The law of the wall in turbulent flow

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The 'law of the wall' for the inner part of a turbulent shear flow over a solid surface is one of the cornerstones of fluid dynamics, and one of the very few pieces of turbulence theory whose results include a simple analytic function for the mean velocity distribution, the logarithmic law. Various aspects of the law have recently been questioned, and this paper is a summary of the present position. Although the law of the wall for velocity has apparently been confirmed by experiment well outside its original range, the law of the wall for temperature seems to apply only to very simple flows. Since the two laws are derived by closely analogous arguments this throws suspicion on the law of the wall for velocity. Analysis of simulation data, for all the Reynolds stresses including the shear stress, shows that law-of-the-wall scaling fails spectacularly in the viscous wall region, even when the logarithmic law is relatively well behaved. Virtually all turbulence models are calibrated to reproduce the law of the wall in simple flows, and we discuss whether, in practice or in principle, their range of validity is larger than that of the law of the wall itself: the present answer is that it is not; so that when the law of the wall (or the mixing-length formula) fails, current Reynolds-averaged turbulence models are likely to fail too.

1. Introduction

'Law of the wall' is the forceful name for the finding that, with certain assumptions, the mean velocity U in constant-property turbulent flow near a smooth impermeable solid surface of negligible curvature can be correlated in terms of the shear stress at the surface τ_w, the distance from the surface y, and the fluid properties ρ (density) and μ (molecular viscosity). The assumptions are at first sight sweeping: that the *only* effect of the outer flow (say, the outer 80% of a boundary layer's thickness) on the inner flow is to determine τ_w. Quantities such as the layer thickness δ or duct diameter, and the edge velocity U_e or other overall mean velocity scale, are supposed not to matter; effects of the upstream history of the flow, and even of local streamwise (x-wise) pressure gradient and x-wise or y-wise shear-stress gradients, are also neglected. Simple dimensional analysis gives

$$\frac{U}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right), \tag{1.1a}$$

and also less-familiar but consequent relations for the turbulence quantities, such as

$$\frac{-\overline{uv}}{u_\tau^2} = g\left(\frac{u_\tau y}{\nu}\right), \tag{1.1b}$$

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where $u_\tau \equiv \sqrt{\tau_w/\rho}$ and $\nu \equiv \mu/\rho$. The part of the layer thickness δ over which the law of the wall is valid is usually called the 'inner layer' or 'wall layer', the rest being called the 'outer layer'. The outer limit of validity is at most $y = 0.2\delta$ and is generally accepted to become smaller in flows in strong pressure gradients, especially those approaching separation. Very close to the surface (typically $u_\tau y/\nu < 3$) the Reynolds shear stress is negligible and the viscous stress law integrates to give $U = \tau_w y/\mu$, so that (1.1*a*) is always true there, in the linear form $U/u_\tau = u_\tau y/\nu$. In § 2, we will derive the well-known logarithmic (hereafter 'log') law

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln \frac{u_\tau y}{\nu} + C, \tag{1.2}$$

with $\kappa \approx 0.41$ and $C \approx 5.0$ from experimental data. It applies from the outer edge of the *viscous wall region* at $u_\tau y/\nu \approx 30$–50, where viscous effects on the turbulent shear stress become small, to the outer limit of validity of (1.1).

Now (1.1*b*) is in principle a necessary condition for the validity of (1.1*a*), because the mean velocity and shear stress are connected by the mean momentum equation, which under the above assumptions requires that the total shear stress shall be independent of y. (Townsend (1956) shows that self-similar shear-stress profiles do not imply exactly self-similar velocity profiles except at infinite Reynolds number, but the errors are certainly not important in the present context.) Thus, to a good approximation, all turbulence quantities that affect $\overline{uv}$ should scale like (1.1*b*) or its equivalents for higher-order statistics. These turbulence quantities necessarily include all those that appear in the Reynolds shear-stress transport equation, or in *their* transport equations. However, it appears from experiments and simulations that turbulence quantities can show remarkably large departures from law-of-the-wall scaling and one of the purposes of this paper is to explore this paradox.

An important use of the law of the wall is in the measurement of surface shear stress, either by fitting (1.1*a*) to measured velocity profiles or by using surface-mounted pitot tubes. It follows from (1.1*a*) that the difference $P - p$ between the pitot pressure and the static pressure, for a family of surface-mounted probes of given shape and a range of sizes d, can be correlated as

$$\frac{(P-p)d^2}{\rho\nu^2} = h\left(\frac{\tau_w d^2}{\rho\nu^2}\right), \tag{1.3}$$

where the argument of the function h is just the square of $u_\tau d/\nu$. The inverse relation gives τ_w from measured $P - p$. The best-known configuration is the Preston tube, a forward-facing circular-section tube whose diameter d should be somewhat smaller than the largest value of y at which (1.1*a*) holds. The geometry is easily reproduced, and the calibration of Patel (1965) is still accepted as accurate to ±2% or better (Zurfluh (1984) gives a piecewise-cubic spline relation for $(P-p)/\tau_w$, based partly on Patel's calibration and partly on his own data: the latter agree with the calibration to ±0.6%). A predecessor of the Preston tube is the Stanton tube, usually a flattened pitot tube or simply a fence with slots in the surface fore and aft to measure a pressure difference, which is intended to reside wholly within the linear part of the *viscous sublayer*, $u_\tau y/\nu \lesssim 3$. In principle, a sufficiently small Stanton tube is independent of law-of-the-wall scaling, though its reading can still be affected by the non-universal velocity fluctuations discussed in § 4, and in practice the flow field of a body with low Reynolds number is so much larger than the body that it is difficult to make a Stanton tube whose flow field is immersed entirely in the linear sublayer.

Most of the above remarks apply to other sublayer devices, such as the near-wall hot wire or pulsed wire, the surface hot film or pulsed film, and electrochemical surface gauges. The Preston tube is preferred because it is easier to reproduce and can, with care, be resited at different positions on the surface. Clearly, the Preston tube cannot be used as an independent check on the law of the wall, and any discrepancy between values of τ_w from a Preston tube and from a *satisfactory* fit to the law of the wall is probably due to simple experimental error. Various workers have used Stanton tubes to check Preston tubes and/or the law of the wall. The most that one should say is that if Preston and Stanton tube measurements of τ_w differ, then either the Preston-tube value, or both, are wrong: there is no real guarantee that if the measurements agree they are right.

Preston tubes and Stanton tubes are almost certainly the most widely used instruments for measuring skin friction. *All the doubts expressed in this paper about the universality of the law of the wall also apply to Preston and Stanton tubes.*

In boundary layers in pressure gradient, and in duct flows, the total (viscous plus turbulent) shear stress τ varies significantly over the inner layer, which raises questions about the use of τ_w to give a velocity scale: surely local τ would be more plausible? (The 'mixing length formula', to be discussed below, does use local τ.) The shear stress at $y/\delta = 0.2$ in fully developed flow in a plane duct or circular pipe is 0.8 of the wall value, but it is generally accepted that the law of the wall is the same as in a boundary layer in zero pressure gradient, where the change in shear stress across the inner layer is smaller. (In fact the mixing-length formula predicts that the shear-stress gradient in a duct produces a difference in U/u_τ of only 0.1 at $y/\delta = 0.2$ and this is almost undetectable in an experiment.) Furthermore, the law of the wall is found to be the same, but with a smaller region of validity, in boundary layers with rather more than ±20% change in shear stress across the inner layer. Deviations from the law of the wall appear at low bulk Reynolds numbers, as was pointed out by Patel & Head (1969), and this has led to difficulties in interpreting the current generation of direct numerical simulations of turbulence, which are limited to Reynolds numbers, based on δ, of about 10^4. The most obvious reason for Reynolds-number dependence is simply that δ is no longer very large compared to the thickness of the viscous wall region, $30\nu/u_\tau$ so that the conditions for the appearance of an extended region of log law are not so well fulfilled. However, in addition, low-Reynolds-number flows in pressure gradient will have large values of shear-stress gradient $\partial\tau/\partial y$ in 'wall units', i.e. $(\nu/\rho u_\tau^3)(\partial\tau/\partial y)_w \equiv \Delta_\tau$, which appear to affect the viscous wall region and thus the value of C in (1.2), in addition to any effects on κ. Note that in a developing flow $\partial\tau/\partial y = \mathrm{d}p/\mathrm{d}x$ only at the wall: in the log law region $\partial\tau/\partial y \approx \frac{1}{2}\mathrm{d}p/\mathrm{d}x$ is a better approximation in a boundary layer (see also the appendix to Spalart 1988). In simple cases—where the bulk Reynolds number is large and the flow is not changing too rapidly with x—the deviations in C, or in the Van Driest 'damping constant' A^+, can be correlated in terms of Δ_τ (Huffman & Bradshaw 1972): see also § 4.

The above comments about range of validity are imprecise, but amount to the statement that the law of the wall for mean velocity appears to hold in most simple turbulent wall flows. 'Tenacious' is the memorable word used by Kline in a position paper for the 'Ringi' group on turbulent boundary layer structure (now administered by Professor W. G. Tiederman, University of Florida). An indication that the law of the wall is less reliable than commonly believed comes from observations of the law of the wall for mean temperature (§ 2 *b*: hereafter the T-law), derived by an analogous dimensional analysis on the basis of analogous assumptions. Again, this law is well confirmed by measurements in flat-plate boundary layers and simple duct flows. It is harder to measure temperature and heat-flux rate than velocity and skin friction, and the prestige of the law of the wall for velocity (the U-law) has made us too ready to dismiss observed deviations from the T-law as the result of experimental error: several sets of measurements made in the late 1960s (Perry *et al.* 1966; Thielbar *et al.* 1969; Blackwell *et al.* 1972) showed that the T-law failed in boundary layers in pressure gradient, but they have been largely ignored. However, there is now sufficient evidence that the T-law is comparatively fragile: it breaks down before the U-law in boundary layers with high free-stream turbulence (Maciejewski & Moffat 1990) or streamline curvature (Gibson 1990). The sensitivity to pressure gradient (which does not appear in the temperature-field equations in low-speed flow) is remarkable.

Now since the derivation of the T-law closely parallels that of the U-law, one would expect their ranges of validity to be broadly similar. This raises the question 'Is the tenacity of the U-law just good luck, and if so when does our luck run out?'. A surprising conclusion of the analysis of simulation data in this paper is that, in the *lower* part of the viscous wall region, the Reynolds shear stress fails to obey (1.1*b*)—it depends significantly on bulk Reynolds number as well as flow type. The cumulative effect on the mean velocity at the edge of the viscous wall region is small in duct flows and boundary layers in zero pressure gradient, but significant in flows with large streamwise gradients, at least at low bulk Reynolds number.

Since we are concerned with relatively small quantitative discrepancies, qualitative data on inner-layer turbulence structure are unlikely to be useful. Therefore we will not discuss structure in detail in this paper, though quantities such as streak spacing (Kim *et al.* 1971) or burst rate may be useful diagnostics. For a review, see Robinson (1991).

In § 2 we present the 'bookwork' for the laws of the wall for velocity and temperature, with special attention to the weak points, and in § 3 we do the same for the mixing-length formula on which various extensions of the law of the wall are based. Throughout the paper, 'law of the wall' means the formula in (1.1) or the corresponding result for temperature, including the log laws. For convenience we use 'log law region' for the fully turbulent part of the inner layer, whether or not the log laws actually apply in the flow being discussed. Section 4 deals with the viscous wall region, making use of recent simulation results. Section 5 discusses turbulence models, nearly all of which are calibrated to reproduce the laws of the wall in simple situations. The final question posed by this paper is whether we can expect these models to predict deviations from the law of the wall. The present evidence is only moderately encouraging.

2. The logarithmic laws...

(a) ... *for velocity*

An equivalent of (1.1), obtainable either by differentiating (1.1) or by substituting $\partial U/\partial y$ for U in the list of variables, is

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{y} F\left(\frac{u_\tau y}{\nu}\right). \tag{2.1}$$

For large $u_\tau y/\nu \equiv y^+$, we expect ν not to affect the energy-containing shear-stress-bearing eddies or their relation to the mean flow, because y^+ is a representative

Reynolds number of the energy-containing eddies as well as of the mean flow. Therefore, for large enough y^+, we expect F to become constant, equal to $1/\kappa$, say, so that

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}, \tag{2.2}$$

and, integrating and requiring compatibility with (1.1a), we get

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln \frac{u_\tau y}{\nu} + C, \tag{2.-1}$$

where C is related to the increase in U across the viscous wall region, $y^+ < 30$–50.

By experiment, (1.2) is valid, with $\kappa \approx 0.41$ and $C \approx 5.0$, for $y^+ \gtrsim 30$ but $y/\delta \lesssim 0.1$–0.2, in boundary layers in small pressure gradient or in pipe or duct flows, provided that the bulk Reynolds number is not too small. Note that the outer limit of validity must depend on the outer-layer length scale δ rather than the viscous length scale ν/u_τ, because viscous effects are negligible in the log law region.

The above derivation of the log law is due to Landau (1944 work quoted by Landau & Lifshitz (1993)). Most Western textbooks give (usually abbreviated) versions of the Millikan 'overlap' argument—a formalization of earlier work by Prandtl and von Kármán—which requires the whole of the layer outside the viscous wall region to scale in terms of the local variables u_τ and δ, independent of the viscosity and of upstream history of the flow. The Millikan derivation is appealing and thought-provoking, but in fact adds little to that based on (2.1) alone: in both cases the key is the unimportance of viscosity for $y^+ > 30$–50, which obviously has to be established before the outer layer can be scaled. The reason for working with (2.1) is that outer-layer scaling is certain to depend much more on upstream history than (1.1) or (2.1), simply because the eddies in the outer layer are larger and have longer lifetimes. Indeed the outer layer scales on u_τ and δ only in fully developed duct and pipe flows, in boundary layers in zero pressure gradient, and in so-called 'equilibrium' boundary layers with pressure distributions that are rare in real life: the functional dependence on y/δ is different in each case. It would be very surprising if (2.1) were so restricted: however, this seems to be almost the case for the T-law.

Interestingly, the derivation of (1.1) and (1.2) at the start of this section is conceptually similar to the standard derivation of Kolmogorov's 'universal equilibrium' spectrum, almost the only other piece of turbulence theory that leads to an analytical result. The standard derivation is stated succinctly by Phillips (1991); 'Kolmogorov's first similarity hypothesis is that the statistical structure of the components in the equilibrium range, being independent of the larger scales, can depend only on [ϵ], the rate of energy dissipation and ν which determines the spectral distribution of dissipation, so that, in particular, the energy spectrum is, on dimensional grounds, of the form...' $v^2\eta f(k\eta)$. Here the velocity scale $(\epsilon\nu)^{1/4} \equiv v$ corresponds to u_τ and the length scale $(\nu^3/\epsilon)^{1/4} \equiv \eta$ corresponds to ν/u_τ, and the $-\frac{5}{3}$ power law in the inertial (inviscid) subrange corresponds to the log law. An 'overlap' region is not usually invoked quantitatively in the derivation of the Kolmogorov spectrum, though Mellor (1972) has used it in a derivation based on the spatial-correlation structure functions rather than the spectra.

Alternative formulations of the law of the wall have been discussed recently by Barenblatt and co-workers (Barenblatt 1993; Barenblatt & Prostokishin 1993) and by George & Castillo (1993). Both discuss possible Reynolds-number dependence—that instead of F becoming a constant at large y^+, giving a log law, it continues to change, in such a way that the velocity gradient (and velocity) obey power laws with coefficients that depend on Reynolds number. George & Castillo use an overlap argument for equilibrium (self-preserving) boundary layers, based on allowing different scaling factors for the mean velocity and for the shear stress in the outer layer, instead of using u_τ for both. In most of the present paper the emphasis is on the larger effect of rapid streamwise changes and/or the behaviour of the viscous wall region, which these alternative formulations do not address directly. Therefore we will, without prejudice, base the discussion on the traditional log law: most of what is said would apply to the power laws as well.

(*b*) ... *for temperature*

Kader (1991) discusses the law of the wall for temperature (and other possible similarity laws for temperature profiles). It results from the assumption that for $y \ll \delta$ and $y \ll \delta_T$ (where δ_T is the thickness of the thermal boundary layer, $\leqslant \delta$), the temperature relative to the wall, $T_\mathrm{w} - T$, depends only on τ_w, on the surface heat-transfer rate $\dot{q}_\mathrm{w}$, on the distance from the surface y, and on the fluid properties ρ, μ, the specific heat c_p and the thermal conductivity k (again we assume that all fluid properties are substantially constant). From dimensional analysis we get

$$\frac{T_\mathrm{w} - T}{T_\tau} = f_T\left(\frac{u_\tau y}{\nu},\ \frac{\mu c_p}{k} \equiv Pr,\ \frac{\dot{q}_\mathrm{w}}{u_\tau \tau_\mathrm{w}} \equiv B_q\right). \tag{2.5}$$

The analogy with (1.1) is clear. Here $T_\tau \equiv \dot{q}_\mathrm{w}/(\rho c_p u_\tau)$ is the 'friction temperature' and the last parameter on the right, B_q, represents the ratio of heat transfer from the surface to work done against fluid friction (i.e. energy dissipation): in low-speed flow, energy dissipation is small and this parameter can be ignored. By considering $\partial T/\partial y$ instead of T we obtain an equation analogous to (2.1),

$$\frac{\partial T}{\partial y} = -\frac{T_\tau}{y} F_T\left(\frac{u_\tau y}{\nu}, Pr\right). \tag{2.6}$$

For F_T to become constant we require the effects both of viscosity and of thermal conductivity to be small, requiring both $u_\tau y/\nu \equiv y^+$ and its temperature-field analogue $u_\tau y/(k/\rho c_p) \equiv y^+\,Pr$ to be large. On general grounds of analogy between heat and momentum transfer we expect, and find, that $y^+\,Pr$ must be greater than 30–50: clearly this is a stronger condition than $y^+ > 30$–50 if $Pr < 1$, and it caused confusion in the early days of experiments on turbulent flows of liquid metals, for which $Pr \ll 1$. Given that $F_T = \text{const.}, = 1/\kappa_T$ say, and requiring compatibility with (2.5), we get

$$\frac{T_\mathrm{w} - T}{T_\tau} = \frac{1}{\kappa_T} \ln \frac{u_\tau y}{\nu}, + C_T \tag{2.7}$$

with C_T a function of Pr, equal to about 3.9 for air ($Pr = 0.71$).

The turbulent Prandtl number, Pr_t, is the ratio of the apparent diffusivity of momentum to that of heat, and is defined in terms of *local* variables by

$$Pr_\mathrm{t} \equiv \frac{\overline{uv}/(\partial U/\partial y)}{\overline{vT'}/(\partial T/\partial y)}, \tag{2.8}$$

where $-\rho\overline{uv}$ and $\rho c_p \overline{vT'}$ are the turbulent shear stress and heat flux. In the region of validity of (2.2), and of (2.6) with $F_T = 1/\kappa_T$ (the 'log region'), we find that $Pr_\mathrm{t} = \kappa/\kappa_T$. By experiment (see, for example, Kays & Crawford 1993), (2.7) is valid, for $y^+ Pr > 30$–50, in boundary layers with zero pressure gradient and in pipe flows. The outer limit depends on the thickness of the temperature profile δ_T, which is less than δ if the surface has an 'unheated starting length'. The data show that $\kappa_T \approx 0.48$, implying that the turbulent Prandtl number is approximately 0.85 in the log region. Reynolds' analogy between heat transfer and momentum transfer is of course plausible only if Pr_t is fairly close to unity, though this is not explicitly assumed in the above analysis. Guezennec *et al.* (1990) showed by analysing simulation data that the processes of turbulent diffusion of momentum and of heat are very different in detail. The reason is that pressure fluctuations affect turbulent momentum transfer but not heat transfer: thus, for example, the eruptions of fluid away from the wall in the inner layer have a larger spatial extent in velocity than in vorticity or temperature (or smoke or dye—a warning to practitioners of flow visualization). Therefore, it is somewhat of a coincidence that Pr_t is close to unity in the log region: it is well known that Pr_t falls to as little as 0.5 near free-stream edges. Admittedly there is no obvious reason why these structural differences should result in tenacity of the U-law and fragility of the T-law.

3. Status of the 'mixing length' formulae

(*a*) *Effect of shear-stress gradient*

In boundary layers in pressure gradient—for example—τ varies with y even in the inner layer, and τ_w usually varies more quickly with x than in zero pressure gradient. An obvious extension of the spirit of the law-of-the-wall analyses is to use the local shear stress and heat-transfer rate instead of the surface values. In practice the y-wise heat transfer rate in low-speed flow near a uniformly heated surface, $\dot{q}$, varies only slowly with y in the inner layer so $\dot{q}_w$ may still be an adequate approximation. For the velocity field the generalization of (2.2) is

$$\frac{\partial U}{\partial y} = \frac{\sqrt{\tau/\rho}}{\kappa y}, \tag{3.1}$$

which is of course the 'mixing length' formula. The analogous mixing length formula for temperature, replacing (2.6) (with $F_T = 1/\kappa_T$), is

$$\frac{\partial T}{\partial y} = -\frac{\dot{q}/(\rho c_p)}{\sqrt{\tau/\rho}\kappa_T y}. \tag{3.2}$$

In the above, $\tau = -\rho\overline{uv}$ and $\dot{q} = \rho c_p \overline{vT'}$ (molecular contributions being negligible in the region of validity of these formulae). The mixing length *theory* was in effect an analogy with the kinetic theory of gases, with lumps of fluid exchanging momentum by occasional collisions. The mixing length is analogous to the mean free path: in the log region it is just $\kappa y \approx 0.41y$, and one would not trust continuum (gradient-transport) approximations for gases with a mean free path as large as 40% of the distance from the solid surface! There is an interesting philosophical point: if a theory is dimensionally correct and leads to a result which could be obtained by dimensional analysis alone, the theory need not be *physically* correct—and so it is with the mixing length theory.

The derivation of the mixing length formula presented by Townsend (1961) is based on an analysis of the turbulent kinetic energy equation which at first sight adds little to the dimensional analysis. In fact, Townsend considers the possible effect of y-wise transport of turbulent energy ('diffusion'), which introduces $(y/\tau)\partial\tau/\partial y$ as a parameter, but estimates its effects to be small except very close to separation. If $\tau = \tau_w + \alpha y$, where α is a constant, Townsend shows that the integral of (3.1)—ignoring diffusion—is simply the log law (1.2) with the addition of a function of $\alpha y/\tau_w$, equal to $1/(2\kappa)\alpha y/\tau_w$ if α is small. In principle, the constant C in (1.2) can be a function of $\alpha\nu/(\rho u_\tau^3)$, but except in low-Reynolds-number flows the effect seems to be small in flows not too close to separation.

The 'mixing length' formula (3.1) seems to work for boundary layers with suction or injection (where, in negligible pressure gradient, the momentum equation reduces to $\rho V_w \partial U/\partial y = \partial\tau/\partial y$, leading to $\tau = \tau_w + \rho U V_w$). However, the constant of integration that corresponds to C in (1.2) is allowed by the dimensional analysis to be a function of V_w/u_τ, and can therefore be chosen to give the best fit to given data, which may compensate to some extent for any inaccuracy of (3.1).

In compressible flow, ρ varies with y so that τ/ρ varies even if τ does not, and the Van Driest transformation for the inner layer of compressible boundary layers (e.g. Bradshaw 1977; Huang *et al.* 1993) follows from the 'mixing length' formulae for velocity and for temperature (the latter in the form $Pr_t = \kappa/\kappa_T$). The constants of integration are functions of the 'friction Mach number' $M_\tau \equiv u_\tau/a_w$, where a_w is the speed of sound at the wall, and of the heat-transfer parameter B_q on the right-hand side of (2.5) ..., but this time the 'incompressible' values of the constants give a good fit to data, and there is general confidence in the accuracy of the Van Driest transformation in compressible boundary layers in small pressure gradients. It can be shown that the constant of integration of the temperature formula must be a linear function of B_q for small B_q, in order that C_T in the 'incompressible' T-law (2.7) shall be a true constant, but this is a nice point lost in the data scatter.

Thus, in two cases where the numerator of (3.1) varies significantly with y but streamwise rates of change are small, the mixing length formula for velocity seems to work well, and the compressible-flow results give reasonable confirmation of the constancy of turbulent Prandtl number in the fully turbulent part of the inner layer. We now consider flows with large streamwise rates of change, especially boundary layers in pressure gradient: of course, such flows usually have large $\partial\tau/\partial y$ as well.

(b) *Flows with large streamwise gradients*

In boundary layers in strong favourable or adverse pressure gradients, Patel (1965) found departures from the log law with the sign predicted by the mixing-length formula—but apparently with rather larger magnitude (though Patel (1973) successfully used the mixing length formula to correlate data for boundary layers with severe pressure gradient or transverse surface curvature). In fact, Townsend's result for $\partial\tau/\partial y = \alpha =$ const., quoted above, shows that even where the shear stress has increased to twice the wall value, the value of U/u_τ predicted by the mixing length formula exceeds the log-law value by only about 1.2—typically about 6%. From his measured velocity profiles, Patel (1965) deduced the range of dimensionless pressure gradient $\Delta_p \equiv (\nu/\rho u_\tau^3)\,dp/dx$, within which a Preston tube with $u_\tau d/\nu \ll 200$ could be relied on to 3% accuracy: $-0.005 < \Delta_p < 0.01$. (Strictly the limit should be based on $\partial\tau/\partial y$—i.e. on Δ_τ as defined above—rather than dp/dx.)

Patel's conclusions represented the received wisdom, that departures from the log law in strong pressure gradient were significant but at least roughly in agreement with the mixing-length formula. Huffman & Bradshaw (1972), studying the effect of large $\partial\tau/\partial y$ on the viscous wall region, found that the mixing-length formula adequately described the velocity profile outside that region: however, the flows they studied changed fairly slowly in the x direction.

More recently, several other workers (e.g. Galbraith *et al.* 1977; Rodi & Scheuerer 1986), have found that the plain log law (1.2) usually works *better* than the integral of the mixing-length formula (3.1) in boundary layers in pressure gradients (especially adverse pressure gradient). There is no rigorous explanation for the success of (1.2) outside its expected region of validity, and this is one of the main questions posed by this paper. Evidently, the effect of $\partial\tau/\partial y$ in increasing $\partial U/\partial y$ according to the mixing length formula is approximately cancelled by the effect of (i) y-wise diffusion and/or (ii) x-wise gradients (advection). In their direct numerical simulation (DNS) of a boundary layer in adverse pressure gradient at low Reynolds number, Spalart & Watmuff (1993) found that U fell below the log law, i.e. in the opposite sense to the deviation predicted by the mixing-length formula for positive $\partial\tau/\partial y$, and by roughly the same amount. This agrees qualitatively with the measurements of Nagano *et al.* (1991) in adverse pressure gradient. Gasser *et al.* (1993; see also Hirt & Thomann 1986) compared Preston tubes, Stanton tubes and wall pulsed wires to a floating-element balance and found that all read low in a boundary layer downstream of a small separation bubble (and in boundary layers downstream of a positive shear-stress minimum): the above list of devices is in order of decreasing size and—as expected—also in order of decreasing error. These results are consistent with a velocity profile that falls below the log law in the presence of large positive $\partial\tau/\partial y$. However, Gasser *et al.* (1993) found that the devices tended to read high in boundary layers in strong adverse pressure gradient (decreasing surface shear stress), though the trend of error with device size was not consistent. Finally, Le *et al.* (1993), in a DNS of the reattached flow downstream of a backward-facing step, again found that U fell below the log law, in qualitative agreement with the surface-tube results of Gasser *et al.* (1993) behind a small separation bubble: however, the backstep flow is essentially a reattached mixing layer with a very different structure from a conventional boundary layer.

The above discussion suggests that it is unlikely that departures from the law of the wall (or, indeed, from the mixing-length formula) in rapidly changing flows can be correlated solely in terms of τ/τ_w, $\partial\tau/\partial y$ or of any other local parameter. If y-wise diffusion of Reynolds stresses were important, the mixing-length formula would not work in flows with suction or injection, nor, probably, in compressible flows with large y-wise density gradients. For quantitative purposes, we need to consider the effects of flow history. Specifically, we should investigate the universality or otherwise of the coefficients in turbulence models based on differential equations that allow for flow history (in practice, two-equation models or better). We return to turbulence modelling in § 5.

(c) *The law of the wall for temperature*

The T-law seems to fail spectacularly in flows whose velocity field changes rapidly in the x direction: data are still fairly scarce, but Kays (personal communication, 1994) regards the evidence as overwhelming. It should be recorded that Head (1969) suggested that the T-law analysis was in error, 'either because of some error in the particular assumptions for heat transfer or (more seriously) in the basic assumptions of mixing length theory' (read 'law-of-the-wall analysis'). The T-law is generally well supported by measurements in flows with slowly changing velocity fields.

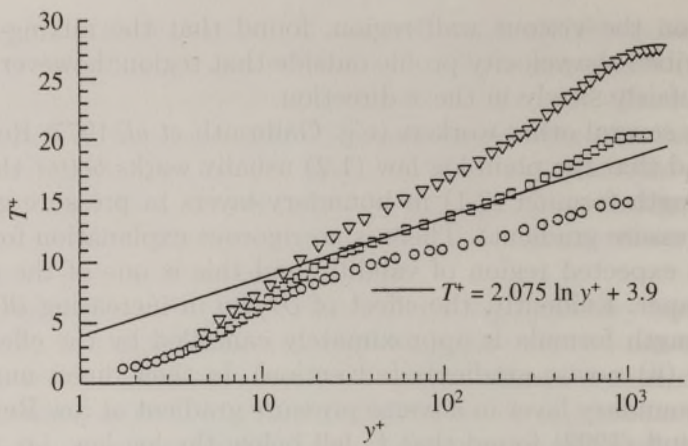

Figure 1. Temperature profiles in favourable (Thielbahr *et al.* 1969; ▽) and adverse (Blackwell *et al.* 1972; ○) pressure gradient. $T^+ \equiv (T_w - T)/T_\tau$. From Kays & Crawford (1993).

Experimental evidence for boundary layers in strong pressure gradients is limited but consistent. Perry *et al.* (1966) show temperature profiles in adverse pressure gradient which run roughly parallel to (2.7) but well below it. At the nearest measurement point to the wall, $(T_w - T)/T_\tau$ is still considerably lower than the pure-conduction limit, $(u_\tau y/\nu)\,Pr$, which must be attained close enough to the wall. This suggests the possibility of consistent measurement error, but the constant-pressure profiles measured by Perry *et al.* (1966) follow (2.7) well. Samples of the temperature profiles measured by Thielbar *et al.* (1969) and by Blackwell *et al.* (1972) are shown in figure 13-11 of Kays & Crawford (1993), which is reproduced here as figure 1. The profiles follow the pure-conduction limit near the wall but show large departures from (2.7) in the log law region, the profile slope being larger in favourable pressure gradient and smaller in adverse pressure gradient (thus, the departure from (2.7) is in the same sense as found by Perry *et al.* (1966)). As expected, τ increases with y in adverse pressure gradient, and conversely, but the profile of $\dot{q}$ is little affected by pressure gradient. The measurements quoted by Kays & Crawford (1993) were at low Reynolds number, but the discrepancies in the T profiles are far larger than in low-Reynolds-number simulations in zero pressure gradient. Integration of (3.2), assuming that $\dot{q} \approx \dot{q}_w$ but with u_τ, in the definition of T_τ, replaced by $\sqrt{(\tau_w + \alpha y)/\rho}$, gives an additional term in (2.7), equal to $-1/(2\kappa_T)\alpha y/\tau_w$ if α is small. In principle, C_T, like C, becomes a function of $\alpha\nu/(\rho u_\tau^3)$, representing the effects of the stress gradient on the viscous/conductive wall region, but these seem to be fairly small. Now $\partial\tau/\partial y$ is positive in adverse pressure gradient, and conversely, while the profile of $\dot{q}$ is little affected by pressure gradient. Therefore, the additional 'mixing length' term is of the right sign, but too small to explain the changes in $(T_w - T)/T_\tau$ (the *maximum* shear stress in the Blackwell run quoted by Kays & Crawford (1993) is about $2.3\tau_w$, so the additional term gives a reduction in $(T_w - T)/T_\tau$ of about 1.5 while the real decrease is at least 3). Volino & Simon (1994) have analysed some additional data in favourable pressure gradient and confirmed the trends found by Kays & Crawford (1993). The changes in $(T_w - T)/T_\tau$ due to pressure gradient are in the same sense as the changes in U/u_τ discussed above, but far larger. By comparison, the U-law is a good approximation.

The turbulent Prandtl number Pr_t has $\partial T/\partial y$ in its numerator and might be expected to decrease strongly in adverse pressure gradient: Pr_t as measured by Blackwell does decrease, but by only about 15% in the run quoted, where $\partial T^+/\partial y^+$ at given y^+ halves. In view of the failure of law-of-the-wall arguments for temperature, the nearest to an explanation that can be provided for this, and for the success of prediction methods which use constant turbulent Prandtl number (outside the viscous/conductive wall region), is as follows.

Outside the viscous/conductive wall region, definition (2.8) is equivalent to

$$Pr_{\mathrm{t}} = \frac{-(\tau/\rho)/(\partial U/\partial y)}{(\dot{q}/\rho c_p)/(\partial T/\partial y)}, \tag{3.3}$$

so if, in the inner layer of a boundary layer in pressure gradient, $\partial U/\partial y$ is still given by the simple log law, equation (1.2), and $\dot{q}$ is still equal to $\dot{q}_{\mathrm{w}}$, then

$$Pr_{\mathrm{t}} = \frac{\tau}{\tau_{\mathrm{w}}}\kappa y^+ \frac{\partial T^+}{\partial y^+}. \tag{3.4}$$

At given y^+, the effect of imposing (say) an adverse pressure gradient is to increase the first factor, while according to experiment the last factor decreases, so Pr_{t} changes less than expected. It has to be admitted that this is a poor justification for assuming constant Pr_{t} in a prediction method!

Indirect evidence for the failure of the T-law comes from the behaviour of the Stanton number in adverse pressure gradient, for which there are rather more data. Qualitative belief in Reynolds's analogy, for which the main support is the law-of-the-wall analysis above, suggests that heat transfer rate (W m^{-2}) and Stanton number should decrease in adverse pressure gradient (though it is well established that heat transfer is not zero at separation or in separated-flow regions, because skin-friction fluctuations and consequent heat transfer persist even when time-average skin friction is small). In the theory for heat transfer in arbitrary pressure gradient in the second edition of Kays & Crawford (1993), it is assumed that Stanton number follows the constant-pressure trend $St \propto Re_{\Delta_2}^{-0.25}$ where Δ_2 is the enthalpy thickness. The separating-flow experiments of Parikh *et al.* (1976) agree with this: obviously, Δ_2 grows faster, and therefore predicted St decreases somewhat faster, in adverse pressure gradient. Measurements ahead of a separation bubble by Rivir *et al.* (1992) do not include skin-friction measurements, but the Stanton number falls significantly only quite close to separation. Recent direct simulations at low Reynolds number by Coleman & Spalart (1993 and personal communication) also show a decrease only very near separation. These results are compatible with a decrease of $T_{\mathrm{w}} - T$ below the wall-law value in adverse pressure gradient.

Pauley & Eaton (1994) present wall-law plots of velocity and temperature in boundary layers with pairs of imbedded longitudinal vortices, one case in which the 'common flow' between the vortices was directed away from the surface and one with common flow towards the surface. These flows change slowly in the streamwise direction but have large spanwise gradients. In both cases the velocity profiles are quite well fitted by the log law (skin friction having been measured by a surface fence with a height of no more than 8 wall units). The heat-transfer surface has constant heat flux after an unheated starting length, and the temperature profiles presented are typically only about half as thick as the velocity profiles. This may be partly, but certainly not wholly, responsible for the very large departure from the T-law in the common-flow-down case. (In the common-flow-up case, the vortices are further from the surface and the inner layer is not so strongly disturbed.) The spanwise variation of the departure roughly follows the variation of surface shear stress, the turbulent

Prandtl number increasing when the normal-component mean velocity V is negative and decreasing when it is positive. Evidently, spanwise, as well as streamwise, changes in velocity field perturb the law of the wall for temperature.

In the above cases, the wall temperature or heat-flux rate was constant, so that the temperature field would have changed slowly but for the effects of the rapidly changing velocity field. Teitel & Antonia (1993) made measurements downstream of a step change in wall heat flux in a fully developed duct flow at low Reynolds number, and found that the temperature profiles fitted the T-law below a value of y roughly equal to 0.02ξ, where ξ is the distance from the step change. This is a fairly large fraction of the total thickness of the temperature profile, compared to the nominal limit of $y < 0.1\text{–}0.2\delta_T$ for validity of (2.7). This work generally corroborates the finding of Hoffmann & Perry (1979). It appears, therefore, that in this rapidly developing temperature field (with slowly changing velocity field) the T-law does at least as well as one could expect. This suggests that failure of the T-law in rapidly changing *velocity* fields is connected with the failure of the the mixing-length formula (3.1), which depends on local equilibrium of the flow (Townsend 1961).

(*d*) *Compressible flows*

In high-speed boundary layers in zero pressure gradient, the Van Driest transformation (§ 3 *a*) gives a good fit to measured velocity profiles in the inner layer, supporting (3.1) and the assumption of constant turbulent Prandtl number for the case of large $\partial\rho/\partial y$. Also, temperature profiles, such as they are, do not contradict the assumption of constant Pr_t. This is useful information because the heat flux in a high-speed boundary layer varies as $\dot{q} \simeq \dot{q}_w + U\tau_w$ (the rate of dissipation of mean plus turbulent kinetic energy into heat being $\tau\partial U/\partial y$ per unit volume): the implication is that quite large values of $\partial\dot{q}/\partial y$ do not disturb the turbulent Prandtl number. The simulations by Coleman *et al.* (1993 and private communication) of flow in a duct with very cold walls at Mach numbers up to 3 ($\sqrt{k}/a$ up to 0.25, close to the wall) appear to show good support for the Van Driest transformation, except that the additive constant does seem to be changed, either by the very large density gradient close to the surface or simply by the low bulk Reynolds number.

There is little reliable information on surface shear stress in high-speed boundary layers in pressure gradients: frequently, the Van Driest transformation is relied on to find surface shear stress. Fernando & Smits (1990) compared results from Preston tubes, using a calibration based on high-speed data but in zero pressure gradient, and two transformations; Van Driest and Carvin *et al.* (1988). The results agreed well, which is encouraging, but, as in low-speed flow, a Preston tube reading and a velocity-profile fit are not independent measurements of surface shear stress.

(*e*) *Three-dimensional flows*

This subject has recently been reviewed by Johnston & Flack (1994), who conclude that the two-dimensional U-law works quite well for the velocity *magnitude* in three-dimensional boundary layers with not-too-strong crossflow. A law for the flow direction is still wanting: there is very little support for the most plausible three-dimensional version of the mixing-length formula (van den Berg 1975), in which U and τ/ρ in (3.1) represent two-dimensional vectors in the y–z plane and the directions of the resultant shear stress and resultant mean shear are supposed to coincide (isotropic eddy viscosity). Most experimental data show a significant difference between the two directions, so that even though the difference apparently goes to zero at the surface it is not negligible in the log law region. Note that there is no reason why the limiting direction of the Reynolds shear stress as $y \to 0$ (where the Reynolds stresses themselves are zero) should be the same as the limiting direction of the *total* shear stress. The viscous wall region is not in local equilibrium, and there is strong transport of turbulent energy and Reynolds stress towards the surface from the region of maximum energy production at $y^+ \approx 12$. Therefore, the direction of the shear-stress vector very close to the surface will tend to follow that at $y^+ \approx 12$.

Degani *et al.* (1993) have carried out an overlap analysis for the three-dimensional boundary layer. The leading-order solution is a quasi-two-dimensional ('collateral') inner layer: the convective and pressure-gradient terms in the momentum equations appear only at higher order. By definition, the latter terms are significant if there is significant crossflow, and the accuracy of a leading-order solution in the presence of large second-order terms is inevitably doubtful. The analysis of Degani *et al.* (1993) predicts that the velocity component in the direction of the surface shear stress (closely equal to the resultant velocity in mildly three-dimensional flows) follows the two-dimensional log law. This is indeed found in practice (see, for example, Johnston & Flack 1994; Pauley & Eaton 1994), probably accurately enough for a Preston tube to be used for skin-friction measurement as long as the flow direction does not change more than a few degrees between the surface and the top of the tube (Preston tubes are considerably more sensitive to yaw than pitot tubes in uniform flow). The direction of the velocity (relative, say, to an x-axis aligned with the local surface shear stress) does not seem to correlate with local $\partial p/\partial z$ or the local gradient of z-wise shear stress, $\partial \overline{vw}/\partial y$ (van den Berg's (1975) mixing-length analysis would lead to correlation with the latter).

It appears that local law-of-the-wall scaling is not adequate to describe the crossflow in typical three-dimensional boundary layers. The difference between the direction of the shear stress and that of the mean shear is probably attributable more to streamwise rates of change than to influence of the outer layer, because, except in the depths of the viscous wall region, information generally diffuses out to larger y and this would tend to thicken the local-equilibrium region.

4. The viscous wall region

Logic suggests that any departures from the laws of the wall should become smaller as y (say y^+) decreases, since the laws for mean velocity and temperature become exact (by definition) as $y^+ \to 0$: in particular, the effects of bulk Reynolds number $u_\tau \delta/\nu$, based on shear layer thickness δ, should be smallest at small y/δ.

Huffman & Bradshaw (1972) reached the opposite conclusion, on rather slender evidence. The present analysis of low-Reynolds-number simulation data shows conclusively that the Reynolds stresses are strongly flow-dependent and Reynolds-number-dependent near the wall, while the logarithmic law of the wall is much more nearly universal. (Recall that the additive constant in the law of the wall depends on the velocity profile in the viscous wall region, which in turn depends on the Reynolds shear stress in that region.) A simple explanation is that when the Reynolds stresses are small they are more susceptible to outside influence, but it is not clear what the outside influence is.

If the mean velocity U and the turbulent fluctuations u, v and w are expanded in powers of the distance from a solid surface, y, then, using the no-slip condition, the no-permeability condition, the continuity equation, and the fluctuating part of the momentum equation at $y=0$,

$$\mu\partial^2 u_i/\partial x_l^2 = \partial p'/\partial x_i, \tag{4.1}$$

we obtain

$$\overline{u^{+^2}} = \overline{\left(\frac{\partial u^+}{\partial y^+}\right)^2} y^{+^2} + \overline{\frac{\partial u^+}{\partial y^+}\frac{\partial p'^+}{\partial x^+}} y^{+^3} + \cdots, \tag{4.2}$$

$$\overline{v^{+^2}} = \frac{1}{4}\overline{\left(\frac{\partial p'^+}{\partial y^+}\right)^2} y^{+^4} + \frac{1}{6}\overline{\frac{\partial p'^+}{\partial y^+}\frac{\partial^2 p'^+}{\partial y^{+^2}}} y^{+^5} + \cdots, \tag{4.3}$$

$$\overline{w^{+^2}} = \overline{\left(\frac{\partial w^+}{\partial y^+}\right)^2} y^{+^2} + \overline{\frac{\partial w^+}{\partial y^+}\frac{\partial p'^+}{\partial z^+}} y^{+^3} + \cdots, \tag{4.4}$$

$$\overline{u^+v^+} = \frac{1}{2}\overline{\frac{\partial u^+}{\partial y^+}\frac{\partial p'^+}{\partial y^+}} y^{+^3} + \cdots, \tag{4.5}$$

where all gradients are evaluated at $y^+ = 0$. The superscript '+' indicates non-dimensionalization of velocities by the friction velocity $u_\tau \equiv \sqrt{\tau_w/\rho}$, pressures by τ_w and lengths by ν/u_τ ('wall units'). If the total (viscous plus turbulent) shear stress is independent of distance from the wall then integration of the expression for total shear stress gives

$$U^+ = y^+ + \frac{1}{8}\overline{\frac{\partial u^+}{\partial y^+}\frac{\partial p'^+}{\partial y^+}} y^{+^4} + \cdots, \tag{4.6}$$

the coefficient being one-quarter of that in the leading term of $\overline{u^+v^+}$. Equations (4.2)–(4.5) are given by, for example, Townsend (1956). Sreenivasan (1989), following Monin & Yaglom (1971) and using later experimental data, gives values for some of the coefficients in the corresponding expansions for root-mean-square values, while Mansour *et al.* (1988) and others give some values from simulation results. In the present paper we give results for mean squares, from which the series for root mean squares may be easily derived.

There are many measurements of the limiting behaviour of $\overline{u^{+^2}}$ in the literature, usually given in the form $\sqrt{\overline{u^2}}/U$ which in the limit is just the square root of the coefficient of y^{+^2} in (4.2). Simulation results are somewhat higher than typical experimental values, probably because of finite spatial resolution of the instruments. An exception appears to be the recent work of Durst *et al.* (1994) using a laser Doppler velocimeter in a circular pipe: they found $\sqrt{\overline{u^2}}/U \approx 0.345 + 7.5\times10^{-5} u_\tau\delta/\nu$, where δ is the radius of the pipe, which is as close as could be expected to the plane duct simulations described below, though with a somewhat smaller trend with Reynolds number. The value for the leading coefficient in $\sqrt{\overline{w^2}}/U$ from Monin & Yaglom is 0.07: values up to about 0.1 have been measured more recently but as Sreenivasan points out this is still significantly less than in the simulations, which show 0.2–0.3. Measurements of $\overline{v^2}$ and $\overline{uv}$ near a solid surface are even more difficult than measurements of w. In this paper we concentrate on the analysis of simulation data: the results are necessarily restricted to low Reynolds number but the effect of flow type is likely to be qualitatively the same at higher Reynolds number.

Figures 2–5 show the Reynolds stresses in the viscous wall region for the two-dimensional duct simulations of Kim *et al.* (1987) at $u_\tau\delta/\nu \equiv \delta^+ = 180$ (where δ is the half-height of the duct) and of Kim (unpublished) at $\delta^+ = 395$, and for the constant-pressure boundary-layer simulation of Spalart (1988) at $\delta^+ = 150$, 325

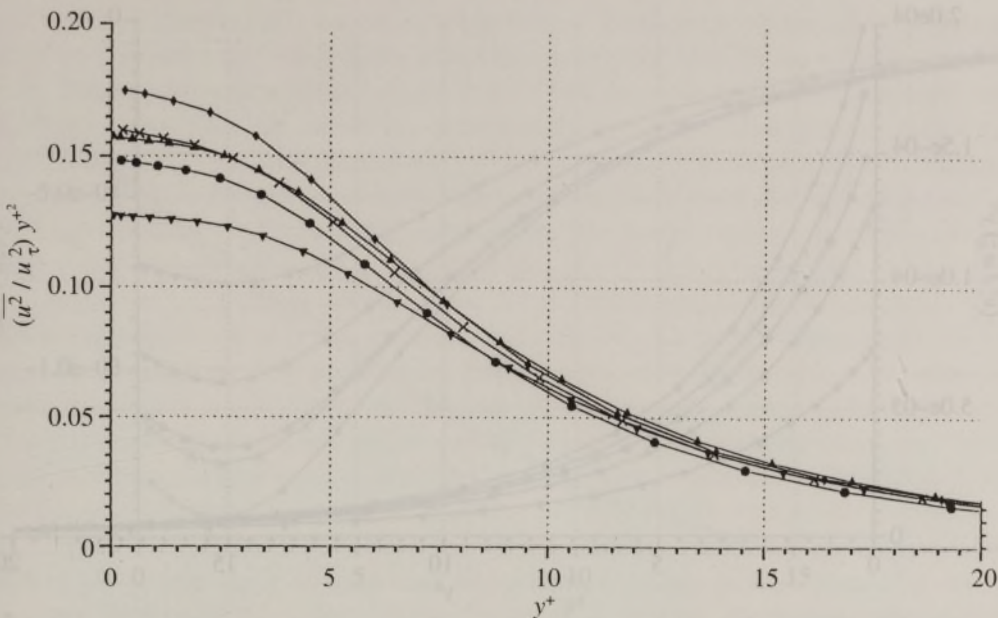

Figure 2. Ratio of $\overline{u^2}$ to y^2, in 'wall units', in a plane duct (Kim *et al.* 1987) and a boundary layer in zero pressure gradient (Spalart 1988): $Re = u_\tau \delta/\nu$ where δ is boundary-layer thickness or duct half-height; ▼, duct, $Re = 180$; ▲, duct, $Re = 395$; •, BL, $Re = 150$; ×, BL, $Re = 325$; ◊, BL, $Re = 650$.

and 650 (where δ is the total thickness of the boundary layer and is not to be compared quantitatively with the half-height of the duct). Each Reynolds stress is normalized by the surface shear stress and divided by the limiting power of y^+ given in (4.2)–(4.5). According to the law-of-the-wall analysis, these plots should be universal functions of y^+, independent of bulk Reynolds number and flow type. In particular, the values at $y^+ = 0$ give the leading coefficients in (4.2)–(4.5), which should be universal according to the law-of-the-wall analysis and obviously are not. Note that the logarithmic law (1.2) is fairly well obeyed in these flows, deviations in U being no more than $\pm 1.0 u_\tau$. Figure 2 shows an increase in dimensionless $\overline{u^2}$ with increasing Reynolds number, for a given flow: experimental results of several workers plotted by Nagano *et al.* (1991) (their figure 6) show a decrease with increasing Re but the authors themselves do not regard the trend as significant.

The normal stresses do not significantly affect the mean motion, but the non-universality of the shear stress is obviously of potential importance. Therefore, before discussing the physics of these results, we investigate the effect of the shear-stress deviations on the mean velocity.

If the variation of total shear stress τ with y is known we can write, with no other assumptions,

$$-\rho\overline{uv} + \mu \partial U/\partial y = \tau. \tag{4.7}$$

The integral of this equation with respect to y, written in wall units for constant-property flow, is

$$\int_0^{y^+} -\overline{u^+v^+}\,\mathrm{d}y^+ + U^+ = \int_0^{y^+} (\tau/\tau_\mathrm{w})\,\mathrm{d}y^+. \tag{4.8}$$

Figure 6 shows $\int_0^{y^+} -\overline{u^+v^+}\,\mathrm{d}y^+$ for the five simulations and it is seen that the per-

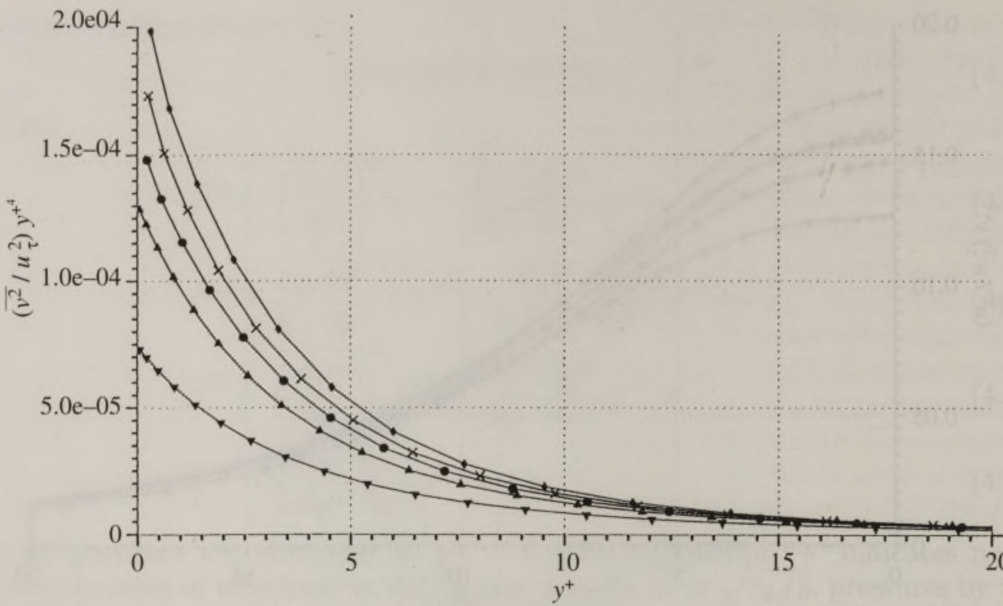

Figure 3. Ratio of $\overline{v^2}$ to y^4, in 'wall units', in a plane duct and a boundary layer: symbols as in figure 2.

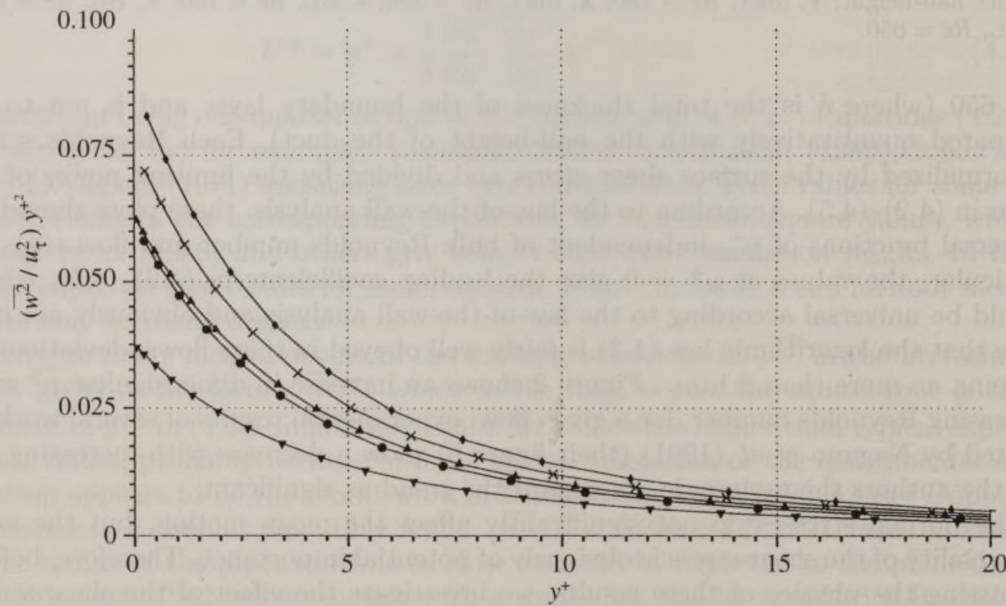

Figure 4. Ratio of $\overline{w^2}$ to y^2, in 'wall units', in a plane duct and a boundary layer: symbols as in figure 2.

centage differences at larger y^+ are considerably less than those near the surface in figure 5.

These flows have significant shear-stress gradients: in the duct at $\delta^+ = 180$, the total shear stress at $y^+ = 30$ is $\frac{5}{6}$ of the wall value, while for the other cases the difference is less but not negligible. One would expect the turbulence to scale more closely on the local (total) shear stress than the wall value—although this is one of the assumptions of the mixing length analysis disparaged above!—and indeed the integrand for the duct at $\delta^+ = 180$ is much lower than the others throughout the

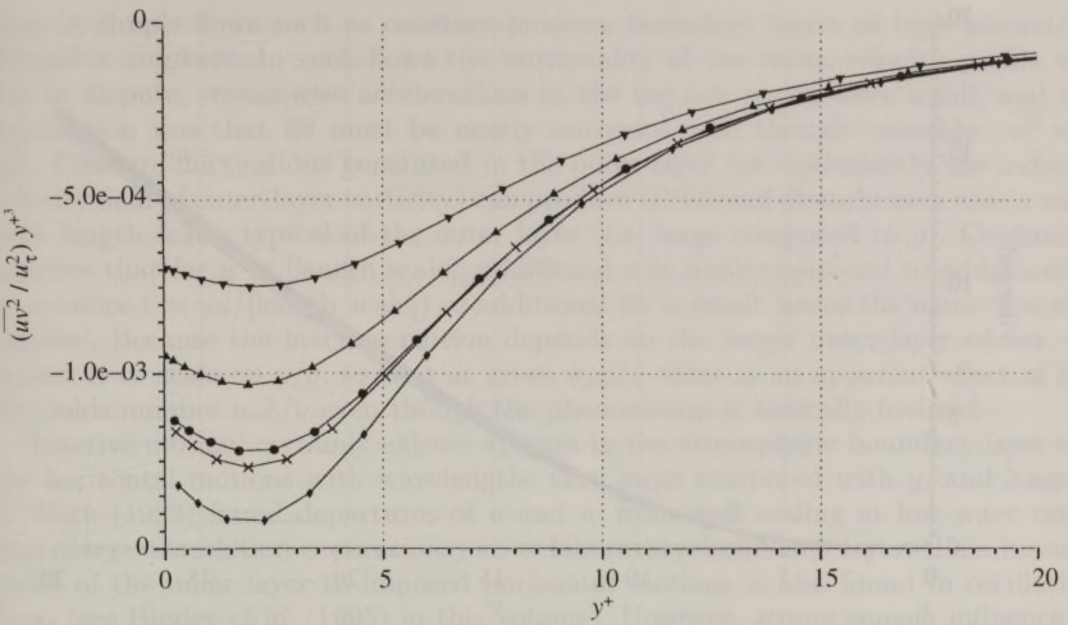

Figure 5. Ratio of $\overline{uv}$ to y^3, in 'wall units', in a plane duct and a boundary layer: symbols as in figure 2.

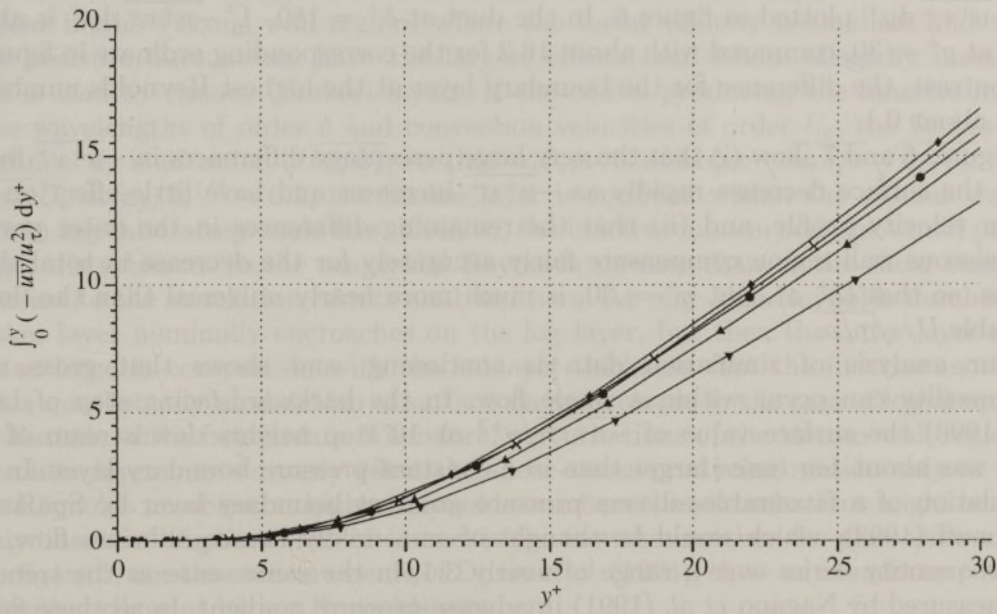

Figure 6. Integral of Reynolds shear stress with respect to y, in 'wall units': symbols as in figure 2.

range shown in figure 6. Now rearranging (4.8) we obtain

$$\int_0^{y^+} -\overline{u^+v^+}\, dy^+ + \int_0^{y^+} (1 - \tau/\tau_w)\, dy^+ = y^+ - U^+ \tag{4.9}$$

and the left-hand side is plotted in figure 7. This modified form of the integral allows the effect of non-universality of $-\overline{u^+v^+}$ on U^+ to be deduced immediately, while if $\tau = \tau_w$ everywhere then the plotted quantity is just the intuitively useful

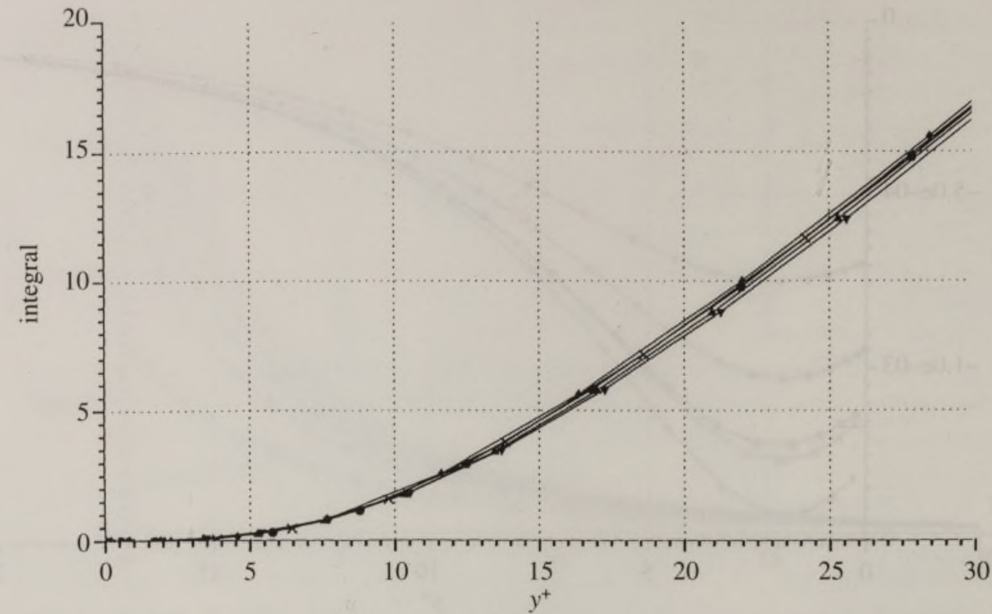

Figure 7. Modified integral of Reynolds shear stress with respect to y, from (4.9), in 'wall units'. Ordinate is equal to $y^+ - U^+$: symbols as in figure 2.

$\int_0^y -\overline{u^+v^+}\,dy^+$ plotted in figure 6. In the duct at $\delta^+ = 180$, $\int_0^y -\overline{u^+v^+}\,dy^+$ is about 13.2 at $y^+ = 30$, compared with about 16.2 for the corresponding ordinate in figure 7: in contrast, the difference for the boundary layer at the highest Reynolds number is only about 0.1.

Figures 6 and 7 show (i) that the very large percentage differences in $-\overline{u^+v^+}$ found near the surface decrease rapidly as $-\overline{u^+v^+}$ increases and have little effect on the mean velocity profile, and (ii) that the remaining differences in the outer part of the viscous wall region compensate fairly accurately for the decrease in total shear stress, so that U^+ at, say, $y^+ = 30$, is much more nearly universal than the 'local' variable $U/\sqrt{\tau/\rho}$.

Our analysis of simulation data is continuing, and shows that gross non-universality can occur within a single flow. In the backward-facing step of Le *et al.* (1993) the surface value of $-\overline{u^+v^+}/y^{+3}$ at 10 step heights downstream of the step was about *ten times* larger than in a constant-pressure boundary layer. In the simulation of a favourable-adverse pressure gradient boundary layer by Spalart & Watmuff (1993), which would be thought of as a mildly non-equilibrium flow, the same quantity varies over a range of nearly 3:1, in the same sense as the trend in $\overline{u^2}$ measured by Nagano *et al.* (1991) in adverse pressure gradient. In all these flows, U^+ departs significantly from the log law.

The mechanism for the very large departures from universality in the lower part of the viscous wall region is not clear. The only purely local quantity that is different in the various cases discussed above is the total shear-stress gradient (or the pressure gradient), and although figure 5 shows that the variation with $\partial\tau^+/\partial y^+$ is in the same sense for the boundary layer as for the duct (where the shear-stress gradient is much larger), it is clear that $\partial\tau^+/\partial y^+$ will not collapse results for both flows. Some non-local mechanism is at work.

'Inactive motion' or 'splat effect' (Townsend 1961; Bradshaw 1967) was an excuse for the failure of the turbulence to scale completely on law-of-the-wall variables, even in simple flows such as constant-pressure boundary layers at high laboratory Reynolds numbers. In such flows the universality of the mean velocity profile was not in dispute, streamwise accelerations in the log law region were small, and the implication was that $\overline{uv}$ must be nearly universal even though—notably—$\overline{u^2}$ was not. Pressure fluctuations generated in the outer layer (or equivalently the induced velocity field of outer-layer vorticity) can produce additional disturbances near a wall, with length scales typical of the outer layer (i.e. large compared to y). Continuity requires that for $y \ll$ [length scale], additional v is small compared to additional u or w (since $v \approx yu/$[length scale]) so additional $\overline{uv}$ is small: hence the name 'inactive motion'. Because the inactive motion depends on the larger outer-layer eddies, we expect it to scale on y/δ, so that at given $u_\tau y/\nu$ there is an apparent effect of the Reynolds number $u_\tau \delta/\nu$ even though the phenomenon is basically inviscid.

'Inactive motion' certainly exists—spectra in the atmospheric boundary layer imply horizontal motions with wavelengths very large compared with y, and Naguib & Wark (1992) found departures of u and w from wall scaling at low wave numbers (large correlation separations) in a laboratory boundary layer. This insensitivity of the inner layer to imposed horizontal motions is also found in oscillating flows (see Binder *et al.* (1995) in this volume). However, *strong enough* influence of the outer layer is bound to affect the shear-stress-producing motion near the wall, e.g. the function g in the vital relationship (1.1*b*) $-\overline{uv}/u_\tau^2 = g(u_\tau y/\nu)$. The inactive motion concept suggests that the largest effects of outer-layer disturbances will appear in the viscous wall region, where the mean velocity is smallest and therefore pressure fluctuations have the largest effects, and where allegedly inactive u and w lead to viscous (Stokes) layers. If the eddies producing the inactive motion have wavelengths of order δ and convection velocities of order U_e, the Stokes-layer thickness δ_s is of order $\sqrt{\nu\delta/U_e}$, so that $u_\tau \delta_s/\nu$ is of order $\sqrt{u_\tau \delta/\nu}\sqrt{u_\tau/U_e}$. Since $\sqrt{u_\tau/U_e} \equiv (c_f/2)^{1/4} \approx 0.2$, the Stokes layer is confined within the viscous wall region in any laboratory-scale flow. Even so, the inactive-motion concept would predict larger disturbances to the tangential Reynolds stresses than to the shear stress, in contrast to the simulation results analysed here. At very high Reynolds number, the Stokes layer nominally encroaches on the log layer, but then the outer-layer scales are so large in comparison with the sublayer thickness that the inactive motion is just a quasi-steady modulation of surface shear stress, as in the case of low-frequency oscillations of the free stream. Therefore, perturbations of the viscous wall region by the outer layer may seriously affect the shear-stress-producing motion only at fairly low Reynolds numbers: at higher Reynolds numbers the perturbations might be more truly 'inactive'. Against this, of course, is the fact that neither the plots shown here, e.g. figure 5, nor the measurements of Durst *et al.* (1994), show any tendency towards a limiting curve at higher Reynolds number.

The behaviour of the law of the wall for temperature in the viscous/conductive wall region is currently controversial. There, the turbulent Prandtl number Pr_t is expected to be a function of y^+ and Pr. The data correlation of Kays (1994) for air flows (with small pressure gradients, so that the T-law is well behaved) shows that Pr_t increases near the wall and asymptotes to approximately 1.7 at the surface. This behaviour is not found in low-*Re* DNS results, where Pr_t is about 1.1 at the surface: for discussion see Huang & Bradshaw (1994). The difference is not too important in practice, because the large values of Pr_t occur where both $\overline{uv}$ and $\overline{vT'}$ are small (so $Pr_t \to 0/0$), but the limiting behaviour of Pr_t is a useful diagnostic for non-universality of the T-law.

It is obvious that if the law of the wall for velocity (and turbulence quantities) is inaccurate in the viscous wall region then the T-law will be inaccurate also, except by good luck. Further analysis of simulation data should clarify matters.

5. Turbulence models

Here 'model' means one or more Reynolds-averaged (usually time-averaged) formulae or differential equations yielding the Reynolds stresses: we do not consider sub-grid-scale models for large-eddy simulations, although these are most severely tested in the near-wall region where the Reynolds-stress-bearing eddies are smallest.

All worthwhile models reproduce the law of the wall in the constant-stress layer. Indeed, it is commonplace that the comparative success of turbulence models in attached flows is due more to the law of the wall than to the turbulence model for the outer layer.

Now most stress-transport models of turbulence, at least, include y-wise diffusion terms and, by definition, also include x-wise transport. However, there seems not to be a series of computations to see whether stress-transport models (mainly descendants of the Launder–Reece–Rodi (LRR) (Launder *et al.* 1975) model) reproduce measurements of the mean velocity and temperature profiles in the inner layer. This is a central difficulty: the LRR family reproduce local equilibrium and the law of the wall in slowly changing flows—but do the constant coefficients in the LRR models really remain constant in flows which are changing rapidly in the x or y directions?

Huang & Bradshaw (1994) have recently explored the consequences of forcing a 'two-equation' turbulence model to reproduce the log law for velocity even in the presence of significant $\partial\tau/\partial y$ (but *neglecting* streamwise transport terms). This requires the eddy (kinematic) viscosity in the log law region to vary as $\nu_{\mathrm{t}} = \kappa y(\tau/\rho)/u_\tau$ rather than $\kappa y\sqrt{\tau/\rho}$ as implied by the mixing-length formula with $l = \kappa y$. Using the definition of turbulent Prandtl number and assuming that $\dot{q} = \dot{q}_{\mathrm{w}}$, this leads to

$$\frac{\partial T}{\partial y} = T_\tau \frac{Pr_{\mathrm{t}}}{\kappa y}\frac{\tau}{\tau_{\mathrm{w}}}, \tag{5.1}$$

which is the mixing-length formula (i.e. (2.6) with the function F equated to $1/\kappa_T \equiv Pr_{\mathrm{t}}/\kappa$) multiplied by a factor τ/τ_{w}. This modified mixing-length formula fits temperature-profile data for boundary layers in pressure gradient quite well, *assuming a constant turbulent Prandtl number*. Obviously the justification for constancy of Pr_{t} relies on the validity of the original 'mixing-length' formulae for U and T, so the result must be regarded as purely empirical.

Huang & Bradshaw (1994) studied the family of two-equation models whose variables are the turbulent kinetic energy, k, and $k^m\epsilon^n$, in search of optimum values of m and n (in the spirit of Spalding 1991): for present purposes the 'optimum' model is that which reproduces the log law for velocity and (5.1) for temperature (taking $Pr_{\mathrm{t}} = \text{const.}$) with $\tau = \tau_{\mathrm{w}} + \alpha y$ and $\dot{q} = \dot{q}_{\mathrm{w}}$. It happens that the optimum is exactly the Wilcox (1993) k, ω model, where $\omega \propto \epsilon/k$, i.e. $m = 1$, $n = -1$ (note that ω is not the root-mean-square vorticity fluctuation, but is nominally related to the vorticity field of the *larger* eddies. Results from the k, ϵ model are poor. Recall that there are essential differences between the different two-equation models, irrespective of the values of the empirical coefficients: if (say) the ω transport equation is transformed into a transport equation for ϵ (using the transport equation for k), the 'diffusion' (turbulent transport) term transforms into a diffusion term plus a source/sink term which has no counterpart in the k, ϵ model itself. (The reason is that the turbulent diffusivity in two-equation models depends on the variables, so that although turbulent transport is modelled as a gradient-diffusion process, it is nonlinear.)

Huang & Bradshaw's (1994) analysis neglects transport of turbulence quantities by the mean flow (the left-hand sides of the transport equations), but these terms are generally too small to account directly for the difference between (say) (2.2) and (3.1): by implication, streamwise gradients affect the turbulence structure (i.e. the empirical coefficients of the terms on the right-hand sides).

So-called 'low Reynolds number' turbulence models have coefficients which are functions of turbulence Reynolds number, typically $k^2/(\epsilon\nu)$, adjusted to reproduce the shear-stress profile (and thus the mean velocity profile) in the standard viscous wall region. There is no guarantee that these models will predict the correct shear stress profile when the standard law of the wall no longer applies. The compelling reason is that the Reynolds-number dependence of the coefficients necessarily has to include compensation for any errors in the high-Reynolds-number form of the model—such as the inability to deal with highly inhomogeneous flows—which become large near the wall.

6. Conclusions

It is a normal characteristic of scientific research that a first-order theory which has served well for many years is found to need second-order corrections. The present state of the law of the wall is more worrying. The velocity profile stays close to the log law in circumstances where the basic law-of-the-wall scaling should not hold, and where the mixing-length formula—which can be defended as a plausible extension of law-of-the-wall concepts—predicts departures from the log law. In some cases at least, the departure from the log law is in the opposite sense to that predicted by the mixing-length formula.

The temperature profile, on the other hand, departs from the log law considerably more than predicted by mixing-length theory (but in the same direction): in fact the law of the wall for temperature breaks down spectacularly in flows where the law of the wall for velocity is still valid. It is especially odd that the T-law is more affected by pressure gradient than the U-law. The turbulent Prandtl number in the log region remains approximately constant, which must be a coincidence.

Now the laws of the wall for velocity and for temperature can be derived by closely analogous arguments: their range of validity ought therefore to be roughly the same. Therefore, the fragility of the T-law, compared to the tenacity of the U-law, casts doubt on the latter.

The mixing-length formula for velocity works well in *some* cases where $\partial(\tau/\rho)/\partial y$ is large, and the mixing-length formula for temperature works well in at least some cases where $\partial\dot{q}/\partial y$ is large, providing that streamwise gradients are small. Large streamwise gradients usually produce large y-wise gradients, but the implication is that failure of the mixing-length formulae is directly due to the effects of upstream history of the flow in the inner layer. The k, ω turbulence model reproduces the U and T profiles observed in strong pressure gradients, but for no clear physical reason: indeed it differs from the k, ϵ two-equation models in its treatment of y-wise, rather than x-wise gradients.

Perhaps the most surprising conclusion of the present paper is that the Reynolds stresses, *including the shear stress*, deviate most strongly from law-of-the-wall scaling deep in the viscous wall region. In simple flows the integrated effect of the shear-stress changes on the mean velocity in the log law region is small, but in rapidly changing flows the changes in the viscous wall region are gross.

Now that serious questions have been raised about the status of the laws of the wall, a closer look at existing data, particularly simulation results, is needed. Combined with this is the need for more tests of existing turbulence models in flows with large streamwise gradients.

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