An enhanced algorithm for calculating the degree of greyness of interval grey numbers and its application

Shuaishuai Geng
College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, China

Yu Feng
Nanjing University of Aeronautics and Astronautics, Nanjing, China

Yaoguo Dang and Junjie Wang
College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, China, and

Rizwan Rasheed
Department of Architecture and Built Environment, University of Nottingham University Park Campus, Nottingham, UK

Received 20 October 2018
Revised 29 May 2019
23 July 2019
29 July 2019
Accepted 31 July 2019

Abstract

Purpose – This paper aims to propose an enhanced algorithm and used to decision-making that specifically focuses on the choice of a domain in the calculation of degree of greyness according to the principle of grey numbers operation. The domain means the emerging background of interval grey numbers, it is vital for the operational mechanism of such interval grey numbers. However, the criteria of selection of domain always remain same that is not only for the calculated grey numbers but also for the resultant grey numbers, which can be assumed as unrealistic up to a certain extent.

Design/methodology/approach – The existence of interval grey number operation based on kernel and the degree of greyness containing two calculation aspects, which are kernel and the degree of greyness. For the degree of greyness, it includes concepts of domain and calculation of the domain. The concepts of a domain are defined. The enhanced algorithm is also comprised of four deductive theorems and eight rules that are linked to the properties of the enhanced algorithm of the interval grey numbers based on the kernel and the degree of greyness.

Findings – Aiming to improve the algorithm of the degree of greyness for interval grey numbers, based on the variation of domain in the operation process, the degree of greyness of the operation result is defined in this paper, and the specific expressions for algebraic operations are given, which is relevant to the kernel, the degree of greyness and the domain. Then, these expressions are used to the algorithm of interval grey numbers based on the kernel and the degree of greyness, improving the accuracy of the operation results.

Originality/value – The enhanced algorithm in this paper can effectively reduce the loss of information in the operation process, so as to avoid the situation where the decision values are the same and scientific decisions cannot be made during the grey evaluation and decision-making process.

Keywords Kernel, Degree of greyness, Interval grey numbers, Domain

Paper type Research paper

Kybernetes
Vol. 49 No. 6, 2020
pp. 1721-1740

0368-492X
DOI 10.1108/K-10-2018-0562

1. Introduction

The grey system theory is a vital method as proposed by Deng (1982), it has been highly effective for solving an uncertain problem for which information is partially known and partly unknown. This method can undoubtedly extract valuable information from a limited amount of data it has been widely applied in many areas such as; agriculture (Liu *et al.*, 2016), agroforestry (Wang *et al.*, 2016), petrochemical projects (Heravi *et al.*, 2016) and economy (Xu *et al.*, 2015). Interval grey numbers is an important component part and key data type in grey system theory, it has been pre-assumed that interval grey numbers are meant to carry a value within a definite range, but exact value is yet unknown. In other words, it is an uncertain number that holds an exact value within an interval or in a general number set. However, there is enormous potential for further exploration of interval grey numbers, their calculation systems and their application towards various policymaking, planning and management theories. While traditional calculation of interval grey numbers may result in loss of information or rendering it more complex to understand that can lead to further lengthy expansion of this numbering system. Likewise, this traditional system of interval grey may further intensify the calculation problems rather simplifying numbers, due to the fact that the all domain parameters are not always similar, which do not only affect credibility of the calculated grey numbers but also to the integrity of the derived findings and results (Yang and Liu, 2011).

Since Yang and Liu (2011) has defined the degree of greyness as a measure of degree of uncertainty of the grey numbers according to the utility of geometric features as a function of possibility. L. Deng (1985) further explained the entropy of the grey numbers, which is used to describe the degree of greyness, and it has discussed the properties of a degree of greyness for grey numbers. F. Liu (1996) highlighted the simplified degree of greyness with respect to the grey interval length and its underlying meanings. However, the degree of greyness of a grey number cannot be determined if the mean of the grey numbers is zero. Based on this finding, the researchers have realized the uncertainty measure to be tackled while working with grey numbers. Then, Liu and Lin proposed the calculation measures, which are research studies can use to find out the degree of greyness, whereas it also needs to relate with the background or domain value for creation of more valid grey numbers (Liu *et al.*, 2016). Meanwhile, four axioms have been presented by Liu and Lin, which are included in the axiomatic system for finding the degree of greyness more conclusively. Then, many scholars have attempted to build the arithmetic system of interval grey numbers, such as F. Liu (1987) tried to improve the grey numbers algorithms by using the mean of grey number system. Later, Fang *et al.* (2005) have devised the standard interval grey number and relevant operational rules, which has solved many calculation problems up to an extent. However, various new rules did not validate all calculation impacts because of the computational complexities. Later, Xie adequately discussed the arithmetic of grey number system, but his findings without any reasonable formulas (Xie and Xin, 2014). While such shortcomings were addressed by Liu *et al.* (2010), who defined and formulated a system called “Kernel”, along-with subsequent degree of greyness, on which a new algorithm of interval grey numbers has been established (Jiang *et al.*, 2017). Later on Yang *et al.*, 2012 further studied and refined the general grey numbers to develop a more sophisticated algebraic system. Prominently, the algorithm introduce by Liu *et al.* is more applied and convenient for many areas of the grey system theory such as the interval grey number ranking, grey incident analysis, interval grey number prediction and multi-attribute decision-making, etc (Zeng *et al.*, 2019; Wang *et al.*, 2017; Ye *et al.*, 2016; Ma and Liu, 2018; Ma *et al.*, 2019). While in formulation of an algorithm of interval grey numbers, the maximum number of the degree of greyness (i.e. among two-interval grey numbers) is taken as degree of greyness of the operational results. This algorithm simplifies the calculation process, but it lacks rigorous mathematic proofing. Therefore, it has been further needed that operational results regarding degree of greyness must be analyzed more deeply i.e. according to the latest development of grey system theory. While it has been assumed that all operational parameters are carried out in the same domain within which interval grey numbers have been created. Although this assumption facilitates the analysis process, it is still irrelevant for the actual circumstances. For example, when two-interval grey numbers are used to estimate and represent the heights (centimeters) of a male and a child respectively. While according to the real situation these should be analyzed separately within the domain of two-interval grey numbers respectively. Hence, it is evident that the domain of two-interval grey numbers is involved in the operational computations may differ from each other. Moreover, the domain about an interval grey number must have a realistic consideration of all possible values for the grey number to its essence. Therefore, when two-interval grey numbers are being brought into an algebraic operation their domain should also be adjusted concisely.

The relative study of interval grey numbers has already become an emerging topic of research and merged as a vital tool for grey decision-making process. However, it is still limited to an imperfect arithmetic system and many scholars prefer to use a precise number rather than the interval grey numbers when constructing a decision model. Indeed, many scholars have succeeded to solve the problem of the decision-making within interval grey numbers from different perspectives. While deeply looking into and analyzing all such phenomenon and limitations, the current analytical study has exclusively focused to address the degree of greyness of operational results and relevant application of interval grey numbers. As such Gholami and Brennan (2016) have proposed a dynamic grey targets decision-making method based on error propagation, and they used the orthogonal projection method to obtain the ranking of alternatives). Then, Wu *et al.*, 2013 ranked the alternatives in accordance with classical TOPSIS method and expressed the criteria values as EGNs. Later, Xie and Xiao (2011) developed a novel grey MADM method by transforming linguistic scale of rating supplier selection attributes into interval grey numbers. Wang (2017) defined a discrete Choquet integral of degree of greyness of interval grey number and used the possibility degree of interval grey numbers to rank and sort them schematically. However, an advanced system is still needed for a complex and lengthy set of calculation.

However, the criteria of selection of domain always remain same in recent papers, that is not only for the calculated grey numbers but also for the resultant grey numbers, which can be assumed as unrealistic up to a certain extent, so further critical analysis a more complex calculation system is needed to be devised. This article has attempted to use the enhanced algorithm model involving of four deductive theorems and eight rules linked to the properties of an enhanced algorithm of the interval grey numbers based on the kernel and the degree of greyness. Moreover, it is focusing on the choice of domain in calculation of degree of greyness according to the principle of grey numbers operation, it would be able to reduce the complexity involved in the decision-making model sufficiently.

Section 2 of this article has incorporated a brief introduction of the interval grey number along-with the provision of its basic concepts. Section 3 has analyzed the degree of greyness of the operational results of interval grey numbers, and some specific expressions are also proposed. Subsequently, the algorithm of interval grey numbers has been derived based on the kernel and the degree of greyness of grey numbers. While in Section 4, a Yellow River's case concerning risk assessment by ice disaster is demonstrated so to assess and depict the most appropriate decision-making model. Finally, the conclusion has been presented in Section 5.

2. Basic concepts of interval grey numbers

The grey number, denoted by "$\otimes$", is a kind of number that has an exact range of values, but a definite value is unknown. While for the sake of application, a grey number can be termed as an indeterminate number that holds a possible value within an interval or a general set of numbers (Liu and Yang, 2017). To depict the foundation of a value range, the grey numbers can be divided into various categories, and out of these, the interval grey number is the most common category. While analyzing the characteristics of interval grey numbers, Liu *et al.* (Jiang *et al.*, 2017) established a novel algorithm, and a series of definitions, which are introduced in the following sections.

Definition 1. Interval grey number:

Let $\otimes$ be an interval grey number having clear upper and lower boundaries, its mathematic expression is depicted as per equation (1) (Liu *et al.*, 2010):

$$\otimes \in \left[a^{-}, a^{+}\right] \tag{1}$$

where a^+ and a^- denote the upper and lower boundaries of $\otimes$, respectively. While only when a finite number or a countable number of potential values are obtained within an interval grey number will be termed as a discrete interval grey number denoted as $\otimes_d$. Meanwhile, if potential values are continuous within an interval grey numbering system that will be expressed as $\otimes_c$.

Definition 2. Kernel of interval grey number:

Let $\otimes \in [a^-, a^+]$, $a^- < a^+$ be an interval grey number. $\hat{\otimes}$ denotes the kernel of $\otimes$, as an expected value. In such case where there is a lack of information regarding the distribution of values of grey numbers $\otimes$, then equations (2) and (3) are applied to find out the kernel:

$$\hat{\otimes}_c = \frac{1}{2}\left(a^- + a^+\right) \tag{2}$$

$$\hat{\otimes}_d = \frac{1}{n}\sum_{i=1}^{n} a_i, \quad \otimes_d \in \{a_i | i = 1, 2, \cdots, n\} \subseteq \left[a^-, a^+\right] \tag{3}$$

Definition 3. Degree of greyness of interval grey number:

Let $\otimes \in [a^-, a^+]$, $a^- < a^+$ be an interval grey number. While Ω denotes the domain of $\otimes$ and $\mu(\otimes)$ as the measure of Ω. The degree of greyness of $\otimes$ denoted by $g^{\circ}(\otimes)$, has been defined by Liu (F. Liu, 1987) as shown in equation (4):

$$g^{\circ}(\otimes) = \mu(\otimes)/\mu(\Omega) \tag{4}$$

While $0 \le g^{\circ}(\otimes) \le 1$ and the algebraic expressions for the two types of grey numbers, which contain the discrete and continuous grey number, is depicted in the following:

(1) Assuming that $\otimes_d \in \{a_i | i = 1, 2, \ldots, n\} \subseteq [a^-, a^+]$, $\Omega_d = \{b_j | j = j = 1, 2, \ldots, N\}$, when $\forall_i \in [1, 2, \ldots, n]$, $\exists\, j \in [1, 2, \ldots, N]$, and the degree of greyness of this discrete interval grey number is defined as per equation (5):

$$g^{\circ}(\otimes_d) = \frac{n}{N} \tag{5}$$

(2) When $\otimes_c \in [a^-, a^+]$, $\Omega_c \in [A^-, A^+]$ and $[a^-, a^+] \subseteq [A^-, A^+]$, equation (6) is used:

$$g^\circ(\otimes_c) = \frac{a^+ - a^-}{A^+ - A^-} \tag{6}$$

From the above definitions and expressions, the kernel is termed as the mean possibility of an interval grey number, representing a value of expectation and the degree of greyness is measuring the possibility. While with the higher the degree of greyness there is, lower of possibility and vice versa. With the kernel and degree of greyness, another reduced form of interval grey number is depicted as per equation (7) (Liu *et al.*, 2010):

$$\otimes = \hat{\otimes}_{(g^\circ)} \tag{7}$$

The interval grey number '$\otimes$' has clearly represented in equation (1) and that there is a one-to-one correspondence while using its reduced form. However, the information expressed by equation (7) is not well aligned with expression presented in equation (1), whereas the kernel and degree of greyness of interval grey numbers are always real numbers. Therefore, there is a lot of potential for inducing the interval grey number operations into the real number operations according to the simplified expression. As for the grey numbers represented by equation (7) the following algorithm had been further derived defined by Liu *et al.* (2010).

Definition 4. Algorithm of interval grey numbers:

Let $\otimes_a = \hat{\otimes}_{a(g_a^\circ)}$ and $\otimes_b = \hat{\otimes}_{b(g_b^\circ)}$ be two-interval grey numbers. Then,

$$\hat{\otimes}_{a(g_a^\circ)} = \hat{\otimes}_{b(g_b^\circ)} \text{ iff } \hat{\otimes}_a = \hat{\otimes}_b \text{ and } g_a^\circ = g_b^\circ \tag{8}$$

$$-\hat{\otimes}_{(g^\circ)} = (-\hat{\otimes})_{(g^\circ)} \tag{9}$$

$$\hat{\otimes}_{a(g_a^\circ)} + \hat{\otimes}_{b(g_b^\circ)} = \left(\hat{\otimes}_a + \hat{\otimes}_b\right)_{(g_a^\circ \vee g_b^\circ)} \tag{10}$$

$$\hat{\otimes}_{a(g_a^\circ)} - \hat{\otimes}_{b(g_b^\circ)} = \left(\hat{\otimes}_a - \hat{\otimes}_b\right)_{(g_a^\circ \vee g_b^\circ)} \tag{11}$$

$$\hat{\otimes}_{a(g_a^\circ)} \times \hat{\otimes}_{b(g_b^\circ)} = \left(\hat{\otimes}_a \times \hat{\otimes}_b\right)_{(g_a^\circ \vee g_b^\circ)} \tag{12}$$

$$\hat{\otimes}_{a(g_a^\circ)} / \hat{\otimes}_{b(g_b^\circ)} = \left(\hat{\otimes}_a / \hat{\otimes}_b\right)_{(g_a^\circ \vee g_b^\circ)}, \ \hat{\otimes}_b \neq 0 \tag{13}$$

$$1/\hat{\otimes}_{b(g^\circ)} = \left(1/\hat{\otimes}_b\right)_{(g^\circ)}, \ \hat{\otimes}_b \neq 0 \tag{14}$$

$$k \cdot \hat{\otimes}_{(g^\circ)} = (k \cdot \hat{\otimes})_{(g^\circ)}, \ k \textit{ is a real number} \tag{15}$$

The algorithm defined in 2.4 is based on a maximum degree of greyness of the operational results. Moreover, it simplifies the computation of interval grey numbers and it can be able to manage the tough problems of establishing the grey algebraic system up-to some extent.

However, there are doubts about the scientific credibility in operations of degree of greyness as represented in equations (10)-(15) that has also been discussed in Section 1. That is why the proceeding sections of this study have focused on designing a new algorithm for the degree of greyness for interval grey numbers.

3. Design of operational result

The existing algebraic operations for the interval grey numbers are defined according to an identical domain and on an assumption that the operational results for the degree of greyness would always remain same (Li *et al.*, 2015). The earlier fact has been described in Section 1 of this study, whereas, for later, it can be described as an unsustainable because a domain of an interval grey number must contain all the values of a grey number, which is likely to be many variables during the operational process. Moreover, other detailed definitions about such domain variations have been presented by different researchers as highlighted in the following sections.

3.1 Domain of the operational results of interval grey numbers

Definition 5. The domain of an interval grey number is the smallest set within which an interval grey number is created.

As such this definition provides a criterion for giving a suitable domain of an interval grey number, e.g. when the height (centimeters) of an adult is considered as; $\otimes_h \in [178, 180]$ i.e. in-terms of visual observation, and if the range of highest to lowest heights of adults from all over the world is taken as 251-54.6 (according to Guinness Book of World Records) then the domain can be calculated as $\Omega_h = [54.6, 251]$. Usually, it is quite difficult to present the exact upper and lower limits of a domain, however, it must be ensured that the smallest set would have been selected according to known information.

As per the definition stated above a specific expression regarding the domain of operations of interval grey numbers can be derived. Which is described in the following mathematical relationship (definition 6) and numerically depicted in equation (16).

Definition 6. Assuming '$\otimes_a$' and '$\otimes_b$' are two interval grey numbers, and 'Ω_a' and 'Ω_b' denote their domains, respectively and * be an arithmetic relationship as $* \in \{+, -, \times, \div\}$.

Therefore, when $\otimes = \otimes_a * \otimes_b$, and Ω is the domain of $\otimes$, then:

$$\Omega = \Omega_1 * \Omega_2 \tag{16}$$

This relationship [as per Equation (16)], can further be proved as $\forall x_a \in \Omega_a$, $x_b \in \Omega_b$, $x_a * x_b \in \Omega$. Where $\exists \Omega'$, s.t. $x_a * x_b \exists \Omega'$, then $\Omega \subseteq \Omega'$ i.e. according to the definition 5. Thus, $\Omega = \Omega_a * \Omega_b$ has been established logically.

Definition 6. When a new interval grey number is obtained by addition, subtraction, multiplication or division of interval grey numbers, their domains would also take a corresponding arithmetic effect that generates a new domain towards the operational results. Based on this definition 6, the specific expressions of degree of greyness for interval grey number operations have been further analyzed in Section 3.2.

3.2 Specific expressions of the degree of greyness for interval grey number operations

Definition 7. Let '$\otimes_a$' and '$\otimes_b$' are two interval grey numbers, and '$\otimes_a$' and 'Ω_b' as their respective domains, while $\otimes = \otimes_a * \otimes_b$. Then, $\Omega = \Omega_a * \Omega_b$, and Ω is denoted as the domain of $\otimes$. Therefore, an equation (17) have been characterized:

$$g^{\circ}(\otimes) = \frac{\mu(\otimes)}{\mu(\Omega)} = \frac{\mu(\otimes_a * \otimes_b)}{\mu(\Omega_a * \Omega_b)} \tag{17}$$

where * is an arithmetic relationship; $* \in \{+, -, \times, \div\}$.

This study also focused on the continuous interval grey numbers. As it has been s supposed that these are more universal than discrete interval grey numbers (i.e. according to the real-life situations). While for the addition and subtraction of continuous interval grey numbers, the following conclusions have been drawn.

Theorem 1. Assuming that $\otimes_a \in [a^-, a^+]$ and $\otimes_b \in [b^-, b^+]$ as two-interval grey numbers, while; $\Omega_a = [A^-, A^+]$ and $\Omega_b = [B^-, B^+]$ as their respective domains, then an expression as depicted in equation (18) can be built:

$$g^{\circ}(\otimes_a \pm \otimes_b) = \frac{\mu(\Omega_a)}{\mu(\Omega_a) + \mu(\Omega_b)} \cdot g^{\circ}(\otimes_a) + \frac{\mu(\Omega_b)}{\mu(\Omega_a) + \mu(\Omega_b)} \cdot g^{\circ}(\otimes_b) \tag{18}$$

It can be further proven by considering the addition of two-interval grey numbers in-terms of:

$$\otimes_a + \otimes_b \in [a^- + b^-, a^+ + b^+], \ \Omega_a + \Omega_b = [A^- + B^-, A^+ + B^+]$$

Then by the application of equation (17), it can be depicted as:

$$\begin{aligned} g^{\circ}(\otimes_a + \otimes_b) &= \frac{\mu(\otimes_a + \otimes_b)}{\mu(\Omega_a + \Omega_b)} = \frac{(a^+ + b^+) - (a^- + b^-)}{(A^+ + B^+) - (A^- + B^-)} = \frac{(a^+ - a^-) + (b^+ - b^-)}{(A^+ - A^-) + (B^+ - B^-)} \\ &= \frac{\mu(\otimes_a) + \mu(\otimes_b)}{\mu(\Omega_a) + \mu(\Omega_b)} = \frac{\mu(\Omega_a)}{\mu(\Omega_a) + \mu(\Omega_b)} \cdot g^{\circ}(\otimes_a) + \frac{\mu(\Omega_b)}{\mu(\Omega_a) + \mu(\Omega_b)} \cdot g^{\circ}(\otimes_b) \end{aligned}$$

In another situation, when two-interval grey numbers are subtracted can be proved similarly. As a special case that addition and subtraction operations are taken between a real number 'k' and an interval grey number '$\otimes_a \in [a^-, a^+]$', it can be expressed in terms of equation (19):

$$g^{\circ}(k \pm \otimes_a) = g^{\circ}(\otimes_a), \ g^{\circ}(-\otimes_a) = g^{\circ}(\otimes_a) \tag{19}$$

Theorem 1. has shown a degree of greyness for addition and subtraction of interval grey numbers, that in fact is not an addition of the degrees of greyness of their operators, so the measure of their domains must be critically re-assessed. However, this is different from the algebraic operations as depicted in equations (10) and (11). Furthermore, according to *Theorem 1*, another expression can be derived as per the following:

Property 1. Let us suppose that $\otimes_a \in [a^-, a^+]$ and $\otimes_b \in [b^-, b^+]$ are two interval grey numbers, and

$$\min\{g^{\circ}(\otimes_a), g^{\circ}(\otimes_b)\} \leq g^{\circ}(\otimes_a \pm \otimes_b) \leq \max\{g^{\circ}(\otimes_a), g^{\circ}(\otimes_b)\}$$

While $g^{\circ}(\otimes_a) = g^{\circ}(\otimes_b)$, then the degree of greyness will remain unchanged.

This can be justified by assuming $g^{\circ}(\otimes_a) \leq g^{\circ}(\otimes_b)$ and further explained as per the following:

$$\begin{aligned}
g^{\circ}(\otimes_a) &= \frac{\mu(\Omega_a)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\mu(\Omega_b)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)\\
&\leq \frac{\mu(\Omega_a)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\mu(\Omega_b)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\\
&\leq \frac{\mu(\Omega_a)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)+\frac{\mu(\Omega_b)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\\
&= g^{\circ}(\otimes_b)
\end{aligned}$$

While Property 1 has clearly depicted that the value of degree of greyness as (derived via addition and subtraction of two-interval grey numbers among the degrees of greyness of the operators. However, when $g^{\circ}(\otimes_a) = g^{\circ}(\otimes_b)$, equation (18) represents the same degree of greyness i.e. denoted by equations (10) and (11).

Furthermore, for the sake of multiplication and division, the following theorems have been deployed.

Theorem 2. Let assume that $\otimes_a \in [a^-, a^+]$ and $\otimes_b \in [b^-, b^+]$ are two interval grey numbers, where $\Omega_a = [A^-, A^+]$ and $\Omega_b = [B^-, B^+]$ are their respective domains. While $\overline{\Omega}_a$ and $\overline{\Omega}_b$ are two numbers they have been satisfying the relationships; $\overline{\Omega}_a = (A^- + A^+)/2$ and $\overline{\Omega}_b = (B^- + B^+)/2$, then following empirical equations (1) and (2) can be used to formulate equations (20)-(21) and (22)-(23), respectively.

(1) If $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^+$, or $\Omega_a \subseteq R^-$ and $\Omega_b \subseteq R^-$, it can be expressed as:

$$g^{\circ}(\otimes_a \times \otimes_b) = \frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b) \tag{20}$$

$$g^{\circ}\left(\frac{\otimes_a}{\otimes_b}\right) = \frac{B^+B^-}{b^+b^-}\cdot\left[\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\right] \tag{21}$$

(2) If $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^-$, or $\Omega_a \subseteq R^-$ and $\Omega_b \subseteq R^+$, it can be represented as:

$$g^{\circ}(\otimes_a \times \otimes_b) = \frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)-\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)-\frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)-\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b) \tag{22}$$

$$g^{\circ}\left(\frac{\otimes_a}{\otimes_b}\right) = \frac{B^+B^-}{b^+b^-}\cdot\left[\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)-\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)-\frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)-\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\right] \tag{23}$$

These relationships can be further verified via *relationship 3*:

(3) If $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^+$, then:

$$\otimes_a \times \otimes_b \in [a^- b^-, a^+ b^+], \Omega_a \times \Omega_b = [A^- B^-, A^+ B^+]$$
$$\frac{\otimes_a}{\otimes_b} \in \left[\frac{a^-}{b^+}, \frac{a^+}{b^-}\right], \frac{\Omega_a}{\Omega_b} = \left[\frac{A^-}{B^+}, \frac{A^+}{B^-}\right]$$

While according to equation (17), it can be depicted that:

$$\begin{aligned} g^\circ(\otimes_a \times \otimes_b) &= \frac{\mu(\otimes_a \times \otimes_b)}{\mu(\Omega_a \times \Omega_b)} \\ &= \frac{a^+ b^+ - a^- b^-}{A^+ B^+ - A^- B^-} = \frac{\frac{1}{2}[(a^+ - a^-)(b^+ + b^-) + (a^+ + a^-)(b^+ - b^-)]}{\frac{1}{2}[(A^+ - A^-)(B^+ + B^-) + (A^+ + A^-)(B^+ - B^-)]} \\ &= \frac{\hat{\otimes}_b \cdot \mu(\otimes_a) + \hat{\otimes}_b \cdot \mu(\otimes_b)}{\overline{\Omega}_b \cdot \mu(\Omega_a) + \overline{\Omega}_a \cdot \mu(\Omega_b)} \\ &= \frac{\hat{\otimes}_b \cdot \mu(\Omega_a)}{\overline{\Omega}_b \cdot \mu(\Omega_a) + \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^\circ(\otimes_a) + \frac{\hat{\otimes}_b \cdot \mu(\Omega_b)}{\overline{\Omega}_b \cdot \mu(\Omega_a) + \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^\circ(\otimes_b) \end{aligned}$$

$$\begin{aligned} g^\circ\left(\frac{\otimes_a}{\otimes_b}\right) &= \frac{\mu\left(\frac{\otimes_a}{\otimes_b}\right)}{\mu\left(\frac{\Omega_a}{\Omega_b}\right)} = \frac{\frac{a^+}{b^-} - \frac{a^-}{b^+}}{\frac{A^+}{B^-} - \frac{A^-}{B^+}} = \frac{B^+ B^-}{b^+ b^-} \cdot \frac{a^+ b^+ - a^- b^-}{A^+ B^+ - A^- B^-} \\ &= \frac{B^+ B^-}{b^+ b^-} \cdot \left[\frac{\hat{\otimes}_b \cdot \mu(\Omega_a)}{\overline{\Omega}_b \cdot \mu(\Omega_a) + \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^\circ(\otimes_a) + \frac{\hat{\otimes}_b \cdot \mu(\Omega_b)}{\overline{\Omega}_b \cdot \mu(\Omega_a) + \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^\circ(\otimes_b)\right] \end{aligned}$$

Furthermore, due to $A^- B^- \le a^- b^- \le a^+ b^+ \le A^+ B^+$ and $A^-/B^+ \le a^-/b^+ \le a^+/b^- \le A^+/B^-$, the values of the specific expressions for multiplication and division operations have been within the range of 0-1, which justify the specific characteristic of degree of greyness. While in another situation, which depicts that $\Omega_a \subseteq R^-$ and $\Omega_b \subseteq R^-$ that can also be justified similarly as per empirical expression 4.

(4) If $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^-$, then:

$$\otimes_a \times \otimes_b \in [a^+ b^-, a^- b^+], \Omega_a \times \Omega_b = [A^+ B^-, A^- B^+]$$
$$\frac{\otimes_a}{\otimes_b} \in \left[\frac{a^+}{b^+}, \frac{a^-}{b^-}\right], \frac{\Omega_a}{\Omega_b} = \left[\frac{A^+}{B^+}, \frac{A^-}{B^-}\right]$$

While as per equation (17), it can be further expressed as per following:

$$g^{\circ}(\otimes_a \times \otimes_b) = \frac{\mu(\otimes_a \times \otimes_b)}{\mu(\Omega_a \times \Omega_b)}$$

$$= \frac{a^-b^+ - a^+b^-}{A^-B^+ - A^+B^-} = \frac{\frac{1}{2}\left[(a^+ + a^-)(b^+ - b^-) - (a^+ - a^-)(b^+ + b^-)\right]}{\frac{1}{2}\left[(A^+ + A^-)(B^+ - B^-) - (A^+ - A^-)(B^+ + B^-)\right]}$$

$$= \frac{\hat{\otimes}_a \cdot \mu(\otimes_b) - \hat{\otimes}_b \cdot \mu(\otimes_a)}{\overline{\Omega}_a \cdot \mu(\Omega_b) - \overline{\Omega}_b \cdot \mu(\Omega_a)}$$

$$= \frac{\hat{\otimes}_a \cdot \mu(\Omega_b)}{\overline{\Omega}_a \cdot \mu(\Omega_b) - \overline{\Omega}_b \cdot \mu(\Omega_a)} \cdot g^{\circ}(\otimes_b) - \frac{\hat{\otimes}_b \cdot \mu(\Omega_a)}{\overline{\Omega}_a \cdot \mu(\Omega_b) - \overline{\Omega}_b \cdot \mu(\Omega_a)} \cdot g^{\circ}(\otimes_a)$$

$$g^{\circ}\left(\frac{\otimes_a}{\otimes_b}\right) = \frac{\mu\left(\frac{\otimes_a}{\otimes_b}\right)}{\mu\left(\frac{\Omega_a}{\Omega_b}\right)} = \frac{\frac{a^-}{b^-} - \frac{a^+}{b^+}}{\frac{A^-}{B^-} - \frac{A^+}{B^+}} = \frac{B^+B^-}{b^+b^-} \cdot \frac{a^-b^+ - a^+b^-}{A^-B^+ - A^+B^-}$$

$$= \frac{B^+B^-}{b^+b^-} \cdot \left[\frac{\hat{\otimes}_b \cdot \mu(\Omega_a)}{\overline{\Omega}_b \cdot \mu(\Omega_a) - \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^{\circ}(\otimes_a) - \frac{\hat{\otimes}_a \cdot \mu(\Omega_b)}{\overline{\Omega}_b \cdot \mu(\Omega_a) - \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^{\circ}(\otimes_b)\right]$$

Moreover, due to $A^+B^- \leq a^+b^- \leq a^-b^+ \leq A^-B^+$ and $A^+B^+ \leq a^+/b^+ \leq a^-/b^- \leq A^-/B^-$, the values of the specific expressions for multiplication and division operations have been in the range of 0-1. In another given situation when $\Omega_a \subseteq R^-$ and $\Omega_b \subseteq R^+$ this can also be justified in a similar manner.

Theorem 2 has indicated that the degree of greyness for multiplication and division of interval grey numbers cannot be the multiplication and division of the degrees of greyness of their operators, and their kernels, as well as the midpoints and measures of their domain, must be considered. This is different from the algebraic operations presented in equations (12) and (13). Therefore, this study has also analyzed the size of the relationships of the degree of greyness among operational result and operational process.

Deduction 1. While assuming; $\otimes_a \in [a^-, a^+]$ and $\otimes_b \in [b^-, b^+]$ as two-interval grey numbers. The relationship of sizing among $g^{\circ}(\otimes_a)$, $g^{\circ}(\otimes_b)$ and $g^{\circ}(\otimes_a \times \otimes_b)$, and by the exclusion of specific values, then this relationship cannot be verified.

Now this can be further verified by a consideration that $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^+$, as there is no loss of generality for an assumption that $g^{\circ}(\otimes_a) \leq g^{\circ}(\otimes_b)$, therefore, the following empirical relationships can be built:

(1) If $\hat{\otimes}_a = \overline{\Omega}_a$ and $\hat{\otimes}_b = \overline{\Omega}_b$, these can be ranked as:

$$g^{\circ}(\otimes_a) \leq g^{\circ}(\otimes_a \pm \otimes_b) \leq g^{\circ}(\otimes_b)$$

(2) If the situation above is unrealistic, then, $g^{\circ}(\otimes_a \times \otimes_b)$ needed to be compared with $g^{\circ}(\otimes_a)$ and $g^{\circ}(\otimes_b)$, respectively.

(3) When $g^{\circ}(\otimes_a \times \otimes_b)$ is being compared with $g^{\circ}(\otimes_a)$ then these can be calculated through following numeric relationship:

$$
\begin{aligned}
&g^{\circ}(\otimes_a\times\otimes_b)-g^{\circ}(\otimes_a)\\
&=\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\\
&\quad-\left[\frac{\overline{\Omega}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\overline{\Omega}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)\right]\\
&=\frac{\left(\hat{\otimes}_b-\overline{\Omega}_b\right)\cdot g^{\circ}(\otimes_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot\mu(\Omega_a)+\frac{\hat{\otimes}_a\cdot g^{\circ}(\otimes_b)-\overline{\Omega}_a\cdot g^{\circ}(\otimes_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot\mu(\Omega_b)
\end{aligned}
$$

While building an association, where $\hat{\otimes}_b-\overline{\Omega}_b<0$ and $\hat{\otimes}_a\cdot g^{\circ}(\otimes_b)-\overline{\Omega}_a\cdot g^{\circ}(\otimes_a)\leq 0$, the situation has indicated that $g^{\circ}(\otimes_a\times\otimes_b)<g^{\circ}(\otimes_a)$ arises logically. When $\hat{\otimes}_b-\overline{\Omega}_b<0$ and $g^{\circ}(\otimes_b)-g^{\circ}(\otimes_a)\rightarrow 0$, then $g^{\circ}(\otimes_a\times\otimes_b)<g^{\circ}(\otimes_a)$ as long as $\left(\hat{\otimes}_a-\overline{\Omega}_a\right)$ is smaller.

(4) While $g^{\circ}(\otimes_a\times\otimes_b)$ is comparable with $g^{\circ}(\otimes_b)$ then these can be further associated by means of another mathematical relationship as depicted in the following:

$$
\begin{aligned}
&g^{\circ}(\otimes_a\times\otimes_b)-g^{\circ}(\otimes_b)\\
&=\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_a)+\frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\\
&\quad-\left[\frac{\overline{\Omega}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)+\frac{\overline{\Omega}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^{\circ}(\otimes_b)\right]\\
&=\frac{\left(\hat{\otimes}_b\cdot g^{\circ}(\otimes_a)-\overline{\Omega}_b\cdot g^{\circ}(\otimes_b)\right)\cdot}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot\mu(\Omega_a)+\frac{\left(\hat{\otimes}_a-\overline{\Omega}_a\right)\cdot g^{\circ}(\otimes_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot\mu(\Omega_b)
\end{aligned}
$$

Now, while determining that $\hat{\otimes}_a-\overline{\Omega}_a>0$ and $\hat{\otimes}_b\cdot g^{\circ}(\otimes_a)-\overline{\Omega}_b\cdot g^{\circ}(\otimes_b)\geq 0$, another obvious situation i.e. $g^{\circ}(\otimes_a\times\otimes_b)>g^{\circ}(\otimes_b)$ arises, which has been factual as;$\hat{\otimes}_a-\overline{\Omega}_a>0$ and $g^{\circ}(\otimes_b)-g^{\circ}(\otimes_a)\rightarrow 0$, $g^{\circ}(\otimes_a\times\otimes_b)>g^{\circ}(\otimes_b)$, however, this would be true when $\left(\hat{\otimes}_a-\overline{\Omega}_a\right)$ remains greater.

Thus, the relationship of sizing among $g^{\circ}(\otimes_a)$, $g^{\circ}(\otimes_b)$ and $g^{\circ}(\otimes_a\times\otimes_b)$ cannot be verifiable until researchers would find out some specific and reliable values i.e. similar to the situation where $\Omega_a\subseteq R^+$ and $\Omega_b\subseteq R^+$. Likewise, the other three situations cases can also be proved. Deduction 1 has analyzed the situation regarding multiplication and operation, whereas similarly the situation for division can also be illustrated.

Furthermore, the analysis of reciprocal and scalar multiplication operations has involved specified system of division and multiplication operations. Therefore, this study has also presented the empirical expressions regarding the degree of greyness for reciprocal and scalar multiplication of the interval grey number as depicted in the following sections.

Theorem 3. While assuming that $\otimes_a\in[a^-,a^+]$ is an interval grey number, $\Omega_a=[A^-,A^+]$ as its domain and $\Omega_a\subseteq R^+$ or $\Omega_a\subseteq R^-$, then a new empirical relationship can be established as per equation (24).

$$
g^{\circ}\left(\frac{1}{\otimes_a}\right)=\frac{A^+A^-}{a^+a^-}\cdot g^{\circ}(\otimes_a) \tag{24}
$$

This equation can be proved according to the condition; $\Omega_a\subseteq R^+$ or $\Omega_a\subseteq R^-$and then another expression (as per following) can be derived:

$$\frac{1}{\otimes_a} \in \left[\frac{1}{a^+}, \frac{1}{a^-}\right], \frac{1}{\Omega_a} = \left[\frac{1}{A^+}, \frac{1}{A^-}\right]$$

While as per equation (17), a further relationship can be represented as per following:

$$g^\circ\left(\frac{1}{\otimes_a}\right) = \frac{\mu\left(\frac{1}{\otimes_a}\right)}{\mu\left(\frac{1}{\Omega_a}\right)} = \frac{\frac{1}{a^-} - \frac{1}{a^+}}{\frac{1}{A^-} - \frac{1}{A^+}} = \frac{A^+A^-}{a^+a^-} \cdot \frac{a^+ - a^-}{A^+ - A^-} = \frac{A^+A^-}{a^+a^-} \cdot g^\circ(\otimes_a)$$

While factually the relationship of sizing i.e. $\left[\frac{A^+A^-}{a^+a^-}, 1\right]$ is dependent upon specific values. Therefore, the larger value among $g^\circ\left(\frac{1}{\otimes_a}\right)$ and $g^\circ(\otimes_a)$ cannot be determined as no specific value range has been provided for $\otimes_a$ and Ω_a.

Theorem 4. While assuming $\otimes_a \in [a^-, a^+]$ as an interval grey number, $\Omega_a = [A^-, A^+]$ being its domain and $k \in R$ where $k \neq 0$, then a relationship as per equation (25) can be depicted:

$$g^\circ(k \cdot \otimes_a) = g^\circ(\otimes_a) \tag{25}$$

where $k = 0$ and the degree of greyness (after scalar multiplication operation) = 0.

That can be proved as per the following expressions.

(1) If $k > 0$, then following can be established:

$$k \cdot \otimes_a \in \left[k \cdot a^-, k \cdot a^+\right], k \cdot \Omega_a = \left[k \cdot A^-, k \cdot A^+\right]$$

Now according to 17, the following empirical expression can also be derived:

$$g^\circ(k \cdot \otimes_a) = \frac{\mu(k \cdot \otimes_a)}{\mu(k \cdot \Omega_a)} = \frac{k(a^+ - a^-)}{k(A^+ - A^-)} = g^\circ(\otimes_a)$$

(2) If $k < 0$ this can similarly be proved as per (1)

(3) If $k = 0$ then $k \cdot \otimes_a = 0$. The result of this scalar multiplication operation is a real number, therefore, the degree of greyness will always be zero.

Theorem 4 has illustrated that the degree of greyness for multiplication operation among an interval grey number and a non-zero real number will remain unchanged, which has also been depicted in equation (15) (i.e. as per algebraic operation).

While Theorems 1-4 have demonstrated that the respective expressions of the degree of greyness of addition, subtraction, multiplication, division, reciprocal and outcome of scalar multiplication operation. This analysis indicates that the degree of greyness for addition and subtraction of two interval grey numbers is related to the measure of their domain. Furthermore, the value of the degrees of greyness, the multiplication and division of operations and the degree of greyness must be calculated according to the kernels, the midpoints and measures of the domain. While specifically, the degree of greyness, which has been obtained by addition, subtraction and multiplication between an interval grey number and a non-zero real number, has the same as that of their operation.

While determining the range of values of the interval grey number generated by the multiplication and division operations, the Theorems 2 and 3 have certain limitations for computation of a domain. If the domain of the operators are intersecting a positive real domain and a negative real domain, the expressions of the degree of greyness for the multiplication and division operations cannot be determined by the methodology presented in this study. Moreover, the expressions depicted in Theorems 2 and 3 cannot directly calculate the size of relationships of the degree of greyness as per the results of multiplication operation and division operation except by the inclusion of specific values.

3.3 Revision of algorithm of interval grey numbers

Theorems 1-4 have presented relevant functions to simplify the algebraic operations of the degree of greyness. These are useful for specific calculations of algorithm based on the results of kernel operation and the degree of greyness of interval grey numbers, which improves its scientific credibility. The specific algorithmic rules are depicted in equations (26)-(35) along-with relevant empirical expressions.

Rule 1 (*Equality*):

$$\hat{\otimes}_a(g^\circ{}_a) = \hat{\otimes}_b(g^\circ{}_b) \Longleftrightarrow \hat{\otimes}_a = \hat{\otimes}_b \text{且} g^\circ{}_a = g^\circ{}_b \tag{26}$$

Rule 2 (*Additive operation*):

$$\hat{\otimes}_a(g^\circ{}_a) + \hat{\otimes}_b(g^\circ{}_b) = \left(\hat{\otimes}_a + \hat{\otimes}_b\right)_{\left(\frac{\mu(\Omega_a)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^\circ{}_a + \frac{\mu(\Omega_b)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^\circ{}_b\right)} \tag{27}$$

Rule 3 (Additive inverse):

$$-\hat{\otimes}_{a(g^\circ{}_a)} = \left(-\hat{\otimes}_a\right)_{(g^\circ{}_a)} \tag{28}$$

Rule 4 (Subtraction operation):

$$\hat{\otimes}_{a(g^\circ{}_a)} - \hat{\otimes}_{b(g^\circ{}_b)} = \left(\hat{\otimes}_a - \hat{\otimes}_b\right)_{\left(\frac{\mu(\Omega_a)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^\circ{}_a + \frac{\mu(\Omega_b)}{\mu(\Omega_a)+\mu(\Omega_b)}\cdot g^\circ{}_b\right)} \tag{29}$$

Rule 5 (Multiplication operation):

When $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^+$ or $\Omega_a \subseteq R^-$and $\Omega_b \subseteq R^-$ then the calculations can be depicted as per equation (30):

$$\hat{\otimes}_{a(g^\circ{}_a)} \times \hat{\otimes}_{b(g^\circ{}_b)} = \left(\hat{\otimes}_a \times \hat{\otimes}_b\right)_{\left(\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^\circ{}_a + \frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^\circ{}_b\right)} \tag{30}$$

When $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^-$, or $\Omega_a \subseteq R^-$and $\Omega_b \subseteq R^+$ then equation (31) can be depicted:

$$\hat{\otimes}_{a(g^\circ{}_a)} \times \hat{\otimes}_{b(g^\circ{}_b)} = \left(\hat{\otimes}_a \times \hat{\otimes}_b\right)_{\left(\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)-\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^\circ{}_a - \frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)-\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^\circ{}_b\right)} \tag{31}$$

Rule 6 (Division operation):

When $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^+$, or $\Omega_a \subseteq R^-$ and $\Omega_b \subseteq R^-$ then a new relationship as per equation (32) can be built:

$$\hat{\otimes}_{a(g^\circ{}_a)}/\hat{\otimes}_{b(g^\circ{}_b)} = \left(\hat{\otimes}_a/\hat{\otimes}_b\right)_{\left(\frac{B^+B^-}{b^+b^-}\cdot\left[\frac{\hat{\otimes}_b\cdot\mu(\Omega_a)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^\circ{}_a + \frac{\hat{\otimes}_a\cdot\mu(\Omega_b)}{\overline{\Omega}_b\cdot\mu(\Omega_a)+\overline{\Omega}_a\cdot\mu(\Omega_b)}\cdot g^\circ{}_b\right]\right)} \tag{32}$$

While $\Omega_a \subseteq R^+$ and $\Omega_b \subseteq R^-$, or $\Omega_a \subseteq R^-$ and $\Omega_b \subseteq R^+$ then another relationship in the form of equation (33) can be established:

$$\hat{\otimes}_{a(g^\circ{}_a)} / \hat{\otimes}_{b(g^\circ{}_b)} = \left(\hat{\otimes}_a / \hat{\otimes}_b\right)_{\left(\frac{B^+ B^-}{b^+ b^-} \cdot \left[\frac{\hat{\otimes}_b \cdot \mu(\Omega_a)}{\overline{\Omega}_b \cdot \mu(\Omega_a) - \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^\circ{}_a - \frac{\hat{\otimes}_a \cdot \mu(\Omega_b)}{\overline{\Omega}_b \cdot \mu(\Omega_a) - \overline{\Omega}_a \cdot \mu(\Omega_b)} \cdot g^\circ{}_b\right]\right)} \tag{33}$$

Rule 7 (Reciprocal):

If $\hat{\otimes}_a \neq 0$, and $\Omega_b \subseteq R^+$ or $\Omega_b \subseteq R^-$ then an empirical equation (34) can be formed:

$$1/\hat{\otimes}_{a(g^\circ{}_a)} = \left(1/\hat{\otimes}_a\right)_{\left(\frac{A^+ A^-}{a^+ a^-} \cdot g^\circ{}_a\right)} \tag{34}$$

Rule 8 (Scalar multiplication):

While $k \in R$ and $k \neq 0$ then a new relationship as depicted in equation (35) can be established:

$$k \cdot \hat{\otimes}_{a(g^\circ{}_a)} = \left(k \cdot \hat{\otimes}_a\right)_{(g^\circ{}_a)} \tag{35}$$

Meanwhile, when $k = 0$, $k \cdot \hat{\otimes}_{a(g^\circ{}_a)} = 0$ then the algorithm of interval grey numbers established above can be extended to any interval grey numbers and it will be operative. This can further be realized by the following example i.e. presented and described according to relevant algorithms (rule 1-8).

Example:

Assuming $\otimes_1 \in [72, 76]$ and $\otimes_2 \in [84,90]$ are two interval grey numbers and their respective domains are defined as; $\Omega_1 = [20, 100]$ and $\Omega_2 = [20, 100]$. While the simplified representations are depicted as per the following:

$$\otimes_1 = 74_{(0.05)}, \ \otimes_2 = 87_{(0.075)}$$

While according to equations (10)-(15), it can further be expressed as:

$$\otimes_1 + \otimes_2 = (74 + 87)_{(0.05 \vee 0.075)} = 161_{(0.05)}$$

$$\otimes_1 - \otimes_2 = (74 - 87)_{(0.05 \vee 0.075)} = -13_{(0.075)}$$

$$\otimes_1 \times \otimes_2 = (74 \times 87)_{(0.05 \vee 0.075)} = 6438_{(0.075)}$$

$$\otimes_1 / \otimes_2 = (74/87)_{(0.05 \vee 0.075)} = 0.85_{(0.075)}$$

$$1/\otimes_1 = (1/74)_{(0.05)} = 0.0135_{(0.05)}$$

$$10 \times \otimes_1 = (10 \times 74)_{(0.05)} = 740_{(0.05)}$$

Now by the application of equations (28)-(35), the following can also be denoted:

$$\otimes_1 + \otimes_2 = (74 + 87)_{\left(\frac{80}{80+80} \times 0.05 + \frac{80}{80+80} \times 0.075\right)} = 161_{(0.0625)}$$

$$\otimes_1 - \otimes_2 = (74 - 87)_{\left(\frac{80}{80+80}\times 0.05 + \frac{80}{80+80}\times 0.075\right)} = -13_{(0.0625)}$$

$$\otimes_1 \times \otimes_2 = (74 \times 87)_{\left(\frac{87\times 80}{60\times 80+60\times 80}\times 0.05 + \frac{74\times 80}{60\times 80+60\times 80}\times 0.075\right)} = 6438_{(0.0825)}$$

$$\otimes_1 / \otimes_2 = (74/87)_{\left(\frac{100\times 20}{90\times 84}\cdot\left[\frac{87\times 80}{60\times 80+60\times 80}\times 0.05 + \frac{74\times 80}{60\times 80+60\times 80}\times 0.075\right]\right)} = 0.85_{(0.0218)}$$

$$1/\otimes_1 = (1/74)_{\left(\frac{100\times 20}{76\times 72}\cdot 0.05\right)} = 0.0135_{(0.0183)}$$

$$10 \times \otimes_1 = (10 \times 74)_{(0.05)} = 740_{(0.05)}$$

The comparative analysis of two sets of equations has concluded that the relative differences among the degrees of greyness have been minimized. While the complexity of operational processing is not intensified by applying the equations (28)-(35).

A case study of Yellow River has further clarified the application of enhanced algorithm of the degree of greyness and its subsequent advantages.

4. Case study

The Yellow River's segment among Ningxia to Inner Mongolia, China is a site of frequent ice flooding disaster. Such natural disaster happens due to the special geographical location, landform and prevalent hydrological and meteorological conditions of the site. The site also features the flow of upper reach through the Loess Plateau and a desert edge, along-with a characteristically large concentration of sediments in the lower reach, which causes an uplift of the riverbed. While the morphology of the river gradually changes from narrow-deep to wider-shallow and ice thawing process in such narrower curved paths gives rise to natural ice dams. The global climate change and extreme weathering conditions of recent years have made such ice disasters more threatening towards health, safety, quality of life and livelihoods. As such the likelihood and risk assessment of ice flood disasters in the Yellow River has been critically analyzed in the present case study as per the need of the time.

4.1 Original data

This study has incorporated the data regarding the river level (b_1) in meters (m), the flow rate (b_2) in m^3/sec second, the temperature (b_3) in Celsius (°C) and the river width (b_4) in m to determine the probability of ice-flooding due the formation of ice-dams. While five sections including of Xiaheyan (a_1), Shizuishan (a_2), Bayangol (a_3), Sanhuhekou (a_4) and Zhaojunfen (a_5) have been taken into account and a temporal jurisdiction of year 2012-2016 is selected as a reference, as portrayed in Table I.

Moreover, for disaster risk assessment by ice-dam formation, specific characteristics of each section of Ningxia to Inner Mongolia reach (during year 2012-2016) have also been determined statistically as per data collected from the Yellow River network (Table II).

It is evident from the data depicted in Table II that Zhaojunfen section of Ningxia-Inner Mongolia reach of the river has the highest likelihood of risk incidence i.e. regarding the times of ice-dam disaster. While the later trend is in the order of Bayangol > Xiaheyan > Shizuishan > Sanhuhekou.

4.2 Application and results of enhanced algorithm

The effectiveness and comparability of enhanced algorithm can be enhanced by eliminating the dimensions so to validate the results via a standardized method of grey range analysis (Guo *et al.*, 2016). During the assessment of current case study, to determine risk and probability of re-occurrence of an ice dam, two types of target values have been set. The river level is taken as a benefit-type target while all other attributes have been accounted in-terms of cost-type target and then subsequently a standardized interval grey number evaluation matrix has been structured as depicted in equation (36):

$$X(\otimes) = \left(x_{ij}(\otimes)\right)_{5\times 4} = \begin{bmatrix} [0.27,\ 0.31] & [0.03,\ 0.73] & [0.32,\ 1.00] & [0.00,\ 0.74] \\ [0.00,\ 0.05] & [0.37,\ 0.69] & [0.11,\ 0.71] & [0.14,\ 0.88] \\ [0.43,\ 0.47] & [0.02,\ 1.00] & [0.14,\ 0.89] & [0.09,\ 0.97] \\ [0.00,\ 0.05] & [0.41,\ 0.73] & [0.02,\ 0.77] & [0.11,\ 0.85] \\ [0.96,\ 1.00] & [0.00,\ 0.84] & [0.00,\ 0.65] & [0.31,\ 1.00] \end{bmatrix} \tag{36}$$

While it is characterized that the results regarding attribute value and the domain both belong to [0, 1] (as per standardized processing). Moreover, as per definition 5, it has been clearly depicted that the interval grey numbers have been represented by their reduced formulations, therefore, the standardized evaluation matrix can be reformulated as given in equation (37):

$$X(\otimes) = \left(\hat{\otimes}_{ij}(g^{\circ}_{ij})\right)_{5\times 4} = \begin{bmatrix} 0.29_{(0.04)} & 0.38_{(0.37)} & 0.66_{(0.68)} & 0.37_{(0.74)} \\ 0.03_{(0.05)} & 0.53_{(0.67)} & 0.41_{(0.61)} & 0.51_{(0.74)} \\ 0.45_{(0.04)} & 0.51_{(0.59)} & 0.51_{(0.75)} & 0.53_{(0.88)} \\ 0.03_{(0.05)} & 0.57_{(0.73)} & 0.39_{(0.74)} & 0.48_{(0.74)} \\ 0.98_{(0.04)} & 0.42_{(0.84)} & 0.32_{(0.65)} & 0.65_{(0.69)} \end{bmatrix} \tag{37}$$

Table I. The river course characteristics

	River level	Flow rate	Temperature	River width
Xiaheyan (a_1)	[43.61,45.94]	[320,780]	[−14.2, −5.2]	[2,900, 7,700]
Shizuishan (a_2)	[27.46,30.28]	[350,560]	[−10.4, −2.3]	[2,000, 6,800]
Bayangol (a_3)	[53.37,55.91]	[146,790]	[−12.7, −2.7]	[1,390, 7,100]
Sanhuhekou (a_4)	[27.22,30.54]	[320,530]	[−11.1, −1.2]	[2,160, 7,000]
Zhaojunfen (a_5)	[85.16,87.83]	[248,800]	[−9.5, −0.9]	[1,200, 5,700]

Table II. Ice-dam disaster characteristics of each section in Ningxia-Inner Mongolia reach during 2012-2016

	The ice dam			
	Time	Proportion (%)	Disaster time	Proportion (%)
Xiaheyan (a_1)	5	15.62	2	18.18
Shizuishan (a_2)	5	15.62	1	9.09
Bayangol (a_3)	7	21.88	3	27.28
Sanhuhekou (a_4)	3	9.38	1	9.09
Zhaojunfen (a_5)	12	37.50	4	36.36
Total	32	100	11	100

Now as the computation of hydrological parameters is not very simple so Delphi method (Ocampoa *et al.*, 2018; Wu *et al.*, 2015) has been applied for more realistic analysis and weighting of each relevant attribute, [Equation (38)]:

$$\omega^* = \{[0.15, 0.25], [0.24, 0.36], [0.12, 0.18], [0.32, 0.38]\}$$
$$= \{0.2_{(0.1)}, 0.3_{(0.12)}, 0.15_{(0.06)}, 0.35_{(0.06)}\} \quad (38)$$

Subsequently, the risk probability of ice flooding in the five sections of the river (Table II) has been determined by using the expected-value criterion.

While the synthetic evaluation vector $C = \{c(a_1), c(a_2), c(a_3), c(a_4), c(a_5)\}$ is integrated y by the enhanced one as done in Section 3.3, respectively. The enhanced algorithms of interval grey numbers vectors are denoted as C_r and the subsequent results are depicted in equation (39):

$$C_r = \{0.4_{(0.15)}, 0.4_{(0.17)}, 0.51_{(0.19)}, 0.4_{(0.26)}, 0.6_{(0.20)}\} \quad (39)$$

Later, the evaluation of supposed values of each algorithm has been computed according to the traditional ranking method of interval grey numbers i.e. according to the mechanism of "kernel" and "degree of greyness". The computational relationship has been built as depicted in equation (40):

$$c_r(a_5) > c_r(a_3) > c_r(a_1) > c_r(a_2) > c_r(a_4) \quad (40)$$

4.3. Relative analytics of enhanced and original algorithm

As mentioned in definition 4, the virtual evaluation vector of the algorithm of interval grey numbers was depicted as; $C = \{c(a_1), c(a_2), c(a_3), c(a_4), c(a_5)\}$, whereas, the vector of original algorithm of interval grey numbers has been denoted as C_i and a relationship is built as per equation (41):

$$C_i = \{0.4_{(0.74)}, 0.4_{(0.74)}, 0.51_{(0.88)}, 0.4_{(0.74)}, 0.6_{(0.84)}\} \quad (41)$$

Later, the evaluation of supposed values of each algorithm has been calculated as per traditional ranking method of interval grey numbers i.e. according to "kernel" and "degree of greyness". Then, a new computational relationship has been structured as per equation (42):

$$c_i(a_5) > c_i(a_3) > c_i(a_1) = c_i(a_2) = c_i(a_4) \quad (42)$$

While considering the Section 4.2 and the subsequent computations for a traditional algorithm, which have been derived in Section 4.3, then the comparative formulations of enhanced algorithm versus traditional algorithm are depicted in Table III.

Apparently, these relative results (Table III) are quite variant. While the analysis of supposed values for a_1, a_2 and a_4 by using original algorithm has depicted an unchanged result. Therefore, it is credible to illustrate that there has been no risk probability variance among Xiaheyan, Shizuishan and Sanhuhekou sites for ice dam formation. However, the further critical investigation has revealed that the risk probability of ice flooding at these three sites could not be effectively determined due to loss of certain data during the operational process. Therefore, the enhanced algorithmic analytics need to be applied for this study, which has clearly identified the differential values regarding variable risk probability of ice flooding disaster at all five sites of the river channels. Conclusively, the Zhaojunfen is the most prone site towards ice damming while the Sanhuhekou being the least prone site.

The comprehensive comparative analysis of the ice dam formation time, disaster time along-with its re-occurrence and risk probability analytics for each selected site of the Yellow River, and subsequent, ranking of results of present study have proved to be effective. That strengthens the significance of the successful application of the enhanced algorithm of the degree of greyness for interval grey numbers.

Moreover, three distinct advantages are determined (as per as from Table III). Firstly, the proposed enhanced algorithm can effectively establish a reasonable algebraic expression to support the logical evidence as depicted under this study. Secondly, a traditional algorithm is always based on an assumption that a domain has been same that is not only for the calculated grey numbers but also for the resultant grey numbers, consequently some information can always be lost during this process. Therefore, it is evident that conventional model cannot depict the actual variable risk-probability analytics regarding formation of an ice dam at three river sites (Xiaheyan, Shizuishan, Sanhuhekou). Furthermore, this study also strengthens the argument that conventional computations regarding domain only address the scope of object rather than determining the overall probability analytics of the domain as per actual operations. Finally, it can be visualized that the calculation of degree of greyness by enhanced algorithm is more practical than the conventional methods. That is, evident from the results depicted in Table III.

Finally, it has also been evident that certain characteristic features of ice flooding in the Yellow River like prolonged duration, sudden and severe flooding make the situation much severe than ordinary storm flooding, which is undeterminable by the application of conventional algorithmic operational rules. That is why this study has performed a relative risk assessment for all selected sites of the river based on a comprehensive past data that enabled a comparative ranking of each site as per the intensity of ice damming. Hence, the algorithm proposed in this study has not only simplified the evaluation process but also ensures the accuracy and validation of results. That would help decision-makers to formulate an evidence-based policy and scientific evidence-based strategies s for disaster risk management. However, despite such utility of enhanced algorithm it still has some disadvantages and limitations e.g. accurate determination of domain for each of the interval grey number can always be tough. Thus, such research questions should be deliberated in the future studies.

5. Conclusions

This study has focused on improving the algorithm of the degree of greyness for interval grey numbers. The variation of domain in the operational process, the degree of greyness

Table III. Relative resultant formulations of enhanced algorithm versus traditional algorithm

category	Ranking of resultant
Enhanced algorithm	$c_r(a_5) > c_r(a_3) > c_r(a_1) > c_r(a_2) > c_r(a_4)$
Traditional algorithm	$c_i(a_5) > c_i(a_3) > c_i(a_1) > c_i(a_2) > c_i(a_4)$

can effectively determine and define operational results. While some specific expressions for algebraic operations have also been presented, which are linked to the kernel, the degree of greyness and the domain. Moreover, these expressions are proved to be practically effective while computing the algorithm of interval grey numbers based on the kernel and the degree of greyness, which has raised the accuracy of the operational results accordingly. Finally, the feasibility of the enhanced algorithm of the degree of greyness is also analyzed with the help of a case study and the results indicated that an enhanced algorithm can effectively reduce the information and data loss in the operational process. That has facilitated the decision-making process because of increased credibility of results.

However, to facilitate the value ranges of the interval grey numbers obtained by the multiplication and division operations, there are certain limitations regarding the domain calculations, which can reduce application range of this enhanced algorithmic method. Therefore, there is still more research followed by this research, which is how to minimize limitations and expansion of application platform of this algorithm, and it still needs to consummate the determination of domain because the domain of interval grey numbers usually has been given by common sense.

References

Deng, J.L. (1982), "Control problems of grey systems", *Systems and Control Letters*, Vol. 1 No. 5, pp. 288-294.

F. Liu, S. (1987), "The Perron-Frobenius theorem of grey nonnegative matrices and its proof", *Journal of Henan Agricultural University*, Vol. 21 No. 4, pp. 101-108.

F. Liu, S. (1996), "Axiom on grey degree", *Journal of Grey System*, Vol. 8 No. 4, pp. 397-400.

Fang, Z.G., Liu, S.F., Lu, F., et al. (2005), "Study on improvement of token and arithmetic of interval grey numbers and its GM(1,1) model", *Chinese Engineering Science*, Vol. 7 No. 2, pp. 57-61.

Gholami, M. and Brennan, R.W. (2016), "Comparing two clustering-based techniques to track mobile nodes in industrial wireless sensor networks", *Journal of Systems Science and Systems Engineering*, Vol. 25 No. 2, pp. 177-209.

Guo, S.D., Liu, S.F. and Fang, Z.G. (2016), "Multi-attribute decision-making model based on kernel and degree of greyness of interval grey numbers", *Control and Decision*, Vol. 31 No. 6, pp. 1042-1046.

Heravi, G., Fathi, M. and Faeghi, S. (2016), "Multi-criteria group decision-making method for optimal selection of sustainable industrial building options focused on petrochemical projects", *Journal of Cleaner Production*, Vol. 142 No. 4, pp. 2999-3013.

Jiang, S.Q., Liu, S.F., Fang, Z.G., et al. (2017), "Study on distance measuring and sorting method of general grey number", *Grey Systems: Theory and Application*, Vol. 7 No. 3, pp. 320-328.

L. Deng, J. (1985), *Grey System: Social and Economic*, National Defense Industry Press, Beijing.

Li, Q.X., Liu, S.F., Forrest, J., *et al.* (2015), "Fundamental definitions and calculation rules of grey mathematics: a review work", *Journal of Systems Engineering and Electronics*, Vol. 26 No. 6, pp. 1254-1267.

Liu, S.F., Dang, Y.G. and Fang, Z.G. (2010), *Grey System Theory and Applications*, *(*5th ed.*)*, Science Press, Beijing.

Liu, S.F., Fang, Z.G. and Xie, N.M. (2010), "Algorithm rules of interval grey numbers based on the 'kernel' and the degree of greyness of grey numbers", *Journal of Systems Engineering and Electronics*, Vol. 32 No. 2, pp. 313-316.

Liu, X.L., Moreno, B. and Garcia, A.S. (2016), "A grey neural network and input-output combined forecasting model. Primary energy consumption forecasts in Spanish economic sectors", *Energy*, Vol. 115, pp. 1042-1054.

Liu, S.F. and Yang, Y.J. (2017), *J. Forrest. Grey Data Analysis*, Springer-Verlag, Singapore.

Liu, S.F., Yang, Y.J., Xie, N.M. and Forrest, J. (2016), "New progress of grey system theory in the new millennium", *Grey Systems: Theory and Application*, Vol. 6 No. 1, pp. 2-31.

Ma, X. and Liu, Z.B. (2018), "The kernel-based nonlinear multivariate grey model", *Applied Mathematical Modelling*, Vol. 56, pp. 217-238.

Ma, X., Xie, M., Wu, W.Q., *et al.* (2019), "The novel fractional discrete multivariate grey system model and its applications", *Applied Mathematical Modelling*, Vol. 70, pp. 402-424.

Ocampoa, L., Ebisab, J.A., Ombeb, J. and Escoto, M.G. (2018), "Sustainable ecotourism indicators with fuzzy delphi method-A philippine perspective", *Ecological Indicators*, Vol. 93, pp. 874-888.

Wang, X. (2017), "Approach for multiple attribute decision-making with interval grey number based on choquet integral", *Journal of Intelligent and Fuzzy Systems*, Vol. 32 No. 6, pp. 4205-4211.

Wang, J.J., Hipel, K.W. and Dang, Y.G. (2017), "An improved grey dynamic trend incidence model with application to factors causing smog weather", *Expert Systems with Applications*, Vol. 87, pp. 240-251.

Wang, C.N., Lin, H.S., Hsu, H.P., et al. (2016), "Applying data envelopment analysis and grey model for the productivity evaluation of vietnamese agroforestry industry", *Sustainability*, Vol. 8 No. 11, p. 1139.

Wu, L.F., Liu, S.F. and Yang, Y.J. (2015), "A model to determine OWA weights and its application in energy technology evaluation", *International Journal of Intelligent Systems*, Vol. 30 No. 7, pp. 798-806.

Wu, J., Sun, J.S. and Song, M.L. (2013), "A ranking method for DMUs with interval data based on DEA cross-efficiency evaluation and TOPSIS", *Journal of Systems Science and Systems Engineering*, Vol. 22 No. 2, pp. 191-201.

Xie, N.M. and Xin, J.H. (2014), "Interval grey numbers based multi-attribute decision making method for supplier selection", *Kybernetes*, Vol. 43 No. 7, pp. 1064-1078.

Xie, M. and Xiao, X. (2011), "Grey decision rules for interval MADA based on rough set theory", *IEEE International Conference on Grey Systems and Intelligent Services, IEEE*, pp. 866-869.

Xu, N., Dang, Y.G. and Cui, J. (2015), "Comprehensive optimized GM (1,1) model and application for short term forecasting of Chinese energy consumption and production", *Journal of Systems Engineering and Electronics*, Vol. 26 No. 4, pp. 794-801.

Yang, Y.J., John, R. and Liu, S.F. (2012), "Some extended operations of grey sets", *Kybernetes*, Vol. 41 Nos 7/8, pp. 860-873.

Yang, Y.J. and Liu, S.F. (2011), "Reliability of operations of grey numbers using kernels", *Grey Systems: Theory and Application*, Vol. 1 No. 1, pp. 57-71.

Ye, J., Dang, Y.G. and Ding, S. (2016), "Grey prediction model of interval grey numbers based on axiom of generalized non-decrease grey degree", *Control and Decision*, Vol. 31 No. 10, pp. 1831-1836.

Zeng, X.Y., Shu, L., Yan, S.L., et al. (2019), "A novel multivariate grey model for forecasting the sequence of ternary interval numbers", *Applied Mathematical Modelling*, Vol. 69, pp. 273-286.

Corresponding author
Junjie Wang can be contacted at: junjie881129@163.com