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Pricing discrete barrier options under jump-diffusion model with liquidity risk

Zhe Li [a], Wei-Guo Zhang [a,*], Yong-Jun Liu [a], Yue Zhang [b]

[a] School of Business Administration, South China University of Technology, Guangzhou, 510640, China
[b] Department of Economics and Finance, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong

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A B S T R A C T

Classical option pricing theories are usually built on the paradigm of competitive and frictionless markets, while ignoring the impact of market liquidity on underlying asset prices. In this paper, the importance of liquidity risk on discrete barrier option pricing is analyzed. First, we propose a new model for describing the asset price dynamics in the presence of jumps and liquidity risks, which is able to capture empirically observed patterns. Based on the COS method, we then derive the analytical approximation formulas for the prices of the discrete barrier options. Numerical experiments demonstrate the accuracy of our proposed pricing model by comparing the analytical approximation solutions with Monte Carlo simulation. Finally, empirically studies are carried out to show the superiority of our model based on SSE 50 ETF options. The numerical and empirical results support our idea of introducing liquidity risk and jumps into the underlying asset price dynamics.

1. Introduction

Classical option pricing theory is usually based on the idealized assumptions: the market is frictionless and the underlying assets are perfectly liquid. However, a large number of empirical studies indicate that the liquidity is an important and significant effect factor in many financial asset prices. It has been shown that the stock returns are influenced by liquidity risk and commonality in liquidity, and the expected returns are positive related to the level of stock illiquidity. Therefore, market liquidity has currently become an issue of high concern in asset pricing and financial risk management.

Naturally, investors demand a premium for bearing the risk of illiquidity. The illiquidity premium was first documented for the stock market in Amihud and Mendelson (1986).[1] Soon afterwards, some researches began to focus on the effects of liquidity on option prices. By using a unique dataset that options are issued by a central bank and are not traded prior to maturity, Brenner, Eldor, and Hauser (2001) examined the effect of illiquidity on the price of currency options. Chou, Chung, Hsiao, and Wang (2011) illustrated the influence of both spot and option liquidity levels on option prices. Nordén and Xu (2012) investigated the dynamic relationship between the steepness of the volatility smirk and relative index option liquidity. Christoffersen, Goyenko, Jacobs, and Karoui (2018) presented a

* Corresponding author.
E-mail addresses: bmzheli@mail.scut.edu.cn (Z. Li), wgzhang@scut.edu.cn (W.-G. Zhang), bmyjliu@scut.edu.cn (Y.-J. Liu), yzhang958-c@my.cityu.edu.hk (Y. Zhang).

[1] For more related studies on illiquidity premia in the stock market, see Amihud and Mendelson (1989), Eleswarapu and Reinganum (1993), Brennan and Subrahmanyam (1996), Datar, Naik, and Radcliffe (1998), Amihud (2002), Pastor and Stambaugh (2003), Brunetti and Caldarera (2004), Acharya and Pedersen (2005), Watanabe and Watanabe (2008), Lee (2011), Bali, Peng, Shen, and Tang (2014) and Ho and Chang (2015).

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Table 1
Payoff functions for different types of discretely monitored knock-out barrier options. Notes: Rb is a rebate and t_i's are the observation dates.

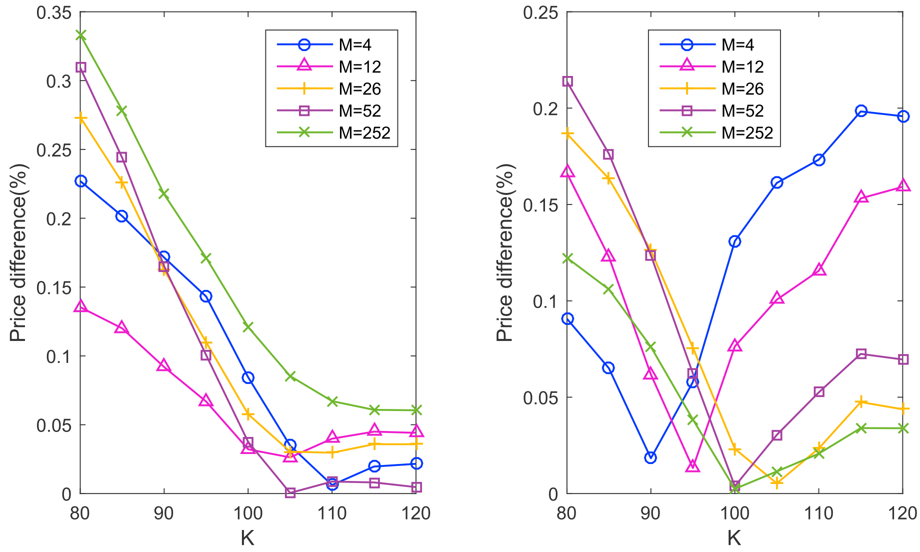
Option type	Payoff
up-and-out call	$[(S_T - K)^+ - Rb]1_{\{S_{t_i} < H\}} + Rb$
up-and-out put	$[(K - S_T)^+ - Rb]1_{\{S_{t_i} < H\}} + Rb$
down-and-out call	$[(S_T - K)^+ - Rb]1_{\{S_{t_i} > H\}} + Rb$
down-and-out put	$[(K - S_T)^+ - Rb]1_{\{S_{t_i} > H\}} + Rb$

Table 2
Model parameters for numerical experiments.

$S_0 = 100, K = 115, T = 1, H = 120, r = 0.05, \sigma = 0.25, \beta = 0.75, \nu_t = 0.5, Rb = 5$	
Jump distribution	
Normal jump:	$\lambda = 3, \mu_J = 0.1, \sigma_J = 0.3$
Double exponential jump:	$\lambda = 3, p = 0.3, \eta_1 = 10, \eta_2 = 5$

Table 3
The prices of discrete up-and-out call barrier options under different jump distributions. Note: The model input parameters values are as in Table 2 except that the monitoring frequencies M are constantly changing. The standard errors for the MC results are presented in brackets.

Monitoring	Normal jumps			Double exponential jumps		
M	Analytical	MC simulation	Relative error	Analytical	MC simulation	Relative error
4	2.1022	2.1020(0.0021)	0.0196%	2.3824	2.3871(0.0034)	0.1983%
12	2.4693	2.4674(0.0023)	0.0450%	2.7727	2.7769(0.0032)	0.1532%
26	2.6233	2.6215(0.0031)	0.0359%	2.9408	2.9422(0.0024)	0.0475%
52	2.7185	2.7174(0.0029)	0.0081%	3.0471	3.0493(0.0027)	0.0725%
252	2.8445	2.8415(0.0025)	0.0608%	3.1886	3.1897(0.0021)	0.0340%



(a) Normal jumps (b) Double exponential jumps

Fig. 1. Relative price differences between analytical solution and Monte Carlo simulation for varying strike prices K and monitoring frequencies M. Notes: The left and right panels shows the discrete barrier option prices computed by the normal jumps and double exponential jumps, respectively. The model input parameters values are as in Table 2 except that the strike price K is constantly changing.

significant positive impact of option illiquidity on expected call option returns by utilizing portfolio sorts and cross-sectional regressions. Theoretically, stock illiquidity is positively related to option illiquidity and thus indirectly affects option prices.

There have been some theoretical researches on how to introduce liquidity into the framework of option pricing. Cetin, Robert, and

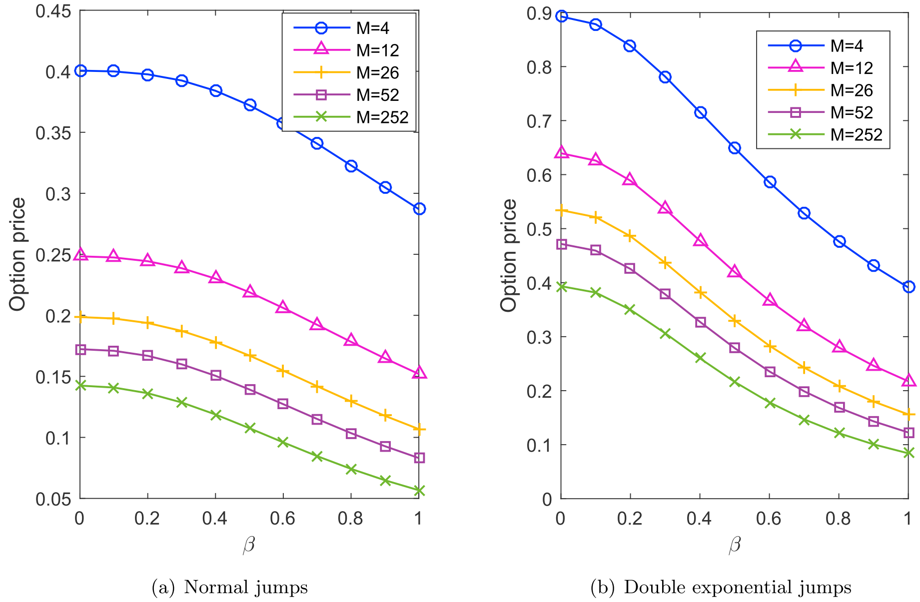


Fig. 2. Discrete up-and-out call barrier option price with respect to β. Notes: The left and right panels shows the discrete barrier option prices computed by the normal jumps and double exponential jumps, respectively. The model input parameters values are as in Table 2 except that the sensitivity of the underlying asset to the market illiquidity level, β, is constantly changing.

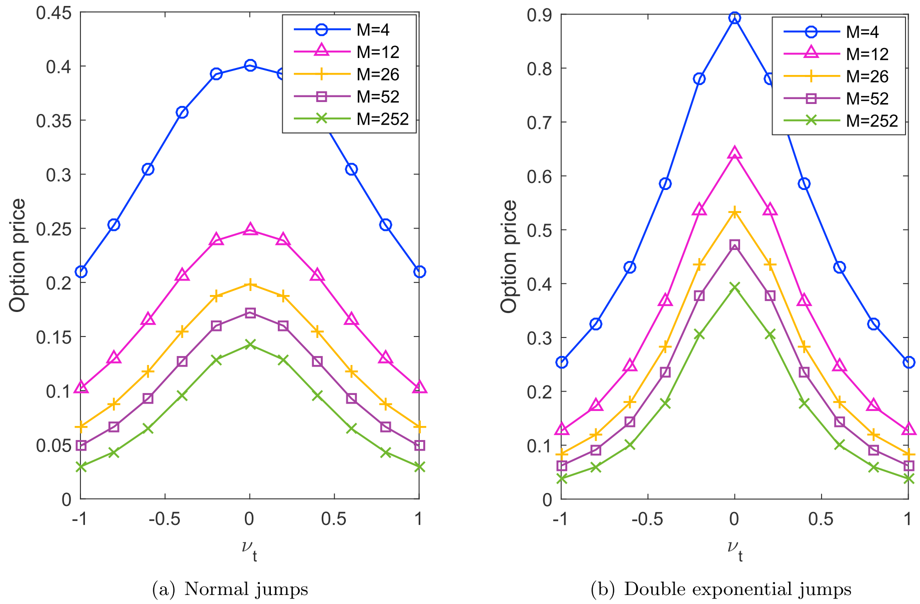
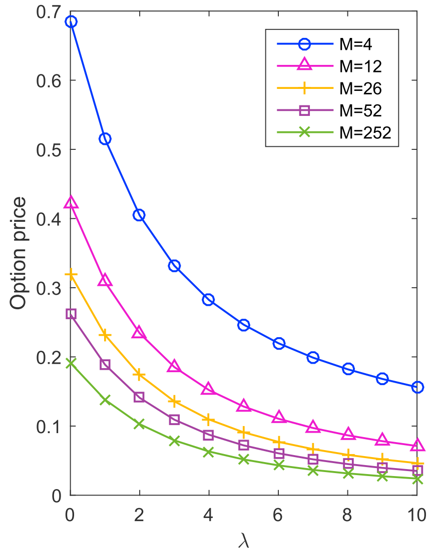


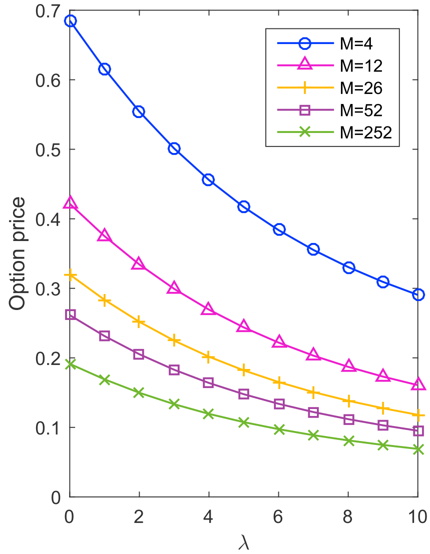
Fig. 3. Discrete up-and-out call barrier option price with respect to ν_t. Notes: The left and right panels shows the discrete barrier option prices computed by the normal jumps and double exponential jumps, respectively. The model input parameters values are as in Table 2 except that the market liquidity level ν_t is constantly changing.

Protter (2004) developed a model for the inclusion of liquidity risk into arbitrage pricing theory by assuming that a stochastic supply curve for a security's price as a function of trade size.[2] Bakstein and Howison (2004) developed a parameterized model for liquidity effects arising from the traded assets, where the liquidity was defined by means of a combination of a trader's individual transaction cost and a price slippage impact. Based on this model, options can be priced and hedged under the risk-neutral world. In contrast to the

[2] Following the framework of Cetin et al. (2004), Cetin, Jarrow, Protter, and Warachka (2006, 2010) studied the problems of option pricing and super-replication in an extended Black Scholes economy, where the underlying asset is imperfectly liquid.



(a) Normal jumps



(b) Double exponential jumps

Fig. 4. Discrete up-and-out call barrier option price with respect to λ. Notes: The left and right panels shows the discrete barrier option prices computed by the normal jumps and double exponential jumps, respectively. The model input parameters values are as in Table 2 except that the jump intensity λ is constantly changing.

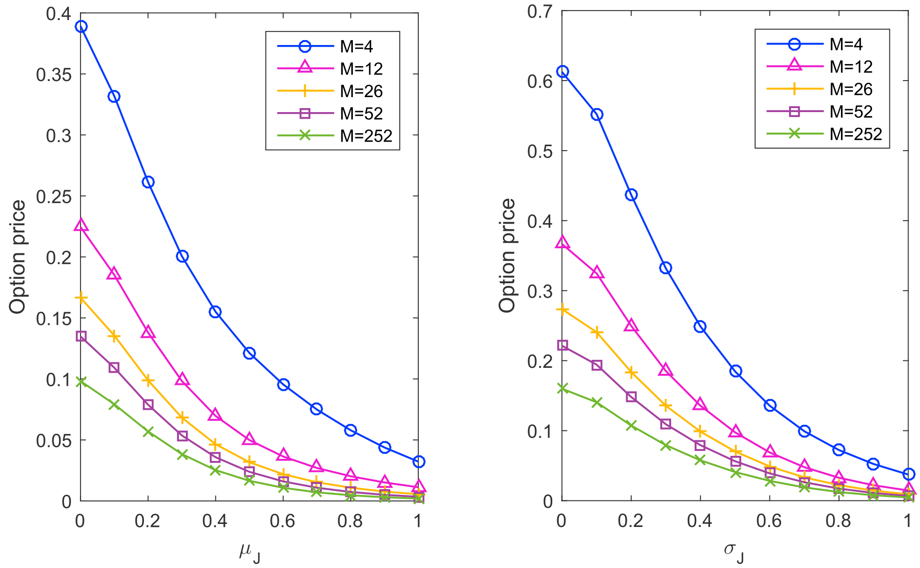


Fig. 5. Discrete up-and-out call barrier option price with respect to the parameters of the normal jump distribution. Notes: The left and right panels shows the discrete barrier option prices computed by the different values of μ_J and σ_J, respectively. The model input parameters values are as in Table 2 except that the mean value μ_J and variance σ_J^2 of the norm jump size are constantly changing.

standard framework and consistent with a market with imperfect liquidity, Liu and Yong (2005) examined the effect of stock liquidity on the replication of a European contingent claim under the assumption that the investors's trading had a direct impact on the stock price. As the liquidity was understood as a nonlinear transaction cost incurred as a function of rate of change of portfolio, Rogers and Singh (2010) proposed a model for the effects of illiquidity and explored some of its consequences for the hedging of European options in the model of Black and Scholes (1973). Working within a Markovian regime-switching setting, Ludkovski and Shen (2013) defined illiquidity as the inability to trade in a timely way and then studied the pricing and hedging problems of European options with liquidity shocks. On the basis of the price dynamics proposed by Brunetti and Caldarera (2004), Feng, Hung, and Wang (2014, 2016) derived the corresponding pricing formulas for European options and investigated the importance of the stock liquidity on option pricing. Their empirical results provided strong evidence to support that incorporating a stock liquidity into the framework of option pricing can

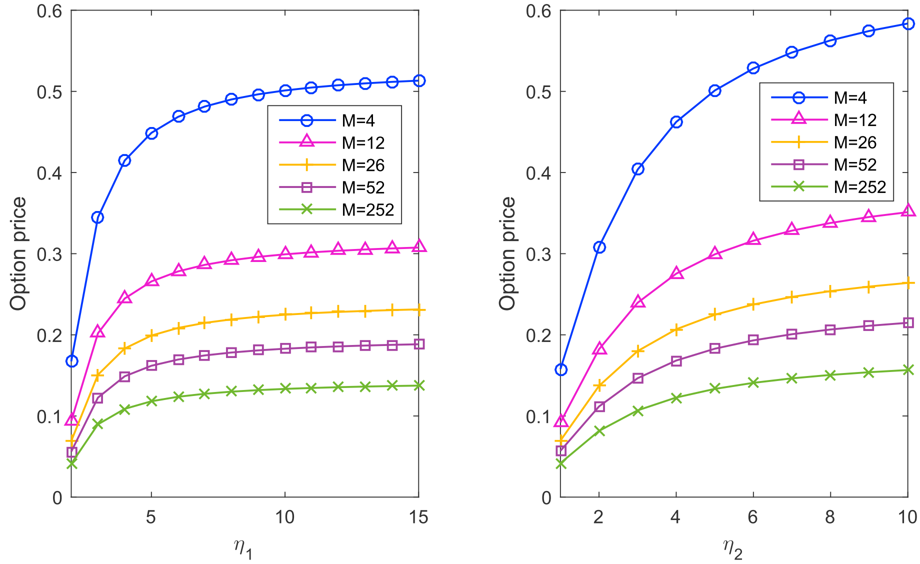
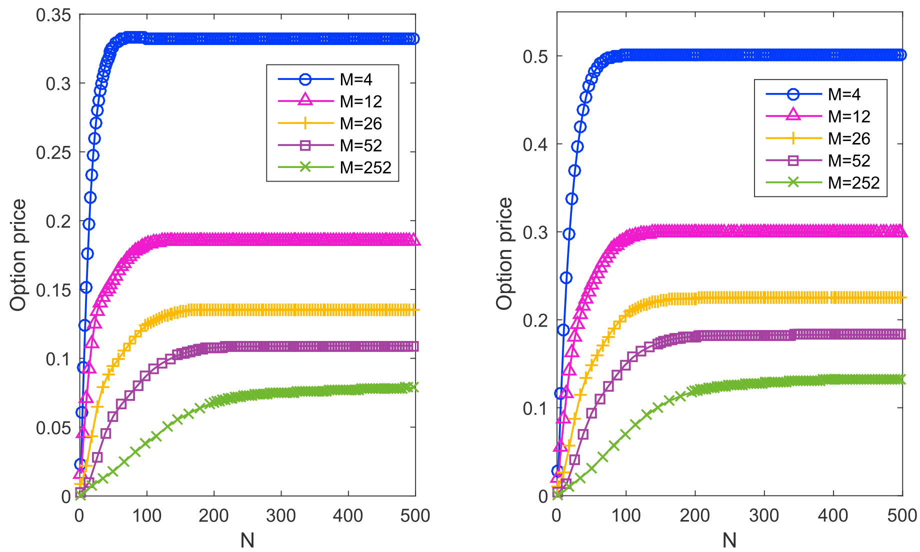


Fig. 6. Discrete up-and-out call barrier option price with respect to the parameters of the double exponential jump distribution. Notes: The left and right panels shows the discrete barrier option prices computed by the different values of η_1 and η_2, respectively. The model input parameters values are as in Table 2 except that the parameters of the double exponential jump size, η_1 and η_2, are constantly changing.



(a) Normal jumps (b) Double exponential jumps

Fig. 7. The convergence of our formula in terms of Fourier-cosine series expansion for pricing discrete barrier options under both normal and double exponential jump distributions. Notes: The left and right panels shows the discrete barrier option prices computed by the normal jumps and double exponential jumps, respectively. The model input parameters values are as in Table 2 except that the terms of the Fourier-cosine series N are constantly changing.

produce smaller pricing errors. Within the framework of conic finance, Leippold and Scharer (2017) studied the discrete time option pricing with stochastic liquidity model, in which the liquidity measure was not directly defined by observable variables in the financial market such as bid-ask spreads or trading volume, but inferred from a comparison of market and model implied bid-ask spreads.

The previous literature mentioned above is limited to the liquidity-adjusted diffusion model without jumps. However, many financial assets, such as stocks, currencies and commodities, usually exhibit jumps, which are particularly prominent in the market after the financial crisis or major incident. To incorporate the jump behavior in option pricing, a variety of modified Black-Scholes models have been proposed. Merton (1976) firstly considered a normal jump-diffusion model for pricing option, which can incorporate the leptokurtic feature and implied volatility smile. By supposing the jump size followed a double exponential distribution, Kou (2002) proposed a novel jump-diffusion model for pricing option. Especially, the model was simple enough to obtain analytical solutions for the

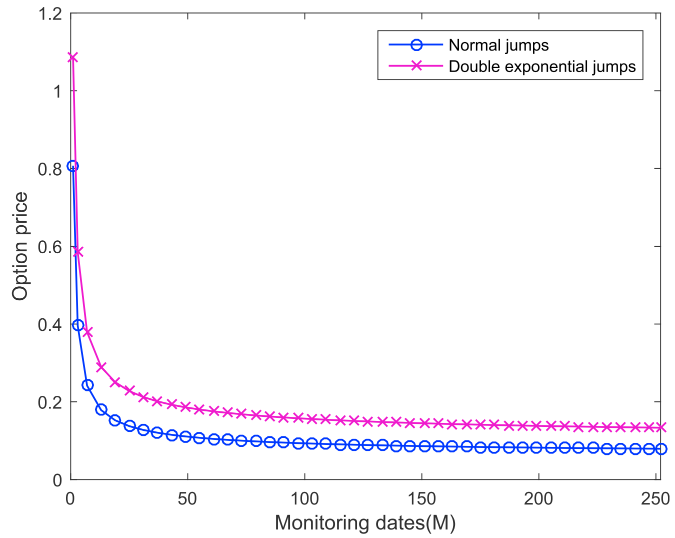
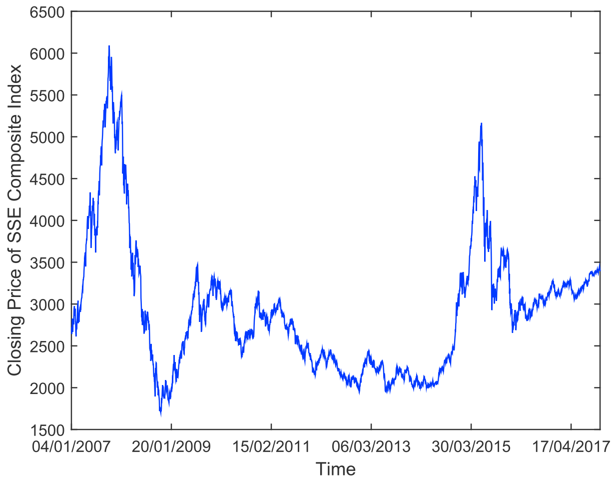


Fig. 8. Discrete up-and-out call barrier option price for different monitoring frequencies. Notes: The purple and blue lines denote the discrete barrier option prices computed by the normal jumps and double exponential jumps, respectively. The model input parameters values are as in Table 2 except that the monitoring dates *M* are constantly changing. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

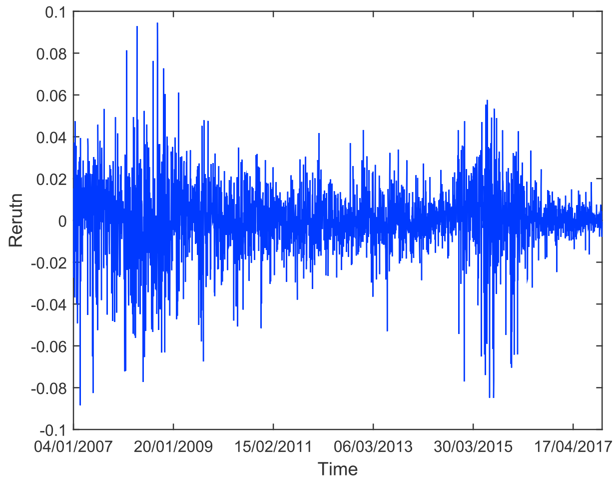
pricing problems of European options, interest rate derivatives and path-dependent options. Carr and Wu (2004) proposed a time-changed Lévy process to model the dynamics of the underlying asset prices. Cai and Kou (2011) extended the double exponential jump-diffusion model to a mixed exponential jump-diffusion model for pricing European, lookback and barrier options. Fu, Li, Li, and Wu (2017) obtained a closed-form solution of the European option under a general jump-diffusion model that can incorporate arbitrary discrete jump size distributions. Using the conditional delta hedging strategy and the minimal mean-square-error hedging, Wang, Li, and Zhuang (2017) presented a closed-form solution of the European option under Student's t noise with jumps. Therefore, given the liquidity effect, it is worth proposing the liquidity-adjusted jump-diffusion model to describe the asset price dynamics. In the following, we will extend the jump-diffusion models of Merton (1976) and Kou (2002) to allow liquidity risk for describing the underlying price dynamics.

So far, we have stated what models can be employed more rationally to describe the asset price dynamics. Nevertheless, we have not yet elaborated on the types of options to be studied in this paper. As we know, the barrier options are among the most actively traded path-dependent options, whose payoff depends not only on the final price of the underlying asset, but also on whether the asset price has breached some barrier level during the life of the option. One important feature of barrier options is that the option values are quite sensitive to whether the extrema are monitored discretely or continuously. Most of the barrier options are discretely monitored in practice. However, there is essentially no analytical solution for discrete barrier options. Therefore, in the literature, the pricing problem of discrete barrier option has become the focus of research in the last decade. Broadie, Glasserman, and Kou (1997) showed that discrete barrier options can be priced with remarkable accuracy using continuous barrier formulas by applying a simple continuity correction to the barrier. Based on Laplace transforms, Petrella and Kou (2004) proposed a new methodology to compute the prices of discretely monitored barrier options. Broadie and Yamamoto (2005) proposed a double-exponential fast Gauss transform algorithm for pricing discrete path-dependent options. The pricing problem of these options can be reduced to evaluation of a series of convolutions of the Gaussian distribution and a known function under the Black-Scholes model. Soon afterwards, under the Black-Scholes framework, Fusai, Abrahams, and Sgarra (2006) derived an analytical solution for pricing discretely monitored barrier options by reducing the valuation problem to a Wiener-Hopf equation which can be solved analytically. Feng and Linetsky (2008) presented a Hilbert transform-based semi-analytical method to price discretely monitored single-and double barrier options. By applying the Fourier-cosine series expansion to the transition probability density function, Fang and Oosterlee (2009) proposed an novel analytical approximation method for pricing early-exercise and discrete barrier options. Green, Fusai, and Abrahams (2010) extended the approach of Fusai et al. (2006) to price discretely monitored barrier options by taking into account more general Lévy processes and double barrier options. Under the assumption that the correction factor did not dependent on the jump part of the price process, Fuh, Luo, and Yen (2013) extended the continuity correction of Broadie et al. (1997) to a double exponential jump-diffusion model for pricing discretely monitored exotic options. Recently, Lian, Zhu, Elliott, and Cui (2017) derived a semi-analytical and fully explicit solution for pricing discrete barrier options under the assumption that the underlying asset was driven by a general Lévy process. Their approach mainly reduced the valuation problem to an integral equation by $\mathscr{Z}$-transform. Chen and Hsu (2018) presented a semi-closed-form valuation model for barrier options whose underlying asset follows a mean-reverting and regime-switching double exponential jump diffusion process, and the interest rate is modulated by a mean-reverting square root model.

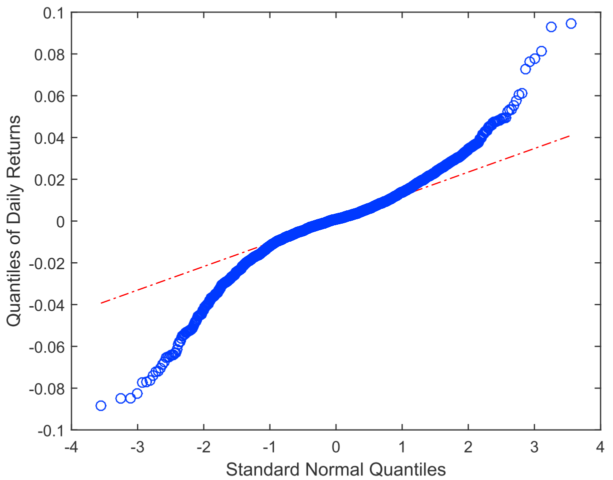
Motivated by the above mentioned insights, we propose to price discretely monitored barrier options when the underlying asset price is driven by a jump-diffusion model with liquidity risk, which allows the underlying asset is not perfectly liquid. In the present paper, we



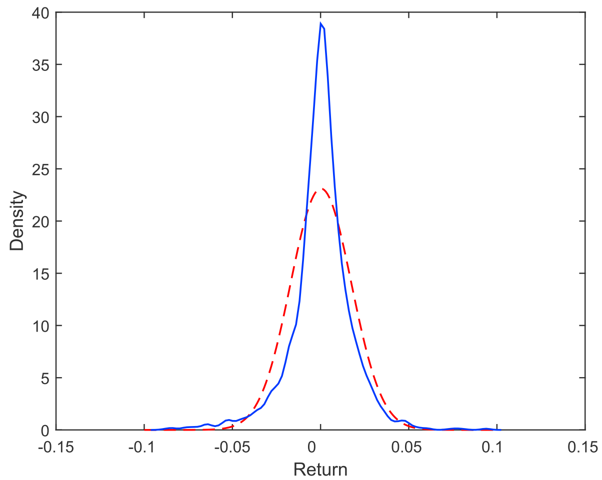
(a) Daily closing prices



(b) Daily log return



(c) QQ plot of sample data versus standard normal



(d) Probability density of the return

Fig. 9. Statistical properties of the SSE Composite Index. *Notes*: These figures describe the closing prices of the SSE Composite Index (a) and the corresponding returns (b). Moreover, figure (c) reports the Quantile-Quantile plot of the returns. In addition, figure (d) shows the kernel density estimation of the returns (solid line), which is contrasted to the normal distribution (dotted line). The latter is specified by the sample mean and variance of the returns. The daily return $R_t = \ln(S_t/S_{t-1})$ and the sample data is from Jan 4, 2007 to Nov 16, 2017.

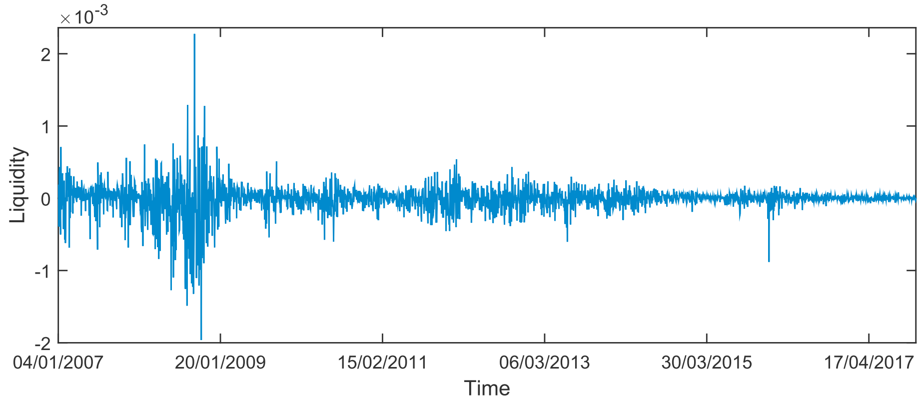


Fig. 10. The time series for the liquidity level of SSE Composite Index from Jan 4, 2007 to Nov 16, 2017.

Table 4
The dependence structure between the return and liquidity of SSE Composite Index. *Note*: The table shows the correlation coefficient ρ, Spearman correlation coefficient r_s and Kendall correlation coefficient R for Gaussian copula and T-copula, respectively. AIC stands for the Akaike information criterion.

	ρ	r_s	R	AIC
Gaussian copula	0.9268	0.9157	0.7479	−5037.8356
T-copula	0.9226	0.9136	0.7579	−5239.8327

Table 5
Estimated parameters. Note: This table shows the daily average of the estimated parameters obtained by minimizing the root mean squared pricing errors between the market price and the model-determined price for each option on Wednesday during the sample period from January 1, 2018 to March 31, 2018. Standard errors are reported in parentheses.

Parameters	L-Kou model	L-Merton model	Kou model	Merton model
σ	0.169537	0.173844	0.172385	0.173966
	(0.033967)	(0.034380)	(0.038831)	(0.037766)
β	0.487700	0.434777	–	–
	(0.202853)	(0.265053)	–	–
λ	0.563904	0.169896	0.539257	0.385482
	(0.279482)	(0.121748)	(0.303038)	(0.575955)
p	0.755682	–	0.739834	–
	(0.339776)	–	(0.284511)	–
η_1	9.718660	–	9.036937	–
	(4.437461)	–	(2.795420)	–
η_2	7.650062	–	6.345066	–
	(4.392633)	–	(2.186580)	–
μ_J	–	0.050964	–	0.021794
	–	(0.167484)	–	(0.233815)
σ_J	–	0.324056	–	0.285118
	–	(0.212699)	–	(0.180763)

Table 6
In-sample pricing errors (root mean squared errors, RMSE) across different moneyness groups. Note: This table shows the in-sample pricing errors across different moneyness categories. Pricing errors are reported as the root mean squared errors of option prices for L-Kou, L-Merton, Kou and Merton models. The moneyness is defined as the underlying price divided by the strike price. The liquidity measure is defined as the ratio of the return to RMB trading volume. The standard errors of the pricing performance are also reported in this table. The sample data is from January 1, 2018 to March 31, 2018.

Moneyness	L-Kou model	Kou model	L-Merton model	Merton model
OTM($<$ 0.97)	0.002926	0.003535	0.003182	0.003868
Std.Error	0.001822	0.001654	0.001699	0.001713
ATM(0.97 $-$ 1.03)	0.004016	0.004564	0.004278	0.004691
Std.Error	0.002370	0.002136	0.002246	0.002104
ITM($>$ 1.03)	0.004412	0.004724	0.004561	0.005280
Std.Error	0.002127	0.002139	0.002131	0.002237
Total	0.003989	0.004260	0.004138	0.004529
Std.Error	0.001946	0.001826	0.001877	0.001722

Table 7

Out-of-sample pricing errors (root mean squared errors, RMSE) across different moneyness groups. Note: This table shows the out-of-sample pricing errors across different moneyness categories. Pricing errors are reported as the root mean squared errors of option prices for L-Kou, L-Merton, Kou and Merton models. The moneyness is defined as the underlying price divided by the strike price. The liquidity measure is defined as the ratio of the return to RMB trading volume. The standard errors of the pricing performance are also reported in this table. The sample data is from January 1, 2018 to March 31, 2018.

Moneyness	L-Kou model	Kou model	L-Merton model	Merton model
OTM(< 0.97)	0.004803	0.005048	0.005354	0.005673
Std.Error	0.003264	0.003238	0.003155	0.003277
ATM($0.97 - 1.03$)	0.007090	0.007429	0.007693	0.007974
Std.Error	0.004274	0.004319	0.004323	0.004343
ITM(> 1.03)	0.006702	0.006983	0.007253	0.007523
Std.Error	0.002779	0.002831	0.002850	0.002979
Total	0.006300	0.006518	0.006713	0.006915
Std.Error	0.002957	0.002988	0.002986	0.003062

investigated the jump distributions of Merton (1976) and Kou (2002), i.e., normal jump and double exponential jump, respectively. Suppose that the liquidity discount factor is connected to both market liquidity and the sensitivity of stock prices to market liquidity, we then derive the analytical approximation formulas for the discrete barrier options based the Fourier-cosine series expansion method (COS method) developed in Fang and Oosterlee (2008).[3]

In addition, we provide the numerical results to test the accuracy of our analytical approximation pricing formulas based on the COS method. Toward this end, we make the performance comparison between the analytical approximation solutions and the Monte Carlo simulation for discretely monitored barrier options across different monitoring frequencies. Then we perform sensitivity analysis to investigate the price changes of the discrete barrier options with respect to model input parameters. Finally, we also present the rates of convergence of the COS discrete barrier option prices with regard to monitoring frequencies and the terms of Fourier-cosine series. In conclusion, the barrier option price is sensitive to the changes in the values of these parameters, hence supporting the use of the liquidity risk and jump process when modeling the underlying asset price dynamic. On the other hand, to better illustrate the reasonability of our proposed pricing model, we also perform statistical analysis by using the sample data of SSE Composite Index traded at the Shanghai Stock Exchange. Due to the shortage of readily available real market data of Barrier options, it is not possible to calibrate related models and then compare the pricing performance of them. To address this problem, we construct the market barrier option data implied by market data of SSE 50 ETF options and the related barrier level. Empirical results confirm that our model generally outperforms the Merton (1976) and Kou (2002) models, and thus it could be an alternative to the jump-diffusion model for some markets.

The remainder of the paper proceeds as follows. Section 2 lays out the liquidity-adjusted jump-diffusion model for the underlying asset. In Section 3, we present the analytical approximation pricing formulas for discrete barrier options based on Fourier-cosine series expansions. Section 4 exhibits the numerical results to investigate the accuracy of analytical approximation pricing formulas. In Section 5, empirical studies are carried out to show the pricing performance of our proposed model. Finally, some conclusions are stated in Section 6.

2. Model description

2.1. Model specification

In this subsection, we propose a new asset price model for pricing discrete barrier options in the presence of liquidity risk and jumps, which incorporates the effects of liquidity costs and jump risk on the underlying asset price. Thus, we assume that the stocks are not perfectly liquid and the imperfect liquidity appears when a surplus or shortage of assets exists in the market.

Given a complete probability space $(\Omega, \mathscr{F}, \mathrm{P})$ with information filtration $\{\mathscr{F}_t\}_{t\geq 0}$, where P is the physical probability measure. We next employ the dynamics of the stock price proposed in Brunetti and Caldarera (2004). In their framework, they postulate the demand function for the illiquid stock takes the form

$$D(S_t, L_t, I_t) = H\left(\frac{I_t^\gamma}{L_t S_t}\right), \tag{1}$$

where H is a smooth and strictly increasing function, γ is a positive constant, I_t is the stock's information process and L_t is the liquidity discount factor. Specifically, the stock's information process I_t follows

[3] The liquidity discount factor successfully captures the effect of the liquidity on the asset price, such as in Subramanian and Jarrow (2001), Brunetti and Caldarera (2004), Longstaff, Mithal, and Neis (2005) and Chen (2012).

$$\frac{dI_t}{I_t} = \mu dt + \eta dW_t^{I,P}, \tag{2}$$

where μ and η are constants, and $W_t^{I,P}$ is a standard Brownian motion. Using Cholesky decomposition, the stock's information process can be decomposed into an idiosyncratic factor (stock-specific information at time t) and a market factor (systematic factor, e.g. information about the market at time t), i.e. $\eta dW_t^{I,P} = \eta(\alpha dW_t^{I,\perp} + \sqrt{1-\alpha^2}\, dW_t^{MKT})$, where $W_t^{I,\perp}$ and W_t^{MKT} are independent standard Brownian motions.

However, many financial assets, such as stocks, currencies and commodities, usually exhibit jumps, which are particularly prominent in the financial market after the financial crisis or major incident. Thus, we extend the information process (2) to a jump-diffusion process in this paper:

$$\frac{dI_t}{I_{t-}} = \mu dt + \eta dW_t^{I,P} + (e^V - 1)dN_t^P, \tag{3}$$

where V denotes the jump magnitudes of the stock's information process upon the arrival of a jump event, N_t^P denotes a Poisson process with jump intensity λ. In the model, all sources of randomness, N_t^P, $W_t^{I,P}$ and V, are assumed to be independent.

Furthermore, the liquidity discount factor L_t is defined as

$$\frac{dL_t}{L_t} = \left(-\beta\nu_t + \frac{1}{2}\beta^2\nu_t^2\right)dt - \beta\nu_t dW_t^{L,P}, \tag{4}$$

where ν_t denotes the level of market liquidity, $\beta > 0$ is the sensitivity of the underlying stock to the level of market illiquidity, $L_0 = 1$ and $dW_t^{I,P} dW_t^{L,P} = 0$. When $\nu_t > 0 (< 0)$, it means that the market is in short supply(surplus); In the case of $\nu_t = 0$, the market liquidity is at the perfect level.[4] Additionally, in this article, we consider only the case of time-dependent market liquidity parameter, i.e. ν_t is a deterministic function of time.

The supply for the illiquid underlying asset at time t is fixed for simplicity and denoted by $\overline{S_t}$.

The market clearing price S_t is defined implicitly by the market clearing condition

$$H\left(\frac{I_t^\gamma}{L_t S_t}\right) = \overline{S_t}.$$

Under the above market clearing condition, we adopt the method of Brunetti and Caldarera (2004) to derive the dynamics of the imperfectly liquid stock price as follows:

$$dS_t = \left(\varpi + \beta\nu_t + \frac{1}{2}\beta^2\nu_t^2\right)S_{t-}dt + \sigma S_{t-}dW_t^{I,P} + \beta\nu_t S_{t-}dW_t^{L,P} + d\sum_{0\le u\le t}(S_u - S_{u-}), \tag{5}$$

where $\varpi = \gamma\mu + \frac{1}{2}\gamma(\gamma-1)\eta^2$ and $\sigma = \gamma\eta$. Obviously, the jump intensity of the imperfectly liquid stock price is the same as that of the information process. Meanwhile, the jump amplitudes of the imperfectly liquid stock price are associated with that of the information process and determined by the demand function H. However, for simplicity we assume that the jump amplitudes of the imperfectly liquid stock price are given in the following. Therefore, the stock price process (5) can be further expressed as

$$\frac{dS_t}{S_{t-}} = \left(\varpi + \beta\nu_t + \frac{1}{2}\beta^2\nu_t^2\right)dt + \sigma dW_t^{I,P} + \beta\nu_t dW_t^{L,P} + (e^J - 1)dN_t^P, \tag{6}$$

where J denotes the jump magnitudes of the imperfectly liquid stock price upon the arrival of a jump event. Parameter β captures the sensitivity of the stock to market illiquidity. Parameter σ measures the volatility of the stock return. As expected, the dynamics of the stock price with imperfect liquidity are adjusted based on information related to the stock price, the level of market liquidity ν_t, and the stock's sensitivity to the level of market illiquidity β.

2.2. Risk-neutral dynamics

The market described in this paper is incomplete. Thus, there is more than one equivalent martingale measure. As is well known, the Esscher transform is an effective method to obtain a reasonable equivalent martingale measure. The pioneer work by Gerber and Shiu (1994) extended the implication of the Esscher transform to option pricing by providing a pertinent solution to the pricing model in incomplete markets. Elliott, Chan, and Siu (2005) showed that under certain conditions, the minimum entropy martingale measure and Esscher transform point at the same martingale measure. One advantage of the Esscher-type change of measure is its ease of use. Next, we will present how to determine an equivalent martingale measure by the Esscher transform.

[4] The relationship between the prices of the stock with imperfect liquidity and perfect liquidity can be expressed as $S_t = \frac{S_{Liquidi,t}}{L_t}$, where $S_{Liquidi,t}$ denotes the price of the perfectly liquid stock (i.e., $\nu_t = 0$ and $L_t = 1$).

Solving the stochastic differential equation (6) gives the dynamics of the imperfectly liquid stock price:

$$\begin{aligned} S_t &= S_0 \exp\left\{ \left(\varpi + \beta\nu_t - \frac{1}{2}\sigma^2 \right) t + \sigma W_t^{I,P} + \beta\nu_t W_t^{L,P} + \sum_{i=1}^{N_t^P} J_i \right\} \\ &= S_0 \exp\{X(t)\} \end{aligned} \tag{7}$$

where $X(t)$ can be interpreted as the continuously compounded rate of return over the t period.

Let $\mathscr{T}$ denote the time index set $[0,\infty)$ and $\mathscr{B}(\mathscr{T})$ the Borel field on $[0,\infty)$. Let $\mathscr{BM}(\mathscr{T})$ denote a collection of $\mathscr{B}(\mathscr{T})$-measureable non-negative functions with compact support on $\mathscr{T}$. Then assume process $\theta \in \mathscr{BM}(\mathscr{T})$ is integrable with respect to the process $X(t)$. For each $t \in \mathscr{T}$, let $\{\Lambda_t^\theta\}_{t\in\mathscr{T}}$ denote a $\mathscr{F}_t$-adapted sotchstic process such that

$$\Lambda_t^\theta = \frac{e^{\theta X(t)}}{\mathbb{E}^P\left[e^{\theta X(t)}\right]}, \quad t \in \mathscr{T}, \tag{8}$$

where $\mathbb{E}^P[\cdot]$ denotes the expectation under the physical probability measure P.

Substituting Eq. (7) into Eq. (8), we get

$$\Lambda_t^\theta = \exp\left\{ -\frac{1}{2}\sigma^2\theta^2 t - \frac{1}{2}\beta^2\nu_t^2\theta^2 t - \lambda t[\varphi_J(\theta) - 1] + \sigma\theta W_t^{I,P} + \beta\nu_t\theta W_t^{L,P} + \theta\sum_{i=1}^{N_t^P} J_i \right\}, \tag{9}$$

where, $\varphi_J(\theta) = \mathbb{E}^P[e^{\theta J}]$ denotes the moment-generating function of the random variable J.

Moreover, it is easy to show that for each $t \in \mathscr{T}$, Λ_t^θ is a P-martingale. Then for each $\theta \in \mathscr{BM}(\mathscr{T})$, we define a new probability measure $Q \sim P$ by the Radon-Nikodym derivative

$$\left.\frac{dQ}{dP}\right|_{\mathscr{F}_t} := \Lambda_t^\theta. \tag{10}$$

Obviously, the new measure Q is defined by the Esscher transform Λ_t^θ associated with $\theta \in \mathscr{BM}(\mathscr{T})$.

According to the fundamental theorem of asset pricing, the absence of arbitrage essentially means that there exists an equivalent martingale measure under which discounted asset prices are local martingales. In other words, the risk-neutral Esscher measure is the Esscher measure of parameter $\theta = \theta^*$ such that the discounted asset price process $\{e^{-rt}S_t\}_{t\ge 0}$ is a martingale (with r the risk-free interest rate). Furthermore, from the martingale condition

$$\mathbb{E}^Q\left[e^{-rt}S_t; \theta^*\right] = S_0 \tag{11}$$

we find

$$\varpi + \beta\nu_t + \frac{1}{2}\beta^2\nu_t^2 + \beta^2\nu_t^2\theta^* + \sigma^2\theta^* + \lambda\mathbb{E}^P\left[e^{\theta^* J}(e^J - 1)\right] - r = 0, \tag{12}$$

from which the parameters θ^* is uniquely determined. Although there is no algebraic solution for θ^* in Eq. (12), one can solve it by numerical method.

Thus, the dynamics of the imperfectly liquid stock price under risk-neutral measure Q can be rewritten as

$$\frac{dS_t}{S_{t-}} = (r - \tilde{\lambda}\vartheta)dt + \sigma dW_t^{I,Q} + \beta\nu_t dW_t^{L,Q} + \left(e^{\tilde{J}} - 1\right)dN_t^Q, \tag{13}$$

where $W_t^{I,Q}$ and $W_t^{L,Q}$ are two independent Brownian motions under risk-neutral measure Q, N_t^Q is a Poisson process with jump intensity $\tilde{\lambda}$ under risk-neutral measure Q, $\{\tilde{J}_i\}$ are independent identically distributed random variables under risk-neutral measure Q, and $\vartheta = \mathbb{E}^Q[e^{\tilde{J}} - 1]$. In the following, the distributions of the jump size are investigated under both normal and double exponential.[5]

Suppose we are now standing at time $t = 0$, and try to price an option with maturity of T. Since the objective is to price discretely monitored barrier options, we consider the underlying asset price being recorded at some regular time interval. Let t_0 denote the initial time and $\mathscr{T}\{t_1, t_2, \ldots t_M\}$ be the set of the monitoring dates with $\triangle t := (t_m - t_{m-1}), t_1 < t_2 < \ldots < t_M = T$. Consequently, we are capable of computing the analytical approximation solutions for discrete barrier options with the help of risk-neutral pricing theory.

2.3. Characteristic function

In order to be able to obtain the analytical approximation pricing formulas of the discretely monitored barrier options, we are

[5] These two distributions were initiatively studied by Merton (1976) and Kou (2002), respectively.

particularly interested in the characteristic function of the logarithm asset price. Hence, we first calculate the characteristic function for the logarithm asset price $X_{t_m} = \ln(S_{t_m})$ given the market information up to time t_{m-1}, for $m = 1, 2, \ldots, M$.

Lemma 1. *If the underlying stock price follows the model (13), then the characteristic function of* $\ln(S_{t_m})$ *given the market information available at time* t_{m-1} *is given by*

$$\begin{aligned}\Phi(\phi; X_{t_{m-1}}, t_{m-1}) &= \mathbb{E}^{Q}\left[e^{i\phi X_{t_m}} \middle| \mathscr{F}_{t_{m-1}}\right] \\ &= e^{A(\phi;\Delta t)X_{t_{m-1}} + B(\phi;\Delta t)},\end{aligned} \tag{14}$$

where $i = \sqrt{-1}$, $A(\phi; \Delta t) = i\phi$, and

$$B(\phi; \Delta t) = \int_{t_{m-1}}^{t_m} \left\{ i\phi\left(r - \tilde{\lambda}\vartheta - \frac{1}{2}\sigma^2 - \frac{1}{2}\beta^2\nu_t^2\right) - \frac{1}{2}(\sigma^2 + \beta^2\nu_t^2)\phi^2 + \lambda\mathbb{E}\left[e^{i\phi\tilde{J}} - 1\right]\right\} dt.$$

Proof: The Feynman-Kac formula states that $\Phi(\phi; X_{t_{m-1}}, t_{m-1})$ is governed by the following partial integro-differential equation (PIDE):

$$\begin{cases} \dfrac{\partial \Phi}{\partial \tau}(\phi; X_t, t) = \left(r - \tilde{\lambda}\vartheta - \dfrac{1}{2}\sigma^2 - \dfrac{1}{2}\beta^2\nu_t^2\right)\dfrac{\partial \Phi}{\partial X_t}(\phi; X_t, t) + \dfrac{1}{2}\left(\sigma^2 + \beta^2\nu_t^2\right)\dfrac{\partial^2 \Phi}{\partial X_t^2}(\phi; X_t, t) \\ \qquad\qquad +\lambda\mathbb{E}\left[\Phi(\phi; X_t + \tilde{J}, t) - \Phi(\phi; X_t, t)\right], \\ \Phi(\phi; X_{t_m}, t_m) = e^{i\phi X_{t_m}}. \end{cases} \tag{15}$$

with $t \in [t_{m-1}, t_m]$ and $\tau = t_m - t$.

Consider the exponential affine form for the characteristic function, $\Phi(\phi; z, t) = e^{A(\phi;\tau)z + B(\phi;\tau)}$ and substitute it into the PIDE (15), we have the following system of ordinary differential equations (ODEs) for $A(\phi; \tau)$ and $B(\phi; \tau)$:

$$\begin{cases} \dfrac{\partial A}{\partial \tau} = 0 \\ \dfrac{\partial B}{\partial \tau} = \left(r - \lambda\vartheta - \dfrac{1}{2}\sigma^2 - \dfrac{1}{2}\beta^2\nu_t^2\right)A(\tau; \phi) + \dfrac{1}{2}(\sigma^2 + \beta^2\nu_t^2)A^2(\phi; \tau) + \lambda\mathbb{E}[e^{A(\phi;\tau)J} - 1], \end{cases} \tag{16}$$

with the initial conditions $A(\phi; 0) = i\phi$ and $B(\phi; 0) = 0$.

It is straightforward to solve $A(\phi; \tau)$ and $B(\phi; \tau)$ by integrating both sides of ODEs (16). □

Remark 1. Once the distribution of J is specified, we can obtain an explicit formula of $B(\phi; \tau)$ in Lemma 1. As mentioned earlier, this article mainly deals with both normal and double exponential jump distributions. So we employ the jump distributions of Merton (1976) and Kou (2002) that $\tilde{J} \sim N(\mu_{\tilde{J}}, \sigma_{\tilde{J}}^2)$ and $\tilde{J} \sim Exp(p, \eta_1, \eta_2)$, respectively.[6] Then the expectation in $B(\phi; \tau)$ can be calculated as follows:

$$\mathbb{E}\left[e^{i\phi\tilde{J}} - 1\right] = \begin{cases} \exp\left\{ i\phi\mu_{\tilde{J}} - \dfrac{1}{2}\phi^2{\sigma_{\tilde{J}}}^2 \right\} - 1, & \text{if } \tilde{J}\tilde{N}\left(\mu_{\tilde{J}}, {\sigma_{\tilde{J}}}^2\right) \\ \dfrac{p\eta_1}{\eta_1 - i\phi} + \dfrac{(1-p)\eta_2}{\eta_2 + i\phi} - 1, & \text{if } \tilde{J}\widetilde{E}xp(p, \eta_1, \eta_2) \end{cases}$$

Now that we have obtained the analytic expression of the characteristic function, we can then compute the cumulants of $\ln(S_{t_T})$, which will be used to determine the truncation of the computational domain in the option pricing. The n-th cumulant of $\ln(S_T)$ is defined by[7]

$$c_n(\ln(S_T)) = \frac{1}{i^n}\frac{d^n \ln(\Phi(\phi; \ln(S_T), T))}{d\phi^n}. \tag{17}$$

Lemma 2. *Under the underlying stock price dynamics (13), the first, second and fourth cumulants of* $\ln(S_T)$, $c_1(\ln(S_T))$, $c_2(\ln(S_T))$ *and* $c_4(\ln(S_T))$, *are respectively given by*

[6] The density of the double exponential distribution is.

$$f_{\tilde{J}}(y) = p\eta_1 e^{-\eta_1 y}1_{\{y\geq 0\}} + (1-p)\eta_2 e^{\eta_2 y}1_{\{y<0\}},$$

[7] Detailed information about the cumulant generating function can be found in Section 2.2.5 of Cont and Tankov (2004).

$$c_1(\ln(S_T)) = \begin{cases} \int_0^T \left[r - \tilde{\lambda}\vartheta - \frac{1}{2}\sigma^2 - \frac{1}{2}\beta^2\nu_t^2 + \tilde{\lambda}\mu_{\tilde{J}} \right] dt, & if\ \tilde{J}\widetilde{N}\left(\mu_{\tilde{J}}, \sigma_{\tilde{J}}{}^2\right), \\ \int_0^T \left[r - \tilde{\lambda}\vartheta - \frac{1}{2}\sigma^2 - \frac{1}{2}\beta^2\nu_t^2 + \tilde{\lambda}\left(\frac{p}{\eta_1} - \frac{1-p}{\eta_2} \right) \right] dt, & if\ \tilde{J}\widetilde{E}xp(p, \eta_1, \eta_2), \end{cases}$$

$$c_2(\ln(S_T)) = \begin{cases} \int_0^T \left[\sigma^2 + \beta^2\nu_t^2 + \tilde{\lambda}\left(\mu_{\tilde{J}}^2 + \sigma_{\tilde{J}}^2\right) \right] dt, & if\ \tilde{J} \sim N\left(\mu_{\tilde{J}}, \sigma_{\tilde{J}}^2\right), \\ \int_0^T \left[\sigma^2 + \beta^2\nu_t^2 + \tilde{\lambda}\left(\frac{2p}{\eta_1^2} + \frac{2(1-p)}{\eta_2^2} \right) \right] dt, & if\ \tilde{J} \sim Exp(p, \eta_1, \eta_2), \end{cases}$$

and

$$c_4(\ln(S_T)) = \begin{cases} T\left[\tilde{\lambda}\left(\mu_{\tilde{J}}^4 + 6\mu_{\tilde{J}}^2\sigma_{\tilde{J}}^2 + 3\tilde{\lambda}\sigma_{\tilde{J}}^4 \right) \right], & if\ \tilde{\mathrm{J}} \sim \mathrm{N}\left(\mu_{\tilde{\mathrm{j}}}, \sigma_{\tilde{\mathrm{j}}}^2\right), \\ T\left[\tilde{\lambda}\left(\frac{24p}{\eta_1^4} + \frac{24(1-p)}{\eta_2^4} \right) \right], & if\ \tilde{\mathrm{J}} \sim \mathrm{Exp}(\mathrm{p}, \eta_1, \eta_2). \end{cases}$$

Proof: By Lemma 1 we can easily verify the above results. □

3. Discrete barrier option pricing with Fourier-cosine series expansions

Barrier options can be classified according to whether the underlying asset price needs to pass or to avoid a certain level to receive a payoff. In the first case they are said to be knock-in options, in the second knock-out. For simplicity, we only investigated the knock-out options in the following. All other barrier options can be priced similarly. If the underlying asset price begins below the barrier and must cross above it to cause the knock-out, the option is called up-and-out. A down-and out option has the barrier below the initial underlying asset price and knocks out when the underlying asset price falls below the barrier level. The payoff functions for different types of discretely monitored knock-out barrier options with strike K, barrier level H and maturity T are shown in Table 1.

For simplicity of notation, we write $x := \ln(S_{t_{m-1}})$ and $y := \ln(S_{t_m})$. Then the prices of the discrete knock-out barrier options with strike K, barrier level H, maturity T and monitoring dates M, satisfy the following recursive formulas:

$$(up-and-out) = \begin{cases} c(x, t_{m-1}) = e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} \upsilon(y, t_m) f(y|x) dy, \\ \upsilon(x, t_{m-1}) = \begin{cases} e^{-r(T - t_{m-1})} Rb, & \text{if } x \geq h, \\ c(x, t_{m-1}), & \text{if } x < h, \end{cases} \end{cases} \tag{18}$$

and

$$(down-and-out) = \begin{cases} c(x, t_{m-1}) = e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} \upsilon(y, t_m) f(y|x) dy, \\ \upsilon(x, t_{m-1}) = \begin{cases} e^{-r(T - t_{m-1})} Rb, & \text{if } x \leq h, \\ c(x, t_{m-1}), & \text{if } x > h, \end{cases} \end{cases} \tag{19}$$

which are followed by

$$\upsilon(x, t_0) = e^{-r(t_1 - t_0)} \int_{\mathbb{R}} \upsilon(y, t_1) f(y|x) dy, \tag{20}$$

where, r denotes the risk-free interest rate; $c(x, t)$ and $\upsilon(x, t)$ are the continuation value and option value at time t, respectively; x and y are state variables at times t_i and t_{i+1}, respectively; $f(y|x)$ is the transition probability density function of y from x under the risk-netural measure Q; $h = \ln(H)$ and $m = M, M-1, \ldots, 2$.

In general, the transition probability density function $f(y|x)$ is unknown. By utilizing Fourier-cosine series expansion and conditional characteristic function, Fang and Oosterlee (2008) proposed an approximation of the transition probability with a truncated region $[a,b]$, i.e.,

$$f(y|x) \approx \frac{2}{b-a} {\sum_{k=0}^{N-1}}' \Re\left\{ \Phi\left(\frac{k\pi}{b-a}; x, t \right) e^{-ik\pi\frac{a}{b-a}} \right\} \cos\left(k\pi \frac{y-a}{b-a} \right), \tag{21}$$

where $\sum'$ indicates that the first term of the summation is multiplied by $1/2$, $\Re\{\cdot\}$ denotes a real part of a complex number, $\Phi(\phi; x, t)$ is the conditional characteristic function of $f(y|x)$, and $[a, b]$ denotes the integration range in the original domain. For some given tolerance TOL, i.e., $\left|\int_{\mathbb{R}\setminus[a,b]} f(y|x)dy < \text{TOL}\right|$, the truncated region $[a, b]$ can be determined via the cumulants of $\ln(S_T)$.[8]

Replacing the conditional density function $f(y|x)$ in Eqs.(18)-(20) with its approximation Eq. (21), and interchanging the summation and integration operators, we have the approximation of the continuation value $c(x, t_{m-1})$ as follows:

$$\widehat{c}(x, t_{m-1}) = e^{-r\Delta t}\frac{2}{b-a}\sum_{k=0}^{N-1}{}'\Re\left\{\Phi\left(\frac{k\pi}{b-a}; x, t_{m-1}\right)e^{-ik\pi\frac{a}{b-a}}\right\}V_k(t_m), \tag{22}$$

where $V_k(t_m)$ is the Fourier-cosine series coefficients of the option value, $v(y, t_m)$, on $[a, b]$, i.e.,

$$V_k(t_m) = \frac{2}{b-a}\int_a^b v(y, t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy. \tag{23}$$

Furthermore, for exponential Lévy process, Eq. (18) can be reduced to

$$\widehat{c}(x, t_{m-1}) = e^{-r\Delta t}\sum_{k=0}^{N-1}{}'\Re\left\{\Phi\left(\frac{k\pi}{b-a}; 0, t_{m-1}\right)e^{ik\pi\frac{x-a}{b-a}}\right\}V_k(t_m). \tag{24}$$

According to this approximation, we can also approximate the value of the discrete barrier option at initial time t_0 as

$$\widehat{v}(x, t_0) = e^{-r\Delta t}\sum_{k=0}^{N-1}{}'\Re\left\{\Phi\left(\frac{k\pi}{b-a}; 0, t_0\right)e^{ik\pi\frac{x-a}{b-a}}\right\}V_k(t_1). \tag{25}$$

Next we will reveal that the $V_k(t_m)$ can be recovered from $V_k(t_{m+1})$, for $k = 0, 1, \ldots, N-1$. The result is given in the form of theorem stated below.

In addition, to simplify notation, we first write

$$\chi_k(x_1, x_2) = \int_{x_1}^{x_2} e^x\cos\left(k\pi\frac{x-a}{b-a}\right)dx \tag{26}$$

and

$$\psi_k(x_1, x_2) = \int_{x_1}^{x_2}\cos\left(k\pi\frac{x-a}{b-a}\right)dx. \tag{27}$$

Obviously, these two integrals have the following analytic expressions:

$$\chi_k(x_1, x_2) = \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \cos\left(k\pi\frac{x_1-a}{b-a}\right)e + \frac{k\pi}{b-a}sin\left(k\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \frac{k\pi}{b-a}\sin\left(k\pi\frac{x_1-a}{b-a}\right)e^{x_1}\right]x_1$$

and

$$\psi_k(x_1, x_2) = \begin{cases}\left[\sin\left(k\pi\frac{x_2-a}{b-a}\right) - \sin\left(k\pi\frac{x_1-a}{b-a}\right)\right]\frac{b-a}{k\pi}, & k \neq 0,\\ x_2 - x_1, & k = 0.\end{cases}$$

Theorem 1. *(Recursion formula for coefficients $\widehat{V}_k(t_m)$) For $m = M-1, M-2, \ldots, 1$, the numerical approximation of $V_k(t_m)$ can be expressed as*

$$\widehat{V}_k(t_m) = \begin{cases}\widehat{C}_k(a, h, t_m) + e^{-r(T-t_m)}Rb\frac{2}{b-a}\psi_k(h, b), & up-and-out,\\ e^{-r(T-t_m)}Rb\frac{2}{b-a}\psi_k(a, h) + \widehat{C}_k(h, b, t_m), & down-and-out,\end{cases} \tag{28}$$

where $\widehat{C}_k(x_1, x_2, t_m)$ is approximated by

[8] For more detailed information about the error analysis on the convergence rate, one can refer to Fang and Oosterlee (2008, 2009, 2011).

$$\widehat{C}_k(x_1,x_2,t_m) = e^{-r\Delta t}\Re\left\{\sum_{j=0}^{N-1}{}'\Phi\left(\frac{j\pi}{b-a};0,t_m\right)\widehat{V}_j(t_{m+1})\mathscr{M}_{k,j}(x_1,x_2)\right\}, \tag{29}$$

with the coefficients $\mathscr{M}_{k,j}(a,h)$ defined as

$$\mathscr{M}_{k,j}(x_1,x_2) = \frac{2}{b-a}\int_{x_1}^{x_2} e^{ij\pi\frac{x-a}{b-a}}\cos\left(k\pi\frac{x-a}{b-a}\right)dx. \tag{30}$$

For $m = M$, we have

1)when $H < K$

$$V_k(t_M) = \begin{cases} \frac{2}{b-a}Rb\psi_k(h,b), & for\ up-and-out\ call, \\ \frac{2}{b-a}[Rb\psi_k(a,h)+\chi_k(\ln K,b)-K\psi_k(\ln K,b)], & for\ down-and-out\ call, \\ \frac{2}{b-a}[K\psi_k(a,h)-\chi_k(a,h)+Rb\psi_k(h,b)], & for\ up-and-out\ put, \\ \frac{2}{b-a}[Rb\psi_k(a,h)+K\psi_k(h,\ln K)-\chi_k(h,\ln K)], & for\ down-and-out\ put. \end{cases} \tag{31}$$

2)*when* $H \geq K$

$$V_k(t_M) = \begin{cases} \frac{2}{b-a}[\chi_k(\ln K,h)-K\psi_k(\ln K,h)+Rb\psi_k(h,b)], & for\ up-and-out\ call, \\ \frac{2}{b-a}[Rb\psi_k(a,h)+\chi_k(h,b)-K\psi_k(h,b)], & for\ down-and-out\ call, \\ \frac{2}{b-a}[K\psi_k(a,\ln K)-\chi_k(a,\ln K)+Rb\psi_k(h,b)], & for\ up-and-out\ put, \\ \frac{2}{b-a}Rb\psi_k(a,h), & for\ down-and-out\ put. \end{cases} \tag{32}$$

Proof: From Table 1, we have the following option values, $v(y,T)$, at time T:

$$v(y,T) = \begin{cases} \left[(\alpha(e^y-K))^+ - Rb\right]1_{\{y_{t_i}<h\}} + Rb, & \text{for up}-\text{and}-\text{out}, \\ \left[(\alpha(e^y-K))^+ - Rb\right]1_{\{y_{t_i}>h\}} + Rb, & \text{for down}-\text{and}-\text{out}, \end{cases} \tag{33}$$

where $\alpha = 1$ for a call option and $\alpha = -1$ for a put option.

We begin with an up-and-out option. By Eq. (23) we have

$$\begin{aligned} V_k(t_M) &= \frac{2}{b-a}\int_a^b v(y,t_M)cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^b \{[(\alpha(e^y-K))^+ - Rb]1_{\{y<h\}} + Rb\}cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^h (\alpha(e^y-K))^+ cos\left(k\pi\frac{y-a}{b-a}\right)dy + \frac{2}{b-a}Rb\int_h^b cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^h (\alpha(e^y-K))^+ cos\left(k\pi\frac{y-a}{b-a}\right)dy + \frac{2}{b-a}Rb\psi_k(h,b) \end{aligned} \tag{34}$$

When $H < K$, the first term on the right-hand side of Eq. (34) is equal to

$$\begin{aligned} \int_a^h (\alpha(e^y-K))^+ cos\left(k\pi\frac{y-a}{b-a}\right)dy &= \begin{cases} 0, & \text{for } \alpha = 1, \\ \int_a^h (K-e^y)cos\left(k\pi\frac{y-a}{b-a}\right)dy, & \text{for } \alpha = -1, \end{cases} \\ &= \begin{cases} 0, & \text{for } \alpha = 1, \\ K\psi_k(a,h)-\chi_k(a,h), & \text{for } \alpha = -1. \end{cases} \end{aligned} \tag{35}$$

When $H \geq K$, the first term on the right-hand side of Eq. (34) is equal to

$$\begin{aligned}\int_a^h (\alpha(e^y-K))^+ \cos\left(k\pi\frac{y-a}{b-a}\right)dy &= \int_a^{\ln K} (\alpha(e^y-K))^+ \cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= +\int_{\ln K}^h (\alpha(e^y-K))^+ \cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \begin{cases}\displaystyle\int_{\ln K}^h (e^y-K)\cos\left(k\pi\frac{y-a}{b-a}\right)dy, & \text{for } \alpha=1,\\ \displaystyle\int_a^{\ln K} (K-e^y)\cos\left(k\pi\frac{y-a}{b-a}\right)dy, & \text{for } \alpha=-1,\end{cases} \\ &= \begin{cases}\chi_k(\ln K,h)-K\psi_k(\ln K,h), & \text{for } \alpha=1,\\ K\psi_k(a,\ln K)-\chi_k(a,\ln K), & \text{for } \alpha=-1.\end{cases}\end{aligned} \tag{36}$$

For $m = M-1, M-2, \ldots, 1$, we employ Eqs. (18), (22) and (23) to obtain

$$\begin{aligned}V_k(t_m) &= \frac{2}{b-a}\int_a^b \upsilon(y,t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^h \upsilon(y,t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy + \frac{2}{b-a}\int_h^b \upsilon(y,t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^h c(y,t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy + \frac{2}{b-a}\int_h^b e^{-r(T-t_m)}Rb\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= C_k(a,h,t_m) + e^{-r(T-t_m)}Rb\frac{2}{b-a}\psi_k(h,b),\end{aligned} \tag{37}$$

where

$$C_k(a,h,t_m) = \frac{2}{b-a}\int_a^h c(y,t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy. \tag{38}$$

To compute the coefficients of $V_k(t_m)$, we need to approximate the continuation values $c(x,t_m)$ at time pints $t_{M-1}, t_{M-2}, \ldots, t_1$. Note that at time t_M the coefficients, $V_k(t_M)$, are exact values which can be determined by Eq. (31). Hence, we can get the approximation $\widehat{c}(x,t_{M-1})$ of the continuation value at time t_{M-1} based on Eq. (22). Substituting the approximation $\widehat{c}(x,t_{M-1})$ to Eq. (38) and interchanging summation and integration operators yields

$$\widehat{C}_k(a,h,t_{M-1}) = e^{-r\Delta t}\Re\left\{{\sum_{j=0}^{N-1}}{}'\Phi\left(\frac{j\pi}{b-a};0,t_{M-1}\right)V_j(t_M)\mathscr{M}_{k,j}(a,h)\right\}, \tag{39}$$

where the coefficients $\mathscr{M}_{k,j}(a,h)$ is defined by the Eq. (30).

Finally, we can obtain the approximation $\widehat{V}_k(t_{M-1})$ from Eqs.(37)-(39). By analogy, the coefficients of $\widehat{V}_k(t_m)$ at time points t_m, $m = M-2, M-3, \ldots, 1$, can also be calculated.

For a down-and-out option, the approximation $\widehat{V}_k(t_m)$ can be derived in a similar way. □

Once the recursion formula for the coefficients $\widehat{V}_k(t_m)$ is derived, we can move backwards in time and find the equation that connect $\widehat{V}_k(t_{m-1})$ and $\widehat{V}_k(t_m)$ for $m = M-1, M-2, \ldots, 2$. With the approximation of $\widehat{V}_k(t_1)$, we substitute it into Eq. (25) to calculate the option price $\widehat{\upsilon}(x,t_0)$. Up to now, on the basis of the Fourier-cosine series expansion method we have presented an analytical pricing formulas for the discretely monitored barrier options under the jump-diffusion model with liquidity risk.

Remark 2. In practice, it is easier to implement by writing $\widehat{V}_k(t_m)$ into vector form $\widehat{\mathbf{V}}(t_m)$. However, this matrix-vector costs $O(N^2)$ operations and is so expensive. Based on the FFT algorithm, Fang and Oosterlee (2009) proposed an efficient algorithm with complexity of $O(N\log_2(N))$ to compute $\widehat{\mathbf{V}}(t_m)$. In this paper, we will also apply the algorithm proposed by Fang and Oosterlee (2009) to the calculation of $\widehat{\mathbf{V}}(t_m)$. Meanwhile, the COS method for pricing discretely monitored barrier options is summarized in Appendix A.

4. Numerical analysis

In this section, we will present the numerical results to show the accuracy of our analytical approximation pricing formulas. Then we perform sensitivity analysis to investigate the price changes of the discrete barrier options with respect to model input parameters. In addition, we make a thorough inquiry into the rates of convergence of the COS discrete barrier option prices with regard to monitoring frequencies and the terms of Fourier-cosine series. For the sake of simplicity, we only perform numerical analysis on the discretely monitored up-and-out call barrier option. All other barrier options can be analyzed in the same way. On the other hand, for analytical tractability, we assume that the market liquidity level ν_t is a constant function, which can be viewed as an observed variable.

4.1. Truncation of the computational domain

The choice of the computational domain $[a,b]$ will directly affect the pricing performance of the COS method. A domain size that is too small will lead to low accuracy of option pricing. However, larger domains would require more terms N in the Fourier-cosine series expansion to achieve a certain level of accuracy. We next choose $N = 2^{15}$ in our numerical experiment.

Following Fang and Oosterlee (2009) we select the integral interval $[a,b]$ as

$$[a,b] = \left[c_1 + x_0 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + x_0 + L\sqrt{c_2 + \sqrt{c_4}}\right], \tag{40}$$

where $x_0 = \ln S_0$, $L = 15$ and c_n is the n-th cumulant of $\ln S_T$ defined by Lemma 2. Furthermore, Fang and Oosterlee (2009) had investigated that the COS method had an exponential convergence in N for density functions in $C^\infty[a,b]$.

4.2. Comparison of the analytical approximation solutions against Monte Carlo simulation

In this subsection, we compare the performance of the analytical approximation formulas with that of the Monte Carlo simulation (hereafter, MC). We make the comparison across different monitoring frequencies M, for $M = 4, 12, 26, 52$ and 252, which correspond to quarterly, monthly, biweekly, weekly and daily monitoring setups, respectively. We also decompose the option maturity into $252 \times 13 = 3276$ subintervals in the MC simulation such that the monitoring underlying asset price can always be acquired from the simulation. Moreover, the number of simulation paths is set to $500,000$ in the numerical experiment. The model input parameters are summarized in Table 2.

Table 3 presents the relative price differences between the analytical approximation solution and the simulation price across various jump distributions and monitoring frequencies. Note that our formula, as an analytical approximation solution for discrete barrier option is very close to the simulation results. The relative errors are smaller than 0.2% in all cases of monitoring frequencies M, which indicate that our analytical approximation solution has a high degree of accuracy.[9] In order to show that the results are more robust, we have also plot the relative price differences across different strike prices K and monitoring frequencies M (see Fig. 1). Note from Fig. 1 that the relative price differences are less than 0.35% under both normal and double exponential jumps.

4.3. Sensitivity analysis of option price on liquidity and jumps factors

As previously mentioned, classical option pricing theories are usually built on the paradigm of competitive and frictionless markets, while ignoring the impact of market liquidity on asset prices. Therefore, we advocate the use of liquidity-adjusted jump-diffusion model to describe the underlying asset price dynamics. It is natural to study the impacts of these parameters change in value on the prices of the discrete barrier options. The model input parameter values in Table 2 are served as a base case and altered within a specified range to calculate different barrier option prices. Additionally, we set the rebate $Rb = 0$ for discretely monitored barrier option. The corresponding results are shown in Figs. 2–6.

Fig. 2 plots the discrete up-and-out call barrier option prices with respect to β under the different jump distributions and monitoring frequencies M. Note that, as expected, the option prices decrease with increasing β. The reason for this is that as the value of β increase, the stock's expected return increasingly fluctuates with market illiquidity. This will make the underlying asset price at the monitoring dates more possible to over the barrier level H, and thus the barrier option is invalid.

Fig. 3 plots the discrete up-and-out call barrier option prices with respect to ν_t under the different jump distributions and monitoring frequencies M. It is not surprising that the option prices decrease with the increase of the absolute value of ν_t due to the variance for the asset return increases with the absolute value of ν_t.[10]

Figs. 4–6 plot the discrete up-and-out call barrier option prices with respect to the distribution parameters of both the normal and double exponential jumps. As seen from these figures, the option prices decrease with the jump intensity λ, the mean of the jump magnitudes μ_J, and the variance of the jump magnitudes σ_J. These are also reasonable as increasing the above jump parameters can make more variability into the underlying asset price dynamics, which in turn make the underlying asset price at the monitoring dates more possible to over the barrier level H. However, Fig. 6 shows that the option prices increase with the parameters η_i, for $i = 1, 2$, which are mainly because of the density function of the double exponential distribution becomes more leptokurtosis and heavy tails as η_i increasing.

[9] The relative error are defined as.

$$\text{Relative error} = \frac{|v_a - v_s|}{v_s} \times 100\%,$$

[10] For more detailed information on how the $\nu_t \neq 0$ significantly affect the expected returns, variances and correlations of the asset price processes, one can refer to Brunetti and Caldarera (2004).

In Fig. 7 we plot the convergence of our formula in terms of Fourier-cosine series expansion for pricing discrete barrier options under both normal and double exponential jump distributions. It can be seen clearly from Fig. 7 that the option prices under each of the monitored dates tend to be pretty stable when $N > 200$. Finally, we also report the rate of convergence of the discrete up-and-out call barrier option prices across varying monitoring frequencies in Fig. 8. The prices of discrete barrier option under both normal and double exponential jumps tend to be stable for weekly monitoring dates and above.

5. Empirical studies

In this section, we empirically compare the pricing performance of the proposed liquidity-adjusted models with those of the classical jump-diffusion models of Merton (1976) and Kou (2002) being taken as benchmark models.

5.1. Empirical evidence

First, we will briefly conduct the statistical analysis to illustrate the reasonability of our proposed jump-diffusion model with liquidity risk, which is used to price discrete barrier options. To be more precise, we employ a data set consisting of SSE Composite Index traded at the Shanghai Stock Exchange to perform the empirical analysis. The SSE Composite Index data set spans the time period from Jan 4, 2007 to Nov 16, 2017. All data set are obtained from Chinese Wind Financial Database.

Fig. 9 reports the statistic properties of the sample data of the SSE Composite Index. Specifically, the top left panel of Fig. 9 shows the daily closing prices of the SSE Composite Index in the sample period. The top right panel of Fig. 9 illustrates the returns associated with the time series in the top left panel. The Quantile-Quantile plot in the bottom left panel of Fig. 9 shows that the distribution of the log returns indeed exhibits fat tails and obviously deviates from the normality assumption. Meanwhile, the bottom right panel of Fig. 9 plots the kernel density estimation of the daily returns for the SSE Composite Index and a corresponding normal distribution is depicted by the red dotted line. It also indicates that the distribution of the returns is fat-tailed, slightly negatively skewed and more highly peaked than the normal distribution. Moreover, we can obviously find from Fig. 9(a) that jumps in the sample data, sometimes the SSE Composite Index jumps up, and sometimes down. It is well known that the jump-diffusion model can better capture the leptokurtosis and fat-tail features of the underlying assets. Thus we need to take this financial phenomenon into account in discrete barrier option pricing framework.

Fig. 10 shows the time series for the liquidity of SSE Composite Index from Jun 2, 1995 to Nov 3, 2017. Here, we use a simple measure to describe liquidity level as follow

$$\mathrm{LIQ}_t = \frac{R_t}{DVol_t}, \tag{41}$$

where R_t and $DVol_t$ denote the return rate and RMB trading volume of the underlying stock at time t, respectively. Although this liquidity measure is similar with the ILLIQ measure proposed by Amihud (2002), it can capture a sudden rise or drop of a stock's price with modest trading volume.

From Fig. 10 we can intuitively see that the market liquidity level seems to be related to the return rate (see Fig. 9(b)). Further on, we apply a quantitative research to models the dependence structure between stock return and market liquidity through copula functions, which represent a flexible instrument for modeling the dependence structure between variables.[11] The results on the dependence structure between the return and liquidity of SSE Composite Index are shown in Table 4. Note that the correlation coefficient ρ is 0.9268 and 0.9226 under Gaussian copula and T-copula, respectively. As expected, the stock returns are influenced by liquidity risk and commonality in liquidity, and the expected returns are positive related to the level of stock illiquidity.[12]

5.2. Empirical comparison

Due to the shortage of readily available real market data of Barrier options, it is not possible to calibrate related models and then compare the pricing performance of them. To address this problem, we construct the market barrier option by the market price of European option and related barrier level. More precisely, we will resort to the artificial barrier option data implied by market data on SSE 50 ETF options and the barrier level (set according to the real market price) for the performance comparison. Here, the SSE 50 ETF option is a standardized European option listed on the Shanghai Stock Exchange. For simplicity, we take example of the up-and-out call option and select the monitoring frequencies $M = 252$, i.e. daily monitoring setup, which is actually to be able to use more sample data for analysis. Let $\mathrm{BCall}(t, K, T)$ denote the price of SSE 50 ETF up-and-out call option at time t_m with strike K, barrier level H and maturity T, and then

$$\mathrm{BCall}(t_m, K, T) = \begin{cases} \mathrm{Call}(t_m, K, T), & \text{if } S_{t_i, i\in[m,M]} < H, \\ 0, & \text{if } S_{t_i, i\in[m,M]} \geq H, \end{cases}$$

where $\mathrm{Call}(t_m, K, T)$ is the price of SSE 50 ETF European call option, and S_{t_i} is the price of the underlying asset SSE 50 ETF. Based on the

[11] The concept of copula function was first introduce by Sklar (1959), but it has only gained increasing popularity in economics and finance fields over the last decade.

[12] For in-depth analysis on the effect of market liquidity on expected stock return, one can refer to the literature cited in footnote 1.

sample data of the underlying asset SSE 50 ETF, we select the barrier level $H = 3.5$.

We use a data set consisting of the SSE 50 ETF and the European call options written on SSE 50 ETF traded at the Shanghai Stock Exchange. The data set of SSE 50 ETF European call options spans the time period from January 1, 2018 to March 31, 2018. All option data are from Chinese Wind Financial Database. Moreover, a filtering scheme is applied to construct our empirical option data set before the raw data is used to calibrate the model parameters. Hence, we first follow the existing literature (see, for example, Bakshi, Cao, and Chen (1997), Christoffersen, Jacobs, and Mimouni (2006), Christoffersen, Feunou, and Jeon (2015), He and Zhu (2016), Kiesel and Rahe (2017) and Li, Zhang, and Liu (2018)) and utilize the Wednesday and Thursday options data to perform our empirical analysis. For the in-sample analysis, we use the Wednesday options data to calibrate the model parameters. Meanwhile, the Thursday options data serves as the real market price to be compared with the predicted price calculated by the estimated parameters in-sample. Second, we restrict attention to the options for which the moneyness is between 0.7 and 1.3 and the time to maturity is between 10 and 365 trading days.[13] After these filters, we have a total of 1665 observations. Finally, we select the one-year Shibor as the risk-free interest rate.[14]

In our calibration process, the risk-neutral model parameters are backed out by minimizing a loss function capturing the fit between the theoretical and market proxies quanto prices. We employ the root mean squared errors (RMSE) as the objective function, i.e.

$$\mathrm{RMSE}(t) = \underset{\Theta}{\mathrm{argmin}} \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} \left| \mathrm{BCall}_{i,Market}(t, K_i, T_i) - \mathrm{BCall}_{i,Model}(t, K_i, T_i) \right|^2}, \tag{42}$$

where N_t is the total number of observation at time t, $\mathrm{BCall}_{i,Market}(t, K_i, T_i)$ denotes the market proxy price of the up-and-out call option contract from the sample, $\mathrm{BCall}_{i,Model}(t, K_i, T_i)$ represents the theoretical up-and-out call option price and the vector of model input parameters Θ.

In addition, it is important to determine what liquidity measure is used to describe the market liquidity level in the calibration process. In this paper, the liquidity is defined as the ability of an asset to trade any amount of securities quickly at the market price without additional transaction cost and within a short of time. To be consistent with this definition of liquidity, we adopt the liquidity measure defined as the return divided by the RMB trading volume in empirical studies, i.e. LIQ_t (see Eq. (41)). As mentioned above, this liquidity measure can be viewed as the modified liquidity measure of Amihud (2002) and has also been used and discussed in other literature, such as Brunetti and Caldarera (2004), Cao and Wei (2010), Feng et al. (2014, 2016) and Li et al. (2018). Theoretically, the measure of market liquidity should be calculated by an average of the liquidity measures of all stocks traded in financial market. In view of this, we use the SSE Composite Index as a proxy to compute the market liquidity on account of it is a stock market index of all stocks (including A shares and B shares) that are traded at the Shanghai Stock Exchange.

We next exhibit the empirical results on the pricing performance of our proposed liquidity-adjusted models and two types of classical jump-diffusion models, i.e., Merton (1976) and Kou (2002) models. For simplicity of notation, we denote the liquidity-adjusted normal and double exponential jump-diffusion models as L-Merton and L-Kou models, respectively. On the basis of the above calibration method (see, Eq. (42)), Table 5 shows the estimated daily averaged parameters for L-Kou, Kou (2002), L-Merton and Merton (1976) models, which are obtained by minimizing the root mean squared pricing errors between the market price and the model-determined price for each option on Wednesday during the sample period from January 1, 2018 to March 31, 2018. Meanwhile, standard errors are reported in parentheses. It is worth noting that the calibrated values, β, of 0.487700 and 0.434777 for L-Kou and L-Merton models, respectively, are positively associated with the sensitivity of the stock expected return to the market liquidity, which are consistent with the results in Table 4. In other words, the effect of market liquidity on the underlying asset price takes an important factor in barrier option pricing.

Table 6 reports the results of in-sample pricing performance for four different models in terms of the error metric, RMSE. From the perspective of in-sample pricing errors, the daily averaged RMSE for the liquidity-adjusted double exponential jump-diffusion model (L-Kou model) is only 0.003989, compared with 0.004260 for the Kou (2002) model. Similarly, the daily averaged RMSE for the liquidity-adjusted normal jump-diffusion model (L-Merton model) is only 0.004138, compared with 0.004529 for the Merton (1976) model. It is clear that whether the jump magnitudes follow the normal distribution or the double exponential distribution, the pricing performance of our proposed model is superior to that of the Kou (2002) or Merton (1976) model. On the other hand, options are traded with a wide range of strikes in real financial market and thus it is important to check the pricing errors across by different moneyness groups, which are also shown in Table 6. Note that the pricing error RMSE is reduced not only overall but also for all moneyness groups.

As previously mentioned, to see whether these results hold in a true out-of-sample case, we employ the option prices data in the previous day to calibrate the model parameters, and then utilize them as inputs to calculate the model-based option prices for the current day. The out-of-sample pricing errors are shown in Table 7. It is no surprise that the out-of-sample pricing errors are always larger than that for in-sample pricing errors, which is mainly due to model prices are computed by determined parameters. However, when out-of-sample pricing errors are taken into consideration, a similar pattern emerges; our liquidity-adjusted jump-diffusion models still show a much better performance than that of both Merton (1976) and Kou (2002) models. More precisely, the

[13] The moneyness is defined as the stock price over the strike, i.e. Moneyness $= S_t/K$. Moreover, we refer to a call option as out-of-money (OTM) if $S_t/K < 0.97$; at-the-money (ATM) if $0.97 \le S_t/K \le 1.03$; and in-the-money (ITM) if $S_t/K > 1.03$.

[14] The one-year Shibor can be obtained from the web page: http://www.shibor.org/.

daily averaged RMSE for the L-Kou and L-Merton models are only 0.006300 and 0.006713, compared with 0.006518 and 0.006915 for the Kou (2002) and Merton (1976) models, respectively. To measure the extent to which model is better or worse than another, we define the improvement rate as the average of the relative differences between the pricing errors from the benchmark model and our proposed liquidity-adjusted quanto model.[15] For double exponential jump-diffusion models (i.e., L-Kou and Kou models), there would be 7.100589% of total improvement if our model is used. Meanwhile, for normal jump-diffusion models (i.e., L-Merton and Merton models), there would be 6.494456% of total improvement if our model is used. Additionally, we can see from Table 7 that the out-of-sample pricing errors of the L-Kou model is lower than that of the L-Merton model across all moneyness groups. From the perspective of improvement rate, the pricing performance of the L-Kou model has a 3.923218% improvement over the L-Merton model.

Putting the above empirical results together, we can see that the proposed model extension of incorporating market liquidity factor into the barrier option pricing problem significantly improves the pricing accuracy. In other words, we really should study the barrier option pricing problem in the presence of jumps and liquidity cost. Of course, this conclusion is based on the empirical test of one set of option data. It is quite possible that the pricing performance of these four models may reverse with some other data sets. Nevertheless, our empirical results exhibited here can at least suggest that it may offer as a good competitor of the Merton (1976) or Kou (2002) model for some other option markets, such as commodity and futures exchange markets.

6. Conclusion

The issue of market liquidity has drawn much attention among academic researchers, institutional professionals and financial regulators in various financial markets. However, there are few literature on how to incorporate liquidity costs into option pricing. In this paper, we extend the classical jump-diffusion models of Merton (1976) and Kou (2002) to include liquidity risk. We then present the analytical approximation pricing formulas for discrete barrier options utilizing the Fourier-cosine series expansion method proposed in Fang and Oosterlee (2008). Numerical experiments demonstrate the accuracy of the proposed liquidity-adjusted jump-diffusion model by comparing the analytical approximation solutions with Monte Carlo simulation. Moreover, we also investigate the price sensitivity of the discrete barrier options to the model input parameters. To better illustrate the reasonability of our proposed pricing model, we also briefly perform statistical analysis by using the sample data of SSE Composite Index traded at the Shanghai Stock Exchange. Finally, empirically studies are carried out to show the superiority of our model based on SSE 50 ETF options. The numerical and empirical results support our idea of introducing liquidity risk and jumps into the underlying asset price dynamics. In other words, it is important to take liquidity effect and jump into account in discrete barrier option pricing problem.

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Appendix B. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.iref.2018.10.002.

Appendix A. Algorithm for pricing discrete barrier options

[15] The improvement rate is defined as.

$$\text{Improvement rate} = \frac{1}{M}\sum_{i=1}^{M}\frac{\text{RMSE}_{i,1} - \text{RMSE}_{i,2}}{\text{RMSE}_{i,1}} \times 100\%,$$

Algorithm 1 Pricing discrete barrier options based on the COS method

1: Set $x_1 = a$, $x_2 = h$, $c = h$ and $d = b$ for up-and-out; $x_1 = h$, $x_2 = b$, $c = a$ and $d = h$ for down-and-out.

2: Construct

$$\mathbf{m}_s(x_1, x_2) = [m_0, m_{-1}, \ldots, m_{-(N-1)}, 0, m_{N-1}, m_{N-2}, m_1]$$

and

$$\mathbf{m}_c(x_1, x_2) = [m_{2N-1}, m_{2N-2}, \ldots, m_1, m_0]$$

by

$$m_j = \begin{cases} \dfrac{(x_2 - x_1)\pi}{b-a}\mathrm{i}, & \text{if } j = 0, \\ \dfrac{\exp\left(\mathrm{i}j\frac{(x_2-a)\pi}{b-a}\right) - \exp\left(\mathrm{i}j\frac{(x_1-a)\pi}{b-a}\right)}{j}, & \text{if } j \neq 0. \end{cases}$$

3: Compute $d_1 = \mathrm{DFT}\{\mathbf{m}_s(x_1, x_2)\}$ and $d_2 = \mathbf{sgn} \cdot \mathrm{DFT}\{\mathbf{m}_c(x_1, x_2)\}$, where DFT denotes the discrete Fourier transform and $\mathbf{sgn} = [1, -1, 1, -1, \ldots]_{1\times 2N}$.

4: Compute $\mathbf{G} = \frac{2}{b-a} Rb\{\psi_k(c, d)\}_{k=0}^{N-1}$.

5: Compute $\hat{\mathbf{V}}(t_M) = \{V_k(t_M)\}_{k=0}^{N-1}$ using Eq.(27) or (28).

6: **for** $m = M$ to 2 **do**

7: Compute $\mathbf{u}(t_m) = \{u_j(t_m)\}_{j=0}^{N-1}$ by

$$u_j(t_m) = \begin{cases} \dfrac{1}{2}\Phi(0; x, t_m)V_0(t_m), & \text{if } j = 0, \\ \Phi(\dfrac{j\pi}{b-a}; 0, t_m)V_j(t_m), & \text{if } j \neq 0. \end{cases}$$

8: Construct $\mathbf{u_s} = [\mathbf{u}(t_m), \underbrace{0, 0, \ldots, 0}_{N}]$.

9: $\mathbf{Msu}$ = the first N elements of $\mathrm{iDFT}\{d_1 \cdot \mathrm{DFT}\{\mathbf{u_s}\}\}$, where $\mathbf{u_s}$ denotes the inverse discrete Fourier transform.

10: $\mathbf{Mcu}$ = reverse{the first N elements of $\mathrm{iDFT}\{d_2 \cdot \mathrm{DFT}\{\mathbf{u_s}\}\}$}.

11: $\hat{\mathbf{C}}(t_{m-1}) = \frac{e^{-r(t_m - t_{m-1})}}{\pi}\mathrm{Im}\{\mathbf{Msu} + \mathbf{Mcu}\}$, where $\mathrm{Im}\{\cdot\}$ denotes the imaginary part of a complex number.

12: Update $\hat{\mathbf{V}}(t_{m-1}) = \hat{\mathbf{C}}(t_{m-1}) + e^{-r(T-t_{m-1})} \cdot \mathbf{G}$

13: **end for**

14: Calculate the option price $\hat{v}(x, t_0)$ by Eq.(21).

where $p \in (0, 1)$, $\eta_1 > 1$ and $\eta_2 > 0$.where v_a and v_s denote the discrete barrier option prices calculated by the analytical formula (25) and Monte Carlo simulation, respectively.where M is the total trading days in out-of-sample period, $\mathrm{RMSE}_{i,1}$ and $\mathrm{RMSE}_{i,2}$ denote the RMSE implied by benchmark model and our proposed liquidity-adjusted model, respectively, at ith trading date.

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