) - \phi(y_2(z))} \frac{F_2'(z)}{F_1'(F_1^{-1}(F_2(z)))}.$$

Comparing Eqs. (E_2) and (E_3), and note the initial conditions (2.3), we obtain that

$$\bar{W}_2(z) \geqslant \bar{y}_2(z) \quad \text{and} \quad W_2(z) \leqslant y_2(z)$$

as $z \geqslant p_2$ by differential inequality theory.

Note that $\phi'(y) > 0$. Hence,

$$\phi\big(\bar{W}_2(z)\big) \geqslant \phi\big(\bar{y}_2(z)\big), \qquad \phi\big(W_2(z)\big) \leqslant \phi\big(y_2(z)\big),$$

and

$$\frac{F_2'(z)}{F_2(z) - \phi(W_2(z))} + \frac{F_2'(z)}{\phi(\bar{W}_2(z)) - F_2(z)} > \frac{F_2'(z)}{F_2(z) - \phi(y_2(z))} + \frac{F_2'(z)}{\phi(\bar{y}_2(z)) - F_2(z)}.$$

Therefore, we have

$$\int_{p_1}^{z_1} V\big(F_1(u), y_1(u), \bar{y}_1(u)\big)\, du - \int_{p_2}^{z_2} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\, dz$$
$$= \int_{p_2}^{z_2} V\big(F_2(z), W_2(z), \bar{W}_2(z)\big)\, dz - \int_{p_2}^{z_2} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\, dz > 0.$$

The proof is completed. □

Now we are in a position to prove the uniqueness of limit cycles if it exists. We first prove the case (ii) and divide two cases of y_A to discuss.

(I) If $y_A \geqslant 0$, then $F_1(z_A) - F_1(z) \geqslant 0$ as $0 < z < z_A$ and $F_2(z_B) - F_2(z) \leqslant 0$ as $0 < z < z_B$. From Lemma 2.1, we have

$$\int_0^{z_A} V\big(F_1(z), y_1(z), \bar{y}_1(z)\big)\,dz - \int_0^{z_B} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\,dz > 0.$$

Hence, the integral of divergent of (1.2) along L is negative, which leads that the closed orbit L is stable. Therefore, system (1.2) has at most one limit cycle.

(II) If $y_A < 0$, then we can take points $D(z_D, y_D)$ and $C(z_C, y_C)$ in $\phi(y) = F_1(z)$ and $\phi(y) = F_2(z)$, respectively, such that $F_1(z_D) = F_2(z_C) = F_1(z_A)$. Note that $F_1(z_A) - F_1(z) \geqslant 0$ as $0 < z_D < z < z_A$, $F_2(G(b)) - F_2(z) \leqslant 0$ as $0 < z < G(b)$ and $F_2(z_B) - F_2(z) \leqslant 0$ as $0 < z_C < z < z_B$. From Lemma 2.1, we have

$$\int_{z_D}^{z_A} V\big(F_1(z), y_1(z), \bar{y}_1(z)\big)\,dz \geqslant 0,$$

$$\int_{z_C}^{z_B} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\,dz \leqslant 0,$$

$$\int_0^{G(b)} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\,dz \leqslant 0. \tag{2.4}$$

On the other hand, $F_1(0) = F_2(G(b)) = 0$ and $F_1(z_D) = F_2(z_C)$. From hypothesis (H5) and condition (ii) of Theorem 1.1, and utilizing Lemma 2.2 we obtain

$$\int_0^{z_D} V\big(F_1(z), y_1(z), \bar{y}_1(z)\big)\,dz - \int_{G(b)}^{z_C} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\,dz > 0. \tag{2.5}$$

Summarizing (2.4) and (2.5), we obtain that

$$\int_0^{z_A} V\big(F_1(z), y_1(z), \bar{y}_1(z)\big)\,dz - \int_0^{z_B} V\big(F_2(z), y_2(z), \bar{y}_2(z)\big)\,dz > 0.$$

Hence, the integral of divergent of (1.2) along L is negative, which implies that the closed orbit L is stable. Thus system (1.2) has at most one limit cycle. We complete the proof of conclusion (R1) in Theorem 1.1 for case (ii).

Let $y = -Y$. Then system (E_i) is transformed to

$$\frac{dz}{dY} = -F_i(z) + \phi(-Y), \tag{F_i}$$

and the case (i) in Theorem 1.1 is transformed to the case (ii) for system (F_i). The similar arguments can be applied to the case. We can obtain the uniqueness of limit cycles in Theorem 1.1 for case (i). Therefore, we finish the proof of conclusion (R1) of Theorem 1.1.

We next prove conclusion (R2) of Theorem 1.1. From conclusion (R1), we only need to prove the existence of limit cycles for system (1.2) if we further assume that $G(b_1) = G(a_1) = +\infty$ and $\Phi(\pm\infty) = \pm\infty$. The following lemma implies the existence of limit cycles for system (1.2).

Lemma 2.3. *Assume that system* (1.2) *with* (A1) *and* (A2) *satisfies the hypotheses* (H1), (H2) *and* (H5). *If* $G(b_1) = G(a_1) = +\infty$ *and* $\Phi(\pm\infty) = \pm\infty$, *then system* (1.2) *has at least a limit cycle.*

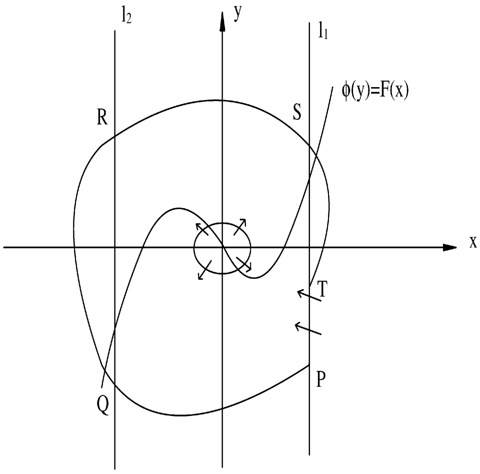

Fig. 3. The annular region of positive invariant set of system (1.2) and the vertical isocline $\phi(y) - F(x) = 0$.

Proof. Let

$$V(x, y) = G(x) + \Phi(y),$$

where $G(x) = \int_0^x g(s)\,ds$ and $\Phi(y) = \int_0^y \phi(s)\,ds$. Then

$$\left.\frac{dV(x, y)}{dt}\right|_{(1.2)} = -g(x)F(x).$$

Taking $0 < r_1 \ll \min\{a, -b\}$, we have

$$\left.\frac{dV(x, y)}{dt}\right|_{(1.2)} > 0, \quad \text{as } V(x, y) = r_1.$$

Thus, the graph of $V(x, y) = r_1$ becomes an intra-boundary of an annular region and the unique equilibrium $(0, 0)$ of system (1.2) is unstable.

On the other hand, since $\Phi(\pm\infty) = \pm\infty$, respectively, and $G(b_1) = G(a_1) = +\infty$, any trajectory of system (1.2) in the region $\{(x, y){:}\ V(x, y) > r_1\}$ spirals as $t \to +\infty$.

From (H2) and (H5), there exists ϵ_0, $0 < \epsilon_0 \ll 1$ such that $F(a + \epsilon_0) = k_1 > 0$, $F(b - \epsilon_0) = k_2 < 0$, and $F(x) \geqslant k_1$ as $a + \epsilon_0 \leqslant x < a_1$ and $F(x) \leqslant k_2$ as $b_1 < x \leqslant b - \epsilon_0$. Let

$$l_1{:}\quad x = a + \epsilon_0, \qquad l_2{:}\quad x = b - \epsilon_0.$$

Taking a point $P(a + \epsilon_0, y_0)$ on line l_1 with $y_0 < 0$, we consider that as time t increases the trajectory $\Gamma(P, t)$ of system (1.2) with the initial point P intersects l_2 at $Q(b - \epsilon_0, y_Q)$ and $R(b - \epsilon_0, y_R)$, respectively, and intersects l_1 again at $S(a + \epsilon_0, y_S)$ and $T(a + \epsilon_0, y_T)$, respectively (see Fig. 3). We claim that $y_0 < y_T < 0$ as $y_0 \to -\infty$. Let

$$V_1(x, y) = \int_0^y \big(\phi(s) - k_2\big)\,ds + \int_0^x g(s)\,ds.$$

We consider the increment of the function $V_1(x, y)$ along the segment of trajectory $\widehat{PQRST}$ of system (1.2)

$$\left.\frac{dV_1}{dy}\right|_{(1.2)} = F(x) - k_2, \tag{2.6}$$

$$\left.\frac{dV_1}{dx}\right|_{(1.2)} = \frac{-g(x)(F(x) - k_2)}{\phi(y) - F(x)}. \tag{2.7}$$

Note that $F(x) \geqslant k_1$ as $a + \epsilon_0 \leqslant x < a_1$ and $F(x) \leqslant k_2$ as $b_1 < x \leqslant b - \epsilon_0$. From (2.6), we have

$$V_1(S) - V_1(T) = \int_{y_T}^{y_S} \big(F(x) - k_2\big)\,dy > (k_1 - k_2)(y_S - y_T) > 0,$$

$$V_1(Q) - V_1(R) = \int_{y_R}^{y_Q} \big(F(x) - k_2\big)\,dy = \int_{y_Q}^{y_R} \big(k_2 - F(x)\big)\,dy > 0.$$

Since the functions $g(x)$ and $F(x)$ are bounded for $x \in (b - \epsilon_0, a + \epsilon_0)$, for arbitrary small $\epsilon > 0$ there exists a sufficiently large $M > 0$, when $-y_0 > M$,

$$\left| \frac{dV_1}{dx}\bigg|_{(1.2)} \right| < \epsilon$$

by (2.7). Hence,

$$\big|V_1(P) - V_1(Q)\big| < \epsilon(a - b + 2\epsilon_0), \qquad \big|V_1(R) - V_1(S)\big| < \epsilon(a - b + 2\epsilon_0).$$

Note that

$$V_1(P) - V_1(T) = V_1(P) - V_1(Q) + V_1(Q) - V_1(R) + V_1(R) - V_1(S) + V_1(S) - V_1(T).$$

Thus,

$$V_1(P) - V_1(T) > \frac{1}{2}(k_1 - k_2)(y_S - y_T) > 0,$$

which implies that $y_0 < y_T < 0$ as $-y_0$ is sufficiently large.

Therefore, the closed curve by the segment of trajectory $\widehat{PQRST}$ of system (1.2) and the segment $\overline{TP}$ on line l_1 becomes an outer-boundary of an annular region (see Fig. 3). The annular region is a bounded positive invariant set for system (1.2), which does not include any equilibria of system (1.2). Therefore, the existence of limit cycles follows directly from the Poincaré–Bendixson theorem. We finished the proof of the existence of limit cycle of system (1.2). □

Remark 2.1. Let us compare Theorem 1 in [13] with Theorem 1.1 here in case $G(b) > G(a)$. Using the notations in this paper, Theorem 1 in [13] can be stated as follows:

Theorem 1. *(See [13].) Suppose that system* (1.2) *with* (A1) *and* (A2) *satisfies the hypotheses* (H1)–(H4)*, then system* (1.2) *has at most a limit cycle, and it is stable if it exists.*

The differences between Theorem 1 in [13] and Theorem 1.1 here are as follows.

(1) When $z > G(a)$, in Theorem 1, hypothesis (H3) required, that is, $F_1(z)F_1'(z)$ is nondecreasing. However, in Theorem 1.1, hypothesis (H5) is required, that is, $F_1(z)$ is nondecreasing for $z > G(a)$.
(2) Hypothesis (H4) in Theorem 1 requires that $F_1'(z) \geqslant F_2'(u)$ if $F_1(z) = F_2(u)$ for $G(a) < z < u$. That implies that they need to compare the slopes of $F_1(z)$ and $F_2(u)$ outside the interval $(0, G(a))$; but condition (i) in Theorem 1.1 here requires that $F_1'(z) \leqslant F_2'(u)$ if $F_1(z) = F_2(u)$ for $0 < u < z$ and $G(a) < z < z_\triangle$. That implies that we need to compare the slopes of $F_1(z)$ and $F_2(u)$ inside the interval $(0, z_\triangle)$. Hence, roughly speaking, our uniqueness conditions of limit cycles are different from that in [13], and these conditions on the interval $(0, G(b))$ in case (i) and on interval $(0, G(a))$ in case (ii) have strongly geometric intuition.

In general case, the condition (i) or (ii) in Theorem 1.1 is not easy to be verified since the inverse function of $G(x)$ is hardly expressed explicitly for some continuous functions $g(x)$. Hence, in order to apply Theorem 1.1 more conveniently, one hopes to represent conditions in this theorem directly by means of the original functions of system (1.2). We provide an alternative statement of conclusion (R1) of Theorem 1.1 as follows.

Corollary 2.4. *Assume that system* (1.2) *with* (A1) *and* (A2) *satisfies the hypotheses* (H1), (H2) *and* (H5).

(i) *If* $\int_b^a g(x)\,dx = 0$, *then system* (1.2) *has at most one limit cycle* (*cf.* [11, Theorem 2.2]).

(ii) *If* $\int_b^a g(x)\,dx \neq 0$, *then there exists only a pair of* a_0 *and* b_0, $b_1 < b_0 < 0 < a_0 < a_1$ *such that* $\int_{b_0}^{a_0} g(x)\,dx = 0$ *and* $F(b_0) = F(a_0)$. *System* (1.2) *has at most one limit cycle if one of the following conditions hold.*

(ii.1) *If* $\int_b^a g(x)\,dx < 0$ (*i.e.* $G(b) > G(a)$), *then* $\frac{f(x_2)}{g(x_2)} > \frac{f(x_1)}{g(x_1)}$ *for any pair of* (x_1, x_2) *satisfying* $a < x_1 < a_0$, $b_0 < x_2 < 0$ *and* $F(x_1) = F(x_2)$.

(ii.2) *If* $\int_b^a g(x)\,dx > 0$ (*i.e.* $G(a) > G(b)$), *then* $\frac{f(x_2)}{g(x_2)} > \frac{f(x_1)}{g(x_1)}$ *for any pair of* (x_1, x_2) *satisfying* $0 < x_1 < a_0$, $b_0 < x_2 < b$ *and* $F(x_1) = F(x_2)$.

3. Example and discussion

The existence and uniqueness of limit cycle is one of the most delicate problem in studying of mathematical models. There are various techniques to establish the uniqueness of limit cycles of ecological systems (cf. [3,6,7,11] and references therein). It is the main technique to transform an ecological system into a generalized Liénard system. For example, a general predator–prey system in [11] can be transferred into the generalized Liénard system (1.2) in suitable region of phase plane. And the geometric shape of prey isocline of predator–prey systems can be kept in that of $F(x) = \phi(y)$ of the generalized Liénard system (1.2). Some researchers think that the geometric property of prey isocline plays important role in understanding both the globally asymptotically stable of a unique equilibrium and the uniqueness of limit cycles for predator–prey systems. Hwang in [4] showed that local asymptotic stability of a unique positive equilibrium together with the existence of a concave down prey isocline implied that this equilibrium was globally asymptotically stable for certain predator–prey systems. Kuang in [5] pointed out that Freeman conjectured that the existence of a unique unstable equilibrium together with existence of humps of prey isocline implied the existence and uniqueness of the limit cycle for predator–prey systems. The role of our main results (Theorem 1.1 or Corollary 2.4) is to give information about the geometry of $F(x) = \phi(y)$ to ensure existence and uniqueness of limit cycles.

We now give an example to show the application of Corollary 2.4. In [7], Ruan and Xiao studied global dynamics of a predator–prey system with nonmonotonic functional response

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{xy}{c + x^2},$$
$$\dot{y} = y\left(\frac{\mu x}{c + x^2} - D\right) \tag{3.1}$$

in the closed first quadrant of R^2, where r, c, μ, K and D are positive parameters. They gave the conditions that system (3.1) has a unique limit cycle (cf. [7, Theorems 2.4–2.5]). However, there is a gap in the proof of Theorem 2.5. We will point out it in the following discussion of the uniqueness of limit cycle for system (3.1).

It is clear that system (3.1) has a unique positive equilibrium (x_1, y_1) if

$$\mu^2 > \frac{16}{3}cD^2, \qquad x_2 > K > x_3, \tag{3.2}$$

where

$$x_1 = \frac{\mu - \sqrt{\mu^2 - 4cD^2}}{2D}, \qquad y_1 = r\left(1 - \frac{x_1}{K}\right)\left(c + x_1^2\right),$$
$$x_2 = \frac{\mu + \sqrt{\mu^2 - 4cD^2}}{2D}, \qquad x_3 = \frac{2\mu - \sqrt{\mu^2 - 4cD^2}}{2D}.$$

The nontrivial periodic orbits of system (3.1) must be in the domain E_1 if it exists, here

$$E_1 = \left\{(x, y)\colon\ 0 < x < K,\ 0 < y < +\infty\right\}.$$

To study the uniqueness of periodic orbits of system (3.1), we transfer system (3.1) in E_1 into a Liénard system. Let $x - x_1 = -X$, $y - y_1 = y_1(e^Y - 1)$ and $x\,dt = (c + x^2)\,dT$. Then system (3.1) can be written as

$$\frac{dX}{dT} = \phi(Y) - F(X),$$
$$\frac{dY}{dT} = -g(X), \tag{3.3}$$

where $\phi(Y) = y_1(e^Y - 1)$, $F(X) = \frac{rX}{K}(X^2 + (K - 3x_1)X + c + 3x_1^2 - 2Kx_1)$, and $g(X) = \frac{DX(X-x_1+x_2)}{x_1-X}$, here $x_1 - K < X < x_1$ and $-\infty < Y < +\infty$.

It is easy to check the following facts:

(a1) $Xg(X) > 0$ for $X \in (x_1 - K\ \ 0) \cup (0\ \ x_1)$, $G(x_1) = \int_0^{x_1} g(s)\,ds = +\infty$ and $G(x_1 - K) = \int_0^{x_1-K} g(s)\,ds = \frac{D}{2}((K - x_1)(x_1 + 2x_2 - K) - 2x_1x_2 \ln \frac{K}{x_1}) > 0$.
(a2) $\phi(0) = 0$, $\phi'(Y) > 0$ for $-\infty < Y < +\infty$, and $\phi(-\infty) = -y_1$ and $\phi(+\infty) = +\infty$.

If inequalities (3.2) hold, then we have

$$K - 3x_1 > 0, \qquad c + 3x_1^2 - 2Kx_1 = x_1(x_2 + 3x_1 - 2K) < 0.$$

Let $\triangle = (K - 3x_1)^2 - 4(c + 3x_1^2 - 2Kx_1) = K^2 + 2x_1K - 3x_1^2 - 4c$. Hence, there exist a and b, $0 < a < x_1$ and $K - x_1 < b < 0$ such that

$$F(b) = F(0) = F(a) = 0,$$

where

$$a = \frac{-(K - 3x_1) + \sqrt{\triangle}}{2}, \qquad b = \frac{-(K - 3x_1) - \sqrt{\triangle}}{2}.$$

$f(X) \overset{\text{def}}{=} F'(X) = 0$ has only two roots in the interval $(x_1 - K\ \ x_1)$, which implies that the prey isocline $\phi(y) = F(x)$ of system (3.1) has only two humps in the range

$$E_2 = \{(x, y)\colon x1 - K < x < x_1,\ -\infty < y < +\infty\},$$

namely a local maximum and a local minimum. Therefore, $F(X)$ satisfies hypotheses (H1), (H2) and (H5).

In the proof of Theorem 2.5 in [7], authors only proved the uniqueness of limit cycles of system (3.1) if the limit cycles enclose two points $(a, 0)$ and $(b, 0)$ inside, and did not prove the uniqueness of limit cycles if the limit cycles enclose only one point of $(a, 0)$ and $(b, 0)$ inside.

To apply Corollary 2.4 for system (3.1), we can modify Theorem 2.5 in [7] to the following conclusion.

Theorem 3.1. *Suppose that $\mu^2 > \frac{16}{3}aD^2$ and $x_2 > K > x_3$. Then system* (3.1) *has at most one limit cycle in the interior of the first quadrant if one of the following conditions holds.*

(i) $\int_b^a g(X)\,dX = 0$.
(ii) *If $\int_b^a g(X)\,dX < 0$ (i.e. $G(b) > G(a)$), then $\frac{f(X_2)}{g(X_2)} > \frac{f(X_1)}{g(X_1)}$ for any pair (X_1, X_2) satisfying $a < X_1 < a_0$, $b_0 < X_2 < 0$ and $F(X_1) = F(X_2)$, where a_0 and b_0 with $x_1 - K < b_0 < 0 < a_0 < x_1$ are solutions of $\int_{b_0}^{a_0} g(X)\,dX = 0$ and $F(b_0) = F(a_0)$.*
(iii) *If $\int_b^a g(X)\,dX > 0$ (i.e. $G(a) > G(b)$), then $\frac{f(X_2)}{g(X_2)} > \frac{f(X_1)}{g(X_1)}$ for any pair (X_1, X_2) satisfying $0 < X_1 < a_0$, $b_0 < X_2 < b$ and $F(X_1) = F(X_2)$, where a_0 and b_0 with $x_1 - K < b_0 < 0 < a_0 < x_1$ are solutions of $\int_{b_0}^{a_0} g(X)\,dX = 0$ and $F(b_0) = F(a_0)$.*

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