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Le:

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Titre **:** Structured Products Selling Risk, From Pricing to Risk Management

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Structured Products Selling Risk, From Pricing to Risk Management

By Antoine-Bruno SARRANT

A dissertation submitted for the Actuary degree.

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Contents

1. ABSTRACT

"*Risk comes from not knowing what you are doing*"
- Warren Buffet

A new investment product category so-called "*Structured Products*" arises for the past decade and aims to supply more important return than conventional assets by providing a pre-packaged investment strategy. Unfortunately dealing with such complex structures is quite challenging whatever the implied business desk. Indeed this product category introduces lots of discontinuities (options structures with embedded indicator functions, underlying stochastic process, market data ...) which are quite challenging regarding the pricing process.

In this dissertation, we will present how to manage a risk exposure generated by an inventory constituted by different structured products, so-called "*Pipeline Risk*". To do this, we will define and implement a risk measure based on the well-known "*Value-at-Risk*" (VaR) approach. This involves estimating the potential loss on current inventory according the underlying risk factor(s) variations and the structured products' sensitivity factors.

Due to the inner complexity of such products, market practitioners have only one method of reference to evaluate a structured product: the Monte Carlo Approach. However this method produces instable results in presence of discontinuities (especially on sensitivity factors) which is a major flaw in Risk Management. So we will use an alternative pricing approach with the use of a Fourier Transform based approach so-called "*Fourier Space Time-stepping*" (FST) method.

This method solves the partial integral differential equation with help of a backward stochastic differential equation (BSDE) in Fourier Space and a vector of intrinsic values defined in Real Space. This integration process allows the aggregation of new conditioned intrinsic values at intermediate maturities. Finally we produced a complex vector representing the actualized value of the option structure which can supply either the price or the adequate sensitivity factor estimation by multiplying this vector with the adequate factor.

We will show the FST pricing capacities by producing a benchmark constituted by eight classical options and three structured product examples. Through this benchmark we will notice that the FST method supplies accurate, fast and stable results whatever the priced item (Price, first order and second order derivatives) and then constitutes a relevant alternative pricing method for structured products.

The last part of this dissertation will focus on the integration of the FST method into the Pipeline Risk Framework used in Barclays Bank Plc. Thereby we will investigate the influence of Fourier parameters on pricing quality but also how to calibrate this method with market data. Thereby we will use a bootstrap-like approach to calibrate characteristic exponent factors. This will allow solving an important issue, well known from Monte-Carlo practitioners: the integration of complex forward implied volatility nodes during the pricing process.

« Le *Risque surgit quand on ne maîtrise pas ce que l'on fait* »
 - Warren Buffet

Les "Produits Structurés" sont une classe d'investissement qui est apparue depuis une dizaine d'année et qui vise à fournir des rendements plus importants en fournissant une stratégie d'investissement intégrée. Cependant travailler avec de tels produits peut s'avérer assez difficile et ce quelle que soit la facette du métier abordé. En effet, cette catégorie de produits introduit de nombreuses discontinuités (structure optionnelle avec présence d'indicatrices, nature du processus stochastique employé, données de marché …) qui sont très difficiles à intégrer dans le processus de valorisation.

Cette dissertation a pour sujet la gestion d'une exposition générée par un stock constitué de différents produits structurés, que nous appellerons « *Pipeline Risk* ». Pour cela, nous définirons et implémenterons une mesure du risque basée sur une approche de type « *Value at Risk* » (VaR). Ceci implique d'estimer la perte potentielle sur la valeur du stock en fonction des variations des facteurs de risques sous-jacents et des sensibilités respectives (ou croisées) des produits structurés concernés.

La problématique est que les professionnels des marchés financiers ne disposent actuellement que d'une méthode de référence pour valoriser ces produits complexes : l'approche Monte Carlo. Cependant cette méthode est connue pour produire des résultats instables en présence de discontinuités. Ceci est surtout visible dans l'estimation des facteurs de sensibilité, ce qui est un défaut majeur en gestion des risques. Ainsi nous nous intéresserons à une méthode de valorisation alternative dite « *Fourier Space Time-stepping* » (FST) qui se base sur la Transformée de Fourier lors de l'intégration de la valeur « *Temps* ».

Cette méthode résout une équation intégro-différentielle partielle à l'aide d'une équation stochastique différentielle à induction « *Backward* » définie dans l'espace de Fourier et d'un vecteur de valeurs intrinsèques défini dans l'espace Réel. Ce processus d'intégration permet d'agréger dans les étapes temporelles intermédiaires de nouveaux éléments de valeurs intrinsèques en fonction de conditions définies par la structure optionnelle. Ceci nous permet d'obtenir in fine un vecteur complexe représentant la valeur actualisée probable de cette structure et qui nous permettra d'obtenir le prix ou bien les facteurs de sensibilités, à la condition de multiplier au préalable ce vecteur avec le facteur d'ajustement adéquat.

Nous montrerons les capacités de valorisation de cette méthode en produisant un banc d'essais constitué de huit options classiques et de trois produits structurés typiques. Nous verrons ainsi que la méthode FST produit rapidement des résultats stables et précis quel que soit l'élément estimé (Prix, dérivées partielles d'ordre un ou deux) et qu'elle est une méthode alternative à considérer sérieusement lors de la valorisation des produits structurés.

Finalement nous focaliserons dans la dernière partie la manière d'intégrer la méthode FST dans le cadre de gestion du « *Pipeline Risk* » de la Banque Barclays. Ainsi nous étudierons la sensibilité de cette méthode de valorisation aux variations des paramètres de la Transformée de Fourier, mais aussi comment calibrer cette méthode avec des données de Marché. Pour cela, nous développerons une méthode de type « *bootstrap* » pour calibrer les facteurs exposants caractéristiques. Ceci nous permettra de nous affranchir de la problématique des volatilités implicites forward complexes, écueil bien connu des praticiens de la méthode Monte Carlo.

2. OUTLINE

This dissertation will talk about the management risk management associated to structured products selling activity and it will be developed in three parts.

The chapter §3 will introduce the dissertation context in terms of business environment and risk issues raised by such activity.

- The first section will present the business context through a quick description of Barclays Bank Plc. key figures, governance and strategy.
- The second section will focus on risk management in financial activities, how risks are generated, measured and controlled. Next we will present Barclays' Risk philosophy and how it implements its general risk framework among its branches, subsidiaries and business units (BU).
- The third section will present the European Structured Products (SPs) Activity, why such investment products are widely used in most of investment strategies, what are advantages and flaws and how Barclays manages this activity in terms of internal strategy, commercial offer and integration in its internal processes.
- The last section will present the risk generated by such activity. Thus we will present the risk analysis based on the business cycle to identify the most important issues in term of risk management. Next we will integrate the most important constraints produced either by financial regulators, internal policies or technical matters, this to produce an adapted risk framework to manage such risk exposure.

In chapter §4, we will present a pricing methodology called "Fourier Space Time-stepping" (FST), produced by Vladimir Surkov during his thesis (Surkov, 2009). This presentation will be constituted of following sections:

- In the first section, we will present the evolution of option pricing knowledge since the Black & Scholes approach. Thus we will highlight the increase of complexity either in term of payoff, stochastic processes or numerical methodologies but also the transition from $\mathbb{R}$ to $\mathbb{C}$ since the Heston work (Heston, 1993).
- The second section is dedicated to readers unfamiliar with the core principles and properties of Fourier Transform (FT). We will start with the basic definitions used in Physic Field and we will present the FT's extensions in statistic and probabilistic fields.
- The third section will be the continuity of previous one and will present an overview of Carr and Madan's work (Carr, et al., 1999) on Option Pricing with help of FT. This fundamental paper highlights the pros, the cons and the numerical principles when implementing an FT-base option model.
- The fourth part will present the FST Methodology, its principles, its properties and how it extends the work done by Carr and Madan especially with complex option structures.

The chapter §5 will focus on the application of FST methodology in option pricing and how it will be integrated into the Pipeline Risk Framework:

- The first section will be related to the technical implementation of FST Pricing Model and how to move from a continuous to a discrete space. Moreover we will present the basis of iterative integration process of FST Method.
- In the second section, we will produce a benchmark of pricing models according to the option category. The benchmarked options will be divided into four categories: path-independent, path-dependent, multiple exercise and lastly structured product examples. During this benchmark we will compare the accuracy and performance of FST method versus referenced methods.
- The third part will be dedicated to technical implementation of the Pipeline Risk Framework where we will focus on how to estimate risk factors and how to calibrate the FST method according to its pricing sensitivity to FT parameters and the integration of market data. Lastly we will present the production process and the produced report used in BAU management ("Business As Usual").

Finally we will conclude this dissertation by summarizing the pros and cons of designed approach and defining the future steps to follow.

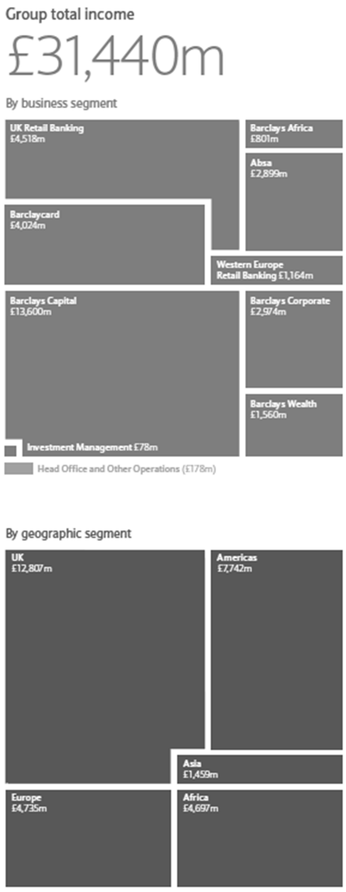
3. INTRODUCTION

3.1 BARCLAYS' CONTEXT

3.1.1 Barclays Group at glance

Barclays is a major global financial services provider engaged in retail banking, credit cards, corporate and investment banking and wealth management with an extensive international presence in Europe, the Americas, Africa and Asia. With over 300 years of history and expertise in banking, Barclays operates in over 50 countries and employs 147,500 people. Barclays moves, lends, invests and protects money for customers worldwide.

Barclays is made up of two 'Clusters': Retail Business Banking (RBB), and Corporate and Investment Banking and Wealth Management (CIBW), each of which has a number of Business Units. The third major area of the business is Group Centre, which comprises all our essential support functions.



UK Retail Banking – £4,518m total income
UK Retail Banking is a leading UK high street bank providing current account and savings products and Woolwich branded mortgages. UK Retail Banking also provides unsecured loans, protection products and general insurance as well as banking and money transmission services to small and medium enterprises.

Barclaycard – £4,024m total income
Barclaycard is an international payments business which manages about £200bn in annual payment value and offers a broad range of payment solutions to consumer and business customers in 22 countries throughout the world.

Western Europe Retail Banking – £1,164m total income
Western Europe Retail Banking provides retail banking and credit card services in Spain, Italy, Portugal and France. The business is building a differentiated proposition providing banking services to retail and mass affluent customers through a variety of distribution channels.

Barclays Africa – £801m total income
Barclays Africa provides retail, corporate and credit card services across Africa and the Indian Ocean. It provides tailored banking (including mobile banking and Sharia-compliant products) to over 2.7m customers and has a top 3 position in 8 of the 10 countries in which we operate.

Absa – £2,899m total income
Absa provides a full range of retail banking services and insurance products through a variety of distribution channels. It also offers customised business solutions for commercial and large corporate customers. It is part of one of South Africa's largest financial services organisations.

Barclays Capital – £13,600m total income
Barclays Capital is the investment banking division of Barclays. It provides large corporate, government and institutional clients with a full spectrum of solutions to meet their strategic advisory, financing and risk management needs. Barclays Capital has a global presence providing advisory services and distribution power to meet the needs of issuers and investors worldwide.

Barclays Corporate – £2,974m total income
Barclays Corporate provides integrated banking solutions to large corporates, financial institutions and multinationals in the UK & Ireland, Continental Europe and New Markets.

Barclays Wealth – £1,560m total income
Barclays Wealth is the wealth management division of Barclays. It focuses on private and intermediary clients worldwide, providing international and private banking, investment management, fiduciary services and brokerage. It has offices in Europe, North America, Asia and Africa.

Investment Management – £78m total income
Investment Management manages the Group's 19.9% economic interest in BlackRock, Inc. and the residual elements relating to Barclays Global Investors, which was sold on 1st December 2009.

Head Office and Other Operations – £178m total loss
Head Office Functions and Other Operations comprise head office and central support functions, businesses in transition and consolidation adjustments.

Barclays Group Chairman is Marcus Agius, and the Chief Executive is Robert E. Diamond, Jr. They are supported by Barclays Executive Committee and the Board of Directors (see panel below).

Panel 1: Barclays' Governance Structure (reference: Barclays 2010 Annual Report)

Barclays' 2010 key performance indicators are summarized in panel below.

Performance Highlights

Profit before tax
£6,065m
Group profit before tax up 32% on 2009, adjusted profit before tax up 11%

Net income
22% up
Net income of £25,768m, up 22% on 2009

Impairment
30% down
Impairment of £5,672m, down 30%, giving a loan loss rate of 118bps compared to 156bps for 2009

Return on equity
7.2%
Improved returns on average shareholders' equity of 7.2% (2009: 6.7%)

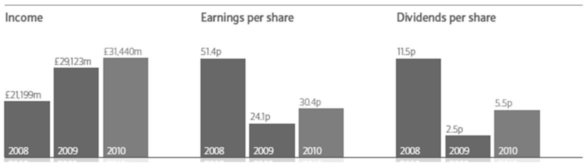
Core Tier 1 ratio
10.8%
Core Tier 1 capital ratio of 10.8% (2009: 10.0%)

Group liquidity pool
£154bn
Group liquidity pool improved by 21% from £127bn in 2009

Net hiring
2,000
Created 2,000 new jobs, 80% of which were in the UK

Global tax paid
£6.1bn
UK tax paid of £2.8bn, including £1.3bn on behalf of employees

Gross new lending to UK
£43bn
Gross new lending to UK households and businesses increased to £36bn, plus £7.5bn from the acquisition of Standard Life Bank



Key Performance Indicators

KPIs	Measures	2010	2009	2008
Capital	Core Tier 1 ratio	10.8%	10.0%	5.6%
	Adjusted gross leverage	20x	20x	28x
Returns	Return on average shareholders' equity (RoE)	7.2%	6.7%	14.3%
	Return on average tangible shareholders' equity (RoTE)	8.7%	9.0%	21.3%
	Profit before tax	£6,065m	£4,585m	£5,136m
	Cost: income ratio	64%	57%	63%
	Loan loss rate	118bps	156bps	95bps
	Dividend	5.5p	2.5p	11.5p
Income growth	Total income	£31,440m	£29,123m	£21,199m
	Income by geography: UK & Ireland	40%	45%	57%
	Europe region	15%	15%	19%
	Americas	25%	22%	0%
	Africa	15%	14%	17%
	Asia	5%	4%	7%
Citizenship	Gross new lending to UK households and businesses	£43bn[a]	£35bn	n/a
	Global investment in our communities	£55.3m	£54.9m	£52.2m
	Colleagues involved in volunteering, regular giving and fundraising initiatives	62,000	58,000	57,000
	Group Employee Opinion Survey (EOS) – Proud to be Barclays	83%	81%	81%
	Percentage of senior managers who are female	24%	24%	25%

Panel 2: Barclays' Key Performance Indicators (source: Barclays 2010 Annual Report).

3.1.2 Barclays Premier Strategy

Present in France since 1917 and with more than 1600 employees, Barclays serves more than 170,000 customers with help of its commercial network (51 branches, 1 Premier Flagship, 64 Premier Clubs in cities of medium size and more than 300 "*home based*" financial advisors).

Barclays acquired an extensive experience and expertise in premium banking field dedicated to affluent customers. Barclays concentrates its development in the premium banking field to provide a banking relationship to its customer based on proximity, quality and innovation. This approach is integrated in a global strategy called "Barclays Premier" and overseen by Tony Blanco, France Country Manager.

This strategy is developed in Western Europe Area with help of dedicated retail pack offers, customer supports but also a set of managed investments with structured underlying solutions. These solutions are provided by a central business unit for Western Europe area which is located in Paris Branch due to its past history.

This dissertation will present the key steps of a risk framework development on Pipeline Risk i.e. the risk borrowed by a financial institution during investment products selling phase dedicated to its affluent customers.

3.2 RISK MANAGEMENT IN FINANCIAL ENVIRONMENTS

3.2.1 What is Risk?

A general definition of Risk can be the volatility of unexpected outcomes which lies in value of assets (equity or earnings).

From a business point of view, firms are exposed to various types of risks which can be classified broadly into business and financial risks:

1. Business Risks are those which the company assumes willingly to create a competitive advantage and to add value for shareholders. It includes the consequences of the business strategy and the business environment in which they operate.
2. Other Risks are usually classified into Financial Risks, which relate to possible losses owing to financial market activities.

Due to their fundamentals, Industrial Corporations are more exposed to Business Risks than Financial Risks. And the situation is reversed in case of Financial Institutions whose core activities are assets and liabilities instruments. Hence both constitute the Business ecosystem which looks like the two sided face of Janus, the Roman God of beginnings and transitions.

A deeper look on Financial Institutions shows that the Risk Management is the financial business keystone. First their core purpose is to assume, intermediate and/or advise on Financial Risks. This implies that Financial Institutions have to measure Financial Risk as precisely as possible in order to price them but also to be credible toward their customers, regulators and shareholders. The last point is very important because any exchange implies Trust, a fragile concept which requires lots of time and endeavour to be built but never acquired. So understanding risks is a concern which must be at the very heart of any financial manager because it allows planning for consequences of adverse outcomes and being better prepared for unavoidable uncertainty. But it also protects shareholders' interests, the customers' business and help to build a better Society.

These are theoretical concerns and unfortunately past history shows lots of failures due to the almost unpredictable nature of Risk. This assertion implies to ask ourselves what are the risk origins and their potential shapes. For instance it can be:

- human-created such as business cycles, inflation, changes in government policies or wars,
- the result of unpredictable natural phenomena (weather, earthquake ...),
- the primary source of long-term economic growth named as technological innovation,
- an unknown source which lie in our infinite universe.

So Risk remains unpredictable and financial managers learned to prevent it by iterations:

- The first step of Risk management is learning from our past experiences. This implies to determine their causes, to classify risks and to study them to determine how it works.

- The second step is to anticipate future risk behaviours and it may be done with help of mathematical tools and methodologies development to evaluate them.
- The third step is to develop a risk system which facilitates exchanges and communication from top to bottom and reverse. This implies that top has to define rules to frame any nature of risk and to channel business developments, the bottom to evaluate and to report current risk exposures, and lastly both have to share their point of view with help of authorizations, reviews and challenges.
- And last but not least, the whole system may function if and only if used by trustworthy people who have an adequate risk education and pragmatic approach.

These elements are the keystone of every Risk Management Framework and we will see how these steps are integrated by financial institutions such as Barclays.

3.2.2 What kind of Risks?

A review of past financial crisis allows classifying risks, and in panel below we presented the Barclays classification of major financial risks.

Figure 1: Classification of Main Financial Risks

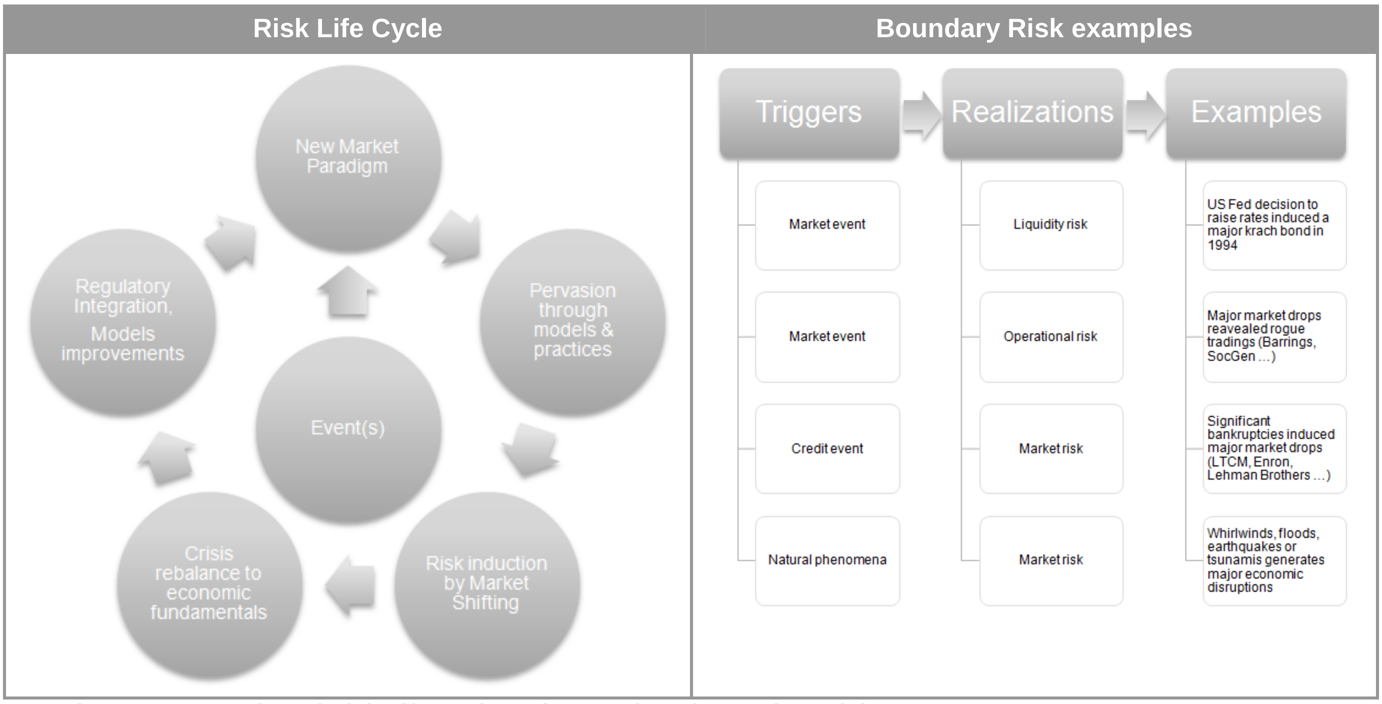
We presented 4 major categories with two or more subcategories which correspond to specialized risks:

- Market Risk is the risk to have losses produced by market movements generated either by level or volatility of market prices
- Liquidity Risk is generated by two sub-risks: asset-liquidity risk and funding risk. The first arises when a transaction cannot be conducted at prevailing market prices owing to the size of the position relative to normal trading lots. The second refers to the inability to meet payment obligations which may force early liquidation thus transforming potential losses to realized losses.
- Credit Risk is the risk of losses owing to the fact that counterparties may be unwilling or unable to fulfil their contractual obligations.
- Operational Risk is the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events.

Now let's have a bigger picture of this classification in a dynamic system as financial markets. In panel below we present our analysis on how risks propagate in financial markets until they trigger a crisis and how their results are integrated by market actors (on left of Panel 3). We also review some crisis to illustrate that our static classification in not self-sufficient to be adapted in dynamic system and we have to introduce the notion of "trigger" and "realization" (on right of Panel 3):

- Risk Cycle (left): 1) exogenous events or endogenous activities induce a change of existing paradigm and so act as new ideas generator, 2) these new ideas will spread across financial market by introducing new models, products and / or practices, 3) This will generate a movement and so a disequilibrium where original yields will decrease and financial actors will increase step by step their risk exposures, 4) the system becomes more and more tightened until a symbolic event will start a crisis and deflate the underlying bubble with the economic consequences, 5) at last most of financial actors will analyse the past crisis and integrate more or less the consequences either in terms of regulation, business practices or risk model developments / improvements.

- Boundary risks (right): we collected some major historical crisis and analysis on how risk propagated among the financial markets and impacted assets values. We introduced two new concepts to illustrate the risk dynamics and we call "*trigger*" the risk event which induces the "*realization*" i.e. the realized risk. We also add a new item to previous risk classification that we called "*boundary*" risk which are risks triggered by a risk event with a different risk nature.



Panel 3: Representation of Risk Life Cycle and Examples of Boundary Risks.

3.2.3 How to Measure Risk?

"*How much may I loose on this investment?*" is a common question among Investors who are in prospective mode or are evaluating the risk embedded in their current position(s) at a given time.

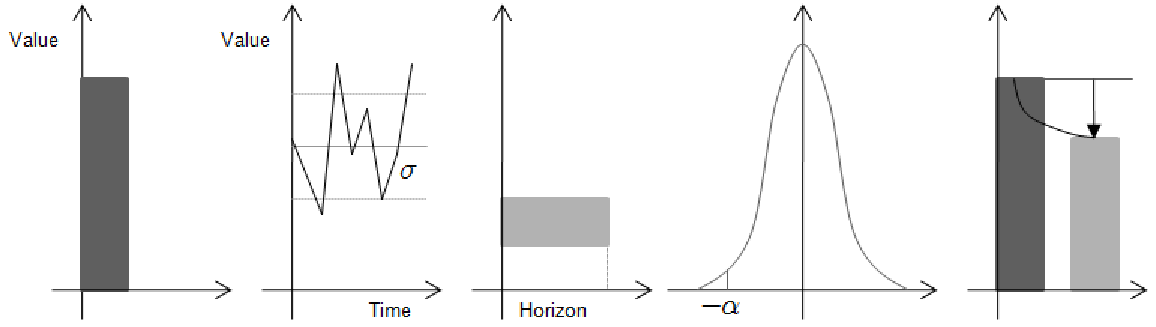
A quick glance on financial history shows that investors developed lots of methodologies to estimate approximately their potential losses according the application field:

- Actuarial Field: Fillip Lundberg developed in 1903 the Ruin theory which aims to estimate insurance companies' insolvency to collective risks. The historical model is based on a compound Poisson risk model and was largely extended nowadays.
- Investment Field: the first mathematical implementation was developed by Harry Markowitz & al. (1953) according to their portfolio theory, though their efforts were directed towards a different end, i.e. devising optimal portfolios for equity investors.
- Risk Management Field: a need for a standard risk methodology arose from the most important crisis of the eighties and nineties. The first sophisticated measure appeared in the Banker's Trust internal document concerning the fixed income portfolios. And next to the Barings bankruptcy, JP Morgan used for the first time the term "*Value-at-Risk*" (VaR) to describe the risk measure that emerged from data. This methodology found a ready audience with commercial and investment banks and became popular enough to be the established measure of risk exposure in regulatory frameworks (e.g. Basel and Solvency Frameworks).

In its most general form, The VaR measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval. In Figure 2 we defined a simple example to illustrate the VaR calculation steps:

1. Mark position exposed to market movements,
2. Measure variability of underlying risk factors from historical data,

3. Set time horizon,
4. Set a confidence level,
5. And finally valuates the potential loss / losses.



$$100M€ \quad x \quad 15\% \quad x \quad \sqrt{(10/252)} \quad x \quad -2.33(\alpha = 95\%) \quad = -7M€$$

Figure 2: A simple VAR example and its calculation steps

A closer look to this example will arise the following technical points:

1. VaR requires using probability distribution to estimate random effects of risk factors and the generation of potential losses. Most of VaR calculations are based on Normal Distribution of log returns however last decade shows its limitations especially to capture rare events.
2. A confidence level must be selected according to VaR uses. For instance, its uses may be used for day-to-day indicators (α = 95.00 %), Economic Capital / Solvency Capital Requirement (α = 99.50 %) or "*Risk Appetite*" scale levels (e.g. Barclays uses α = 99.95 % to estimate its "*risk appetite*" level – see § 3.2.4.2).
3. VaR is strictly oriented to downside risk.
4. VaR requires historical data to estimate the risk factors volatilities but also their potential correlations. This point may be tricky enough and require important technical analysis either on mathematic side or IT side.
5. VaR is mainly used on Market Risks but it may be employed on commodities, electricity prices and so on. However these elements are more volatiles and jumpy so it requires attention on underlying random process.

The Value at Risk can be evaluated with three approaches

1. Parametric VaR:
   Known also as Linear VaR, Variance-Covariance VaR, Greek Normal VaR or Delta-Gamma VaR, it is a parametric approach that assumes a normal probability distribution and requires the calculations of variance/covariance parameters. An important assumption is that the price variations are linear with respect to changes in risk factors.
   a. Pros: Fast calculation, Explicit VaR contributions (split by products, risk factors, business units and so on).
   b. Cons: Lack of nonlinear risk, Poor estimation of extreme events, Constant correlations over time.

2. Historical-simulation VaR,
   Historical simulations represent the simplest way for estimating Value at Risk. The principle is to estimate VaR by creating a hypothetical time series or returns on that portfolio, obtained by running the portfolio through actual historical data and computing changes for each period.
   a. Pros: Easy to communicate results, No required assumptions on underlying risk factors,
   b. Cons: Results are not exhaustive (dominated by most significant recent event), "*Window effect*" on risk estimation (i.e. the VaR decreases significantly when an event pop out from the time window).

3. Monte Carlo VaR
   it estimates VaR by forecasting multiple future risk factors paths and using nonlinear pricing models to estimate value variations for each path. Then VaR is estimated according to the worst cases.
   a. Pros: Use of full pricing model and non-linearity capture, Non exhaustive scenario generation
   b. Cons: Time and IT consumption, numerous technical issues on random process, Communication of results may be tricky.

These elements may be summarized by the following panel:

Indicator	Parametric VaR	Historical VaR	Monte Carlo VaR
Calculation Speed	Very high	High	Low
Mathematic Requirements	Few	Few	High
IT Requirements	Average (according to the portfolio size)	Average	High (Both for developments and IT infrastructures)
Non linearity effects capture	None	Full	Full
Jumps capture	None	Low (according to time window size)	Yes (according to the random process)
Historical data dependence	Average	Average	High

Panel 4: Pros and cons for each VaR approach.

3.2.4 Risk Management in Barclays

Barclays defines clear risk management objectives and has a well-established strategy to deliver them, through core risk management processes (see Barclays' 2010 Annual Report for more details).

The Barclays' approach is to provide direction on:

- Understanding the main risks to achieving Group strategy,
- Establishing Risk Appetite,
- And establishing and communicating the risk management framework.

At a strategic level, risk management objectives are:

- To identify significant risks.
- To formulate Risk Appetite and ensure that business profile and plans are consistent with it.
- To manage risk profile to ensure that specific financial deliverables remain possible under a range of adverse business conditions.
- To optimise risk/return decisions by taking them as closely as possible to the business, while establishing strong and independent review and challenge structures.
- To ensure that business growth plans are properly supported by effective risk infrastructure.
- To help executives to improve the control and co-ordination of risk taking across the business.

The process is then broken down into five steps:

1. Identify,
2. Assess,
3. Control,
4. Report,
5. And manage/challenge.

Each of these steps is broken down further, to establish end to end activities within the risk management process and the infrastructure needed to support it (see panel below).

Steps	Activity
Identify	– Establish the process for identifying and understanding business-level risks.
Assess	– Agree and implement measurement and reporting standards and methodologies.
Control	– Establish key control processes and practices, including limit structures, impairment allowance criteria and reporting requirements. – Monitor the operation of the controls and adherence to risk direction and limits. – Provide early warning of control or appetite breaches. – Ensure that risk management practices and conditions are appropriate for the business environment.
Report	– Interpret and report on risk exposures, concentrations and risk-taking outcomes. – Interpret and report on sensitivities and Key Risk Indicators. – Communicate with external parties.
Manage and Challenge	– Review and challenge all aspects of the Group's risk profile. – Assess new risk-return opportunities. – Advise on optimising the Group's risk profile. – Review and challenge risk management practices.

Please note this approach is consistent with Barclays' Business organization:

- Group defines general objectives, constraints, policies and limits,
- All implementations are delegated to business units, this to allow flexibility and adaption, therefore business units are in charge of controls and reports;
- Group's channels business unit's developments through limits and authorizations.
- Lastly compliance with regulatory and inner policies is insured by regular reviews and challenges.

3.2.4.1 Assigning responsibilities

Responsibility for risk management resides at all levels within the Group, from the Board and the Executive Committee down through the organisation to each business manager and risk specialist. Barclays distributes these responsibilities so that risk/return decisions are taken at the most appropriate level; as close as possible to the business, and subject to robust and effective review and challenge. The responsibilities for effective reviews and challenges reside with senior managers, risk oversight committees, Barclays Internal Audit, the independent Group Risk function, the Board Risk Committee and, ultimately, the Board.

Most important responsibilities are assigned as follow (see also pictures below):

1. The Board is responsible for Risk Appetite and Internal Control and Assurance Framework (Group Control Framework). The Chief Risk Officer (CRO) presents regular reports to the Board summarising developments in the risk environment and performance trends in the key portfolios. Thus the Board oversees the management of the most significant risks through regular review of risk exposures and related key controls. These responsibilities are set in Group's Principal Risks Policy.
2. The "*Board Risk Committee*" (BRC) monitors the Group's risk profile against the agreed appetite, where actual performance differs from expectations; the actions being taken by management are reviewed to ensure that the BRC is comfortable with them. The BRC reports its minutes directly to the Board.
3. The "*Board Audit Committee*" (BAC) receives quarterly reports on control issues of significance and a half-yearly review of the adequacy of impairment allowances, which it reviews relatively to the risk inherent in the portfolios, the business environment, the Group's policies and methodologies and the performance trends of peer banks.
4. The "*Board Remuneration Committee*" receives advice from the Board Risk Committee on the management of remuneration risk, including advice on the setting of performance objectives in the context of incentive packages.
5. The CRO is a member of the "*Group Executive Committee*" and has overall day to day accountability for risk management under delegated authority from the Finance Director. The CRO manages the independent Group Risk function and chairs the Group Risk Oversight Committee, which monitors the Group's risk profile relative to established risk appetite. Reporting to the CRO and working in the Group Risk function are divided among risk-type heads which are retail credit risk, wholesale credit risk, market risk, operational risk, financial crime risk and capital demand. Along with their teams, the risk-type heads

are responsible for establishing a Group wide framework for risk control and oversight. Moreover the risk-type teams liaise with each business as part of monitoring and management processes.

6. Each business unit has an embedded risk management function, headed by a business risk director. Business risk directors and their teams are responsible for assisting business heads in the identification and management of their business risk profiles and for implementing appropriate controls. These teams also assist Group Risk in the formulation of Group policies and their implementation across the businesses. The business risk directors report jointly to their respective business heads and to the Chief Risk Officer.
7. Finally "*Risk-type*" Heads and Business Risk Director report to the CRO and are member of the "*Group Risk Oversights Committee*".

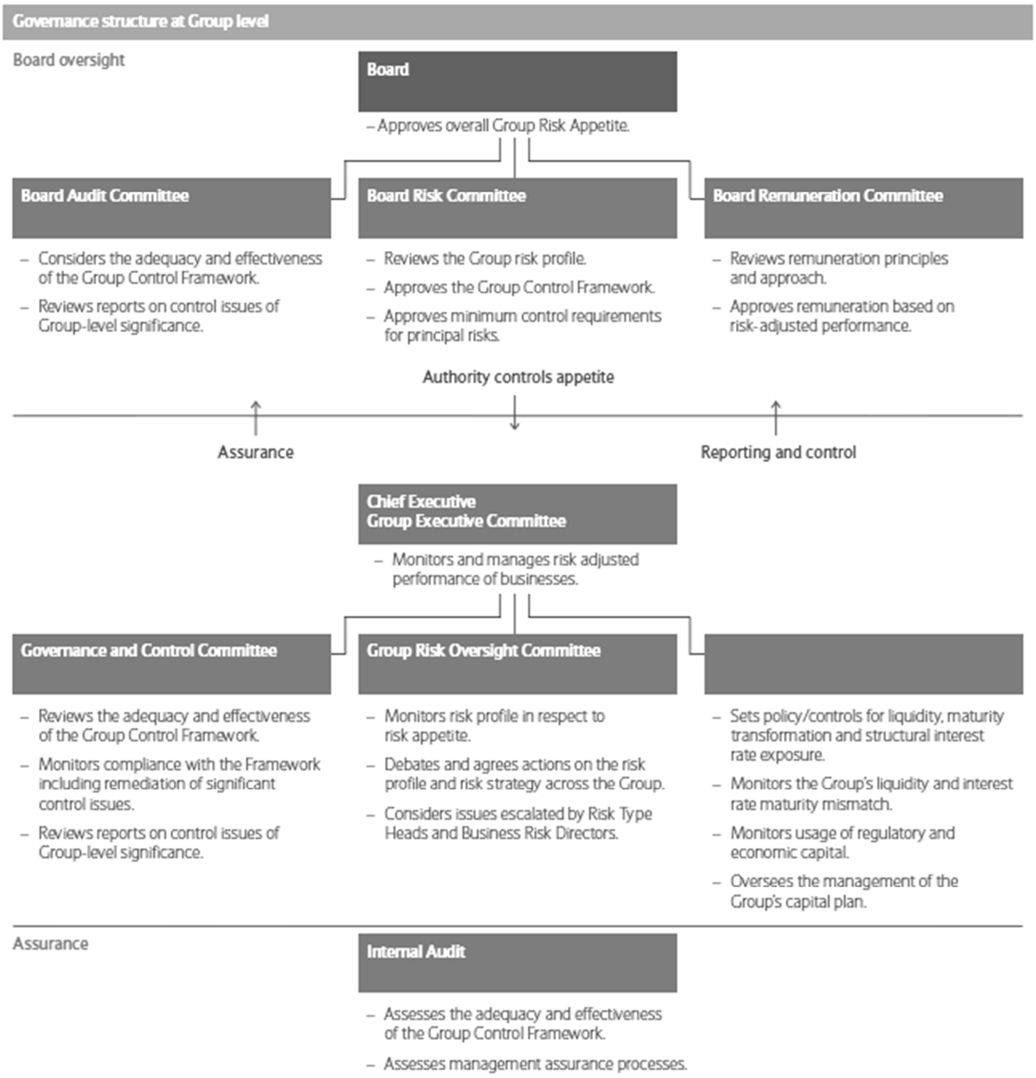


Figure 3: Representation of the Dual Governance Structure of Barclays Bank Plc., divided between the Board and The Group Executive Committee (source: Barclays' 2010 annual report)

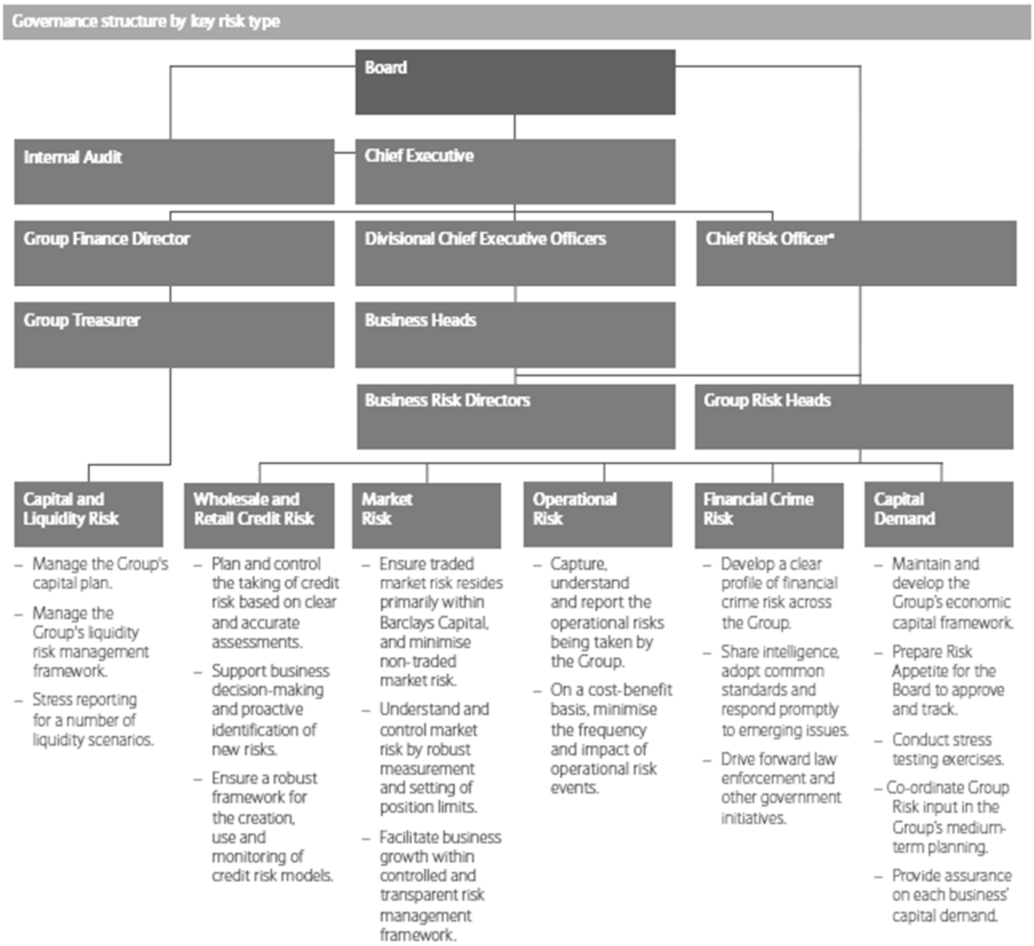


Figure 4: Representation of Barclays' Governance Structure by key risk type. Please note "*Capital and Liquidity Risk*" belongs to the Group Treasurer's management scope.

3.2.4.2 Risk Appetite

Risk Appetite is defined as the level of risk that Barclays is prepared to sustain whilst pursuing its business strategy, recognising a range of possible outcomes as business plans are implemented. Barclays' framework combines a top-down view of its capacity to take risk with a bottom-up view of the business risk profile associated with each business area's medium term plans. The appetite is ultimately approved by the Board.

The Risk Appetite framework consists of three elements: "*Financial Volatility*", "*Mandate & Scale*" and "*Risk Appetite and Stress Testing*". Taken as a whole, the Risk Appetite framework provides a basis for the allocation of risk capacity across Barclays Group.

3.2.4.2.1 Financial Volatility

Financial Volatility is defined as the level of potential deviation from expected financial performance that Barclays is prepared to sustain at relevant points on the risk profile. To measure the risk entailed by the business plans, management estimates the potential earnings volatility from different businesses under various scenarios, represented by severity levels:

- expected loss: the average losses based on measurements over many years
- "*1 in 7*" (moderate) loss: the worst level of losses out of a random sample of 7 years
- "*1 in 25*" (severe) loss: the worst level of losses out of a random sample of 25 years
- "*1 in 100*" (extreme) loss: the worst level of losses out of a random sample of 100 years

These potentially larger but increasingly less likely levels of loss are illustrated in the Risk Appetite concepts chart (see Figure 5). Since the level of loss at any given probability is dependent on the portfolio of exposures in each business, the statistical measurement for each key risk category gives the Group clearer sight and better control of risk-taking throughout the enterprise.

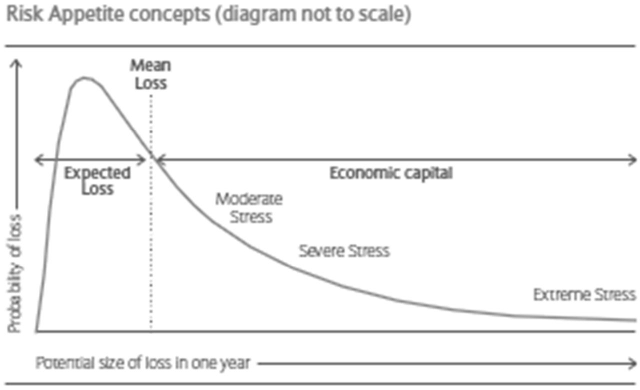


Figure 5: Representation of Barclays' classification of severity levels

The Board sets the Group's financial volatility risk appetite in terms of broad financial objectives (i.e. "*top down*") on through the cycle of "*1 in 7*" and "*1 in 25*" severity levels. The Group's risk profile is assessed through a 'bottom-up' analysis of the Group's business plans to establish the financial volatility. If the projections entail too high a level of risk (i.e. breach the top-down financial objectives at the through the cycle of "*1 in 7*" or "*1 in 25*" or "*1 in 100*" levels), management will challenge each area to rebalance the risk profile to bring the bottom-up risk appetite inline within top-down appetite. Performance against Risk Appetite usage is measured and reported to the Executive Committee and the Board regularly throughout the year.

Specifically, Barclays believes that this framework enables it to:

- Improve management confidence and debate regarding the Group's risk profile
- Re-balance the risk profile of the medium-term plan where breaches are indicated, thereby achieving a superior risk-return profile
- Identify unused risk capacity, and thus highlight the need to identify further profitable opportunities
- Improve executive management control and co-ordination of risk-taking across businesses

3.2.4.2.2 Mandate & Scale

The second element to the setting of risk appetite in Barclays is an extensive system of Mandate & Scale limits, which is a risk management approach that seeks to formally review and control business activities to ensure that they are within Barclays mandate (i.e. aligned to the expectations of external stakeholders), and are of an appropriate scale (relative to the risk and reward of the underlying activities). Barclays achieves this by using limits and triggers to avoid concentrations which would be out of line with external expectations, and which may lead to unexpected losses of a scale that would be detrimental to the stability of the relevant business line or of the Group. These limits are set by the independent Risk function, formally monitored each month and subject to Board-level oversight.

Barclays uses the Mandate & Scale framework to:

- Limit concentration risk
- Keep business activities within Group and individual business mandate
- Ensure activities remain of an appropriate scale relative to the underlying risk and reward
- Ensure risk-taking is supported by appropriate expertise and capabilities

As well as Group-level Mandate & Scale limits, further limits are set by risk managers within each business unit, covering particular portfolios.

3.2.4.2.3 Risk Appetite and Stress Testing

Stress testing occurs throughout the Bank and it helps to ensure that Barclay's medium term plan has sufficient flexibility to remain appropriate over a multi-year time horizon during times of stress.

Stress testing allows us to analyse a specific potential economic scenario or event using defined macro and market based parameters. The results of a stress test, whether at a Group or business level, will produce an output which could be compared to a point in the curve of our Financial Volatility based statistical outcomes, although stress tests are scenario based and as such are not calibrated to a specific confidence level.

Given that the stress testing, Risk Appetite, and medium term planning timelines are all aligned, the outputs of stresses are used by risk functions throughout the Group to inform on Risk Appetite (particularly at a business level). The outputs of stresses also feed into the setting of Mandate & Scale limits. For example, via the use of primary and secondary stresses in Market Risk, we identify and limit the scale of risks that DVaR (i.e. Daily VaR) would not automatically capture.

Reverse stress testing also supports our Risk Appetite framework. Reverse stress testing starts with defining a worst case set of metrics and deduces a scenario that could theoretically cause that situation to occur. This will help to ensure that we understand the tail risks across our books and explain what would have to happen to generate a change in strategy. Group reverse stress testing also identifies risks that in one business alone would not have been sufficient to be a critical event, but are significant at Group level, thereby complementing the Financial Volatility and Mandate & Scale processes.

3.3 <u>"STRUCTURED PRODUCT" RISK MANAGEMENT</u>

3.3.1 What is a Structured Product (SP)?

A Structured Product is generally a pre-packaged investment strategy corresponding to a combination of standard financial investment and derivatives. The whole items are then securitized in one instrument as an independent product by the issuer. Interest in this kind of investment grows these last years and Structured Products are commonly used by investors as a way to diversify their investment portfolios (see Figure 11 for more details regarding Barclays' figures).

Structured Products are not homogeneous because they are "*tailored*" products and their productions are closely linked with market conditions. However they may be classified in the following categories according to the derivatives' underlying(s):

- Interest Rate SP: this category linked the SP's performance to IR variations. The underlying may be based on a single rate reference or a basket of rate references.
- Equity-linked: this category linked the SP's performance to equity market variations. The underlying may be a single stock / index or a basket of stocks / indices.
- FX-linked: the performance is connected to currency variations which must be based on a single or a basket of currencies.
- Credit linked: the performance is linked to credit events of a firm or sector.
- Hybrid linked: this kind of SP combines a set of multiple underlying types.

The most important features of Structured Products according to the offered strategy are:

1. An optional principal protection (in general at maturity),
2. A potential enhanced return,
3. A stabilized volatility within an investment (known as "*Volatility target*" product).

Figure 6 illustrates a typical structured product composition:

- Initial investment is borrowed by a so-called "*Liquidity Support*" which depends essentially if the SP proposes a principal guarantee or not. Usually the support will be a debt note (bond) with a similar maturity when a principal guarantee is offered. Otherwise the support may be an equity support (index or stock basket) or a bond combined with a "*Down and In*" Put through the structured leg of the swap.
- The SP issuer will exchange cash flows generated by Liquidity Support (coupons, dividends) in exchange of a structured leg also called "*formula*". This exchange will be settled with help of swap contract which will define all the derivative features of the structured leg.

During the *"structuring phase*" (i.e. the SP development phase in financial lingo), the issuer will have a particular attention on Liquidity Support selection and Structured Leg composition due to the impact on SP Pricing. Thus the issuer will select bond / equity support on a compromise basis between price and potential credit risk and adjust the structured leg composition by selecting, adding and / or retrieving options to find the more accurate formula. Hence the issuer will try to generate a spread between paid and received legs to generate an upfront fee according to the obtained conditions on liquidity support and structured leg prices, and its inner funding conditions.

Financial creativity expressed itself essentially in structured leg composition and may involve:

- More or less complex payoffs from simple European to digital payoffs,
- Time dependant options such as "*auto call*" options (i.e. reaching the strike level at coupon payment may induce early redemption), memorized coupons (i.e. realized coupons are paid at maturity) or delay option (i.e. past coupons recovery in case of event realization),
- Terminal option on coupon(s) and principal may be set such as digital payoff on coupons payment, put DI on principal with leverage,
- And so on...

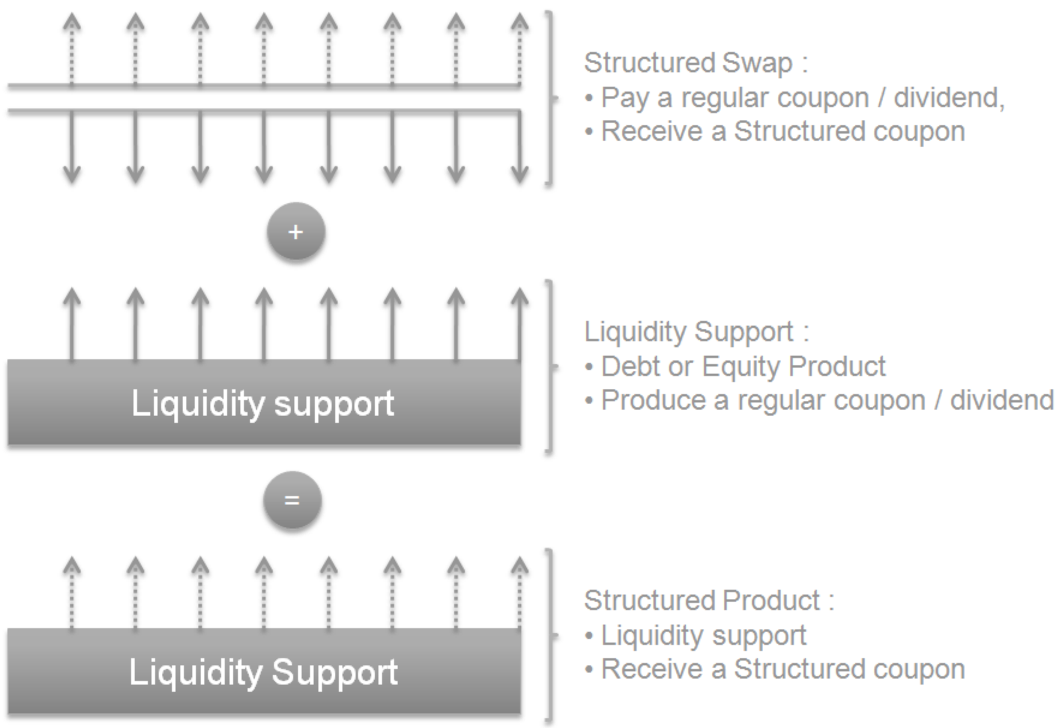


Figure 6: An example of Structured Product composition.

It is important to note that this kind of structures borrows some drawbacks which include:

- A High Credit Risk profile compounded of issuer plus the potential liquidity support credit risks,
- A potential Lack of Liquidity because secondary market is animated only by SP's issuer and so SP's price may be subject to important bid-ask spread,
- A Lack of daily quotes on primary market (only on demand),
- A complex valuation either on primary or on secondary market regarding to the formula complexity and shadow parameters as issuer's funding or complex underlyings (e.g. volatility spread).

Professional investors (e.g. financial institutions, pension funds, insurance companies, corporate firm treasuries ...) show a growing interest to this investment category for past decade because it supplies a packaged investment strategy with high potential yields. Regarding the past two decades, competition increased a lot with more and more aggressive commercial strategy. Moreover assets selection became more and more difficult with the impact of two items: a decreasing trend of riskless assets yield and the growing of assets under management due to free cash availability (see Figure 7). The combination of all these effects increased a business instability either on shortening of recurrent income horizon (past production became more and more sensitive to customers' arbitrage) or a decrease of new business quality due to a risk appetite increase.

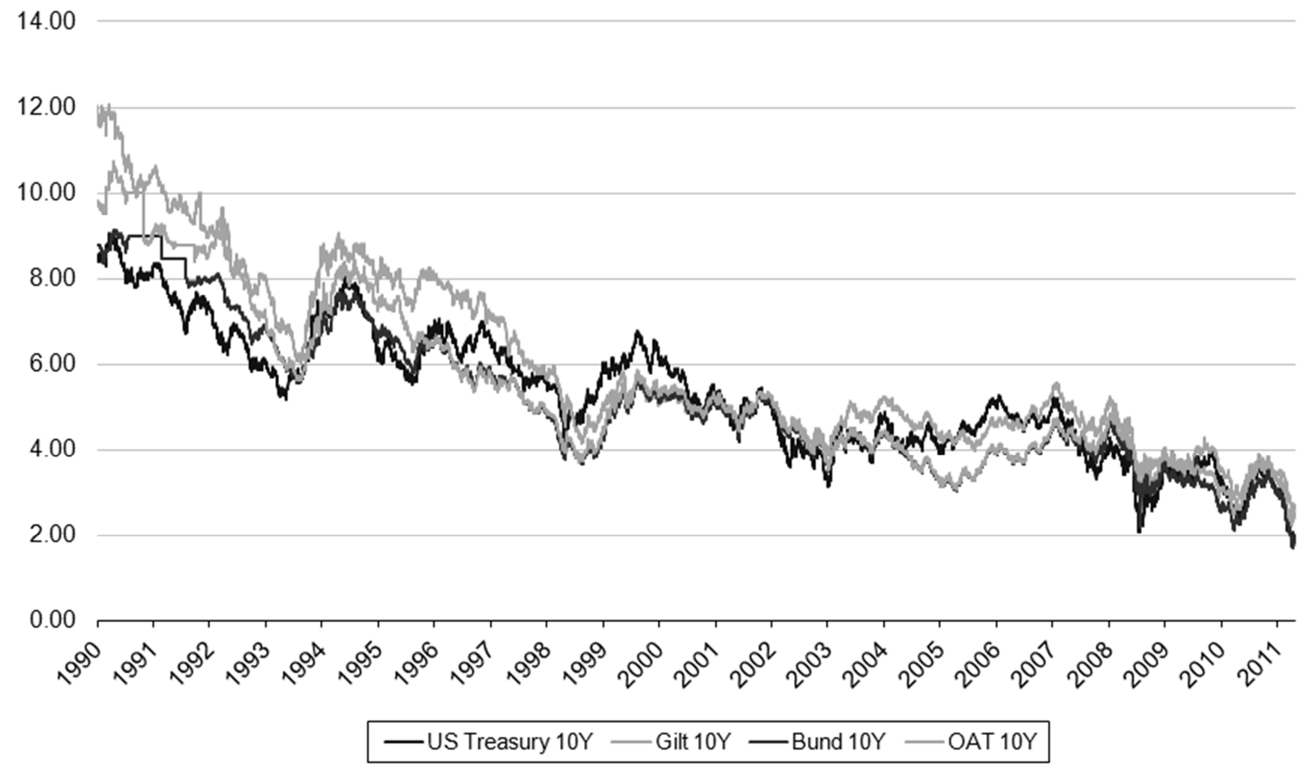


Figure 7: Historical Yields for main "riskless" investments.

In this context, Structured Products supply many commercial advantages such as:

- A tailored investment strategy in adequacy with end customers' appetite,
- A potential yield higher than classical assets' yields,
- A shortening of marketing cycle with help of white-box strategy (i.e. anonymous packaged products),
- An outsourcing of most important technical issues.

Unfortunately as this investment category has some drawbacks and having a good estimation of underlying risks is a must have when they are used to elaborate investment supports dedicated to personal investors. Previously we described the cycle of financial crisis (see §3.2.2) and the impact of creativity on their accelerations.

The recent financial crisis triggered by Lehman Brothers' Bankruptcy highlights the phenomena of unleashed creativity. From a Risk Management point of view, we may state the past crisis cycle as follows:

1. The 2001 crisis triggered the low rate policy for most of central banks which supplied unlimited low-cost cash to every financial contributor,
2. This free cash increase dried out all basic asset pockets by a progressive price increase and so a yield decrease,
3. The investors' risk appetite increased by selecting riskier investments and creating complex investment structures. These new financial supports contaminated gradually end customers' investment such as mutual funds and pension funds.
4. The underlying risks became less and less perceptible and so measurable. Hence financial actors didn't perceive that general rate tightening induced a push back of general disequilibrium, even if several events

punctuated this state of fact such as Bearn Stern' Bailout, BNPP's dynamic monetary fund closing or the major decreases of main equity index since end of July 2007.

5. The Lehman Brothers was the culminant point and acted as catharsis to risks materialization and propagation through most investment supports. Hence most of financial actors endorsed important losses due to major risk endowments.
6. The risk cycle closed by major recapitalizations with support of their local governments by increasing national debt outstanding. This induced a major review of regulatory rules and processes (for instance, the future Basel 3 rules or evolution of Solvency 2 rules on "*Sovereign Risk*" issue).

The last point spread very differently according to the national objective(s). In the case of France, the regulatory authorities reviewed and focused on complex investment products dedicated to individual investors. Regarding the investment wrappers (Life Insurance, PEA), the regulatory authorities stated new rules to limit complexity and to give better explanation on underlying risks (see statements emitted both by AMF and ACP on 15/10/2011 concerning the marketing of complex financial instruments to the public).

These new rules are essential and necessary because every financial transaction is built upon one concept: Trust. And we claim these new constraints are good for business by adding clear rules, improving risk management, and industrializing the processes.

3.3.2 Investment Solutions (IS) in Barclays

For the last two years, Barclays has changed the orientation of its retail business goals to focus essentially on a specific customer category just below Private Banking criteria, named "*affluent*" in most of marketing studies. Regarding internal studies, this category of customers are wealthy enough, educated and looking for increasing their personal assets. Hence they are more eager to Risk and present a more developed Risk Appetite. This category of customers generates a consequent part of Barclays Retail Division's profit and so represents its core business. In current context, Barclays is challenged significantly by its main competitors either on new customers or on its core business. Moreover this customer category is aware of its importance and very challenging on services and products provided.

Therefore Barclays developed a dedicated strategy labelled "*Premier Strategy*" to preserve its current business and seduce new customers. This strategy is declined in three essential key points:

- A set of special retail services and products
- A dedicated customer relationship
- A set of special events regarding the category.

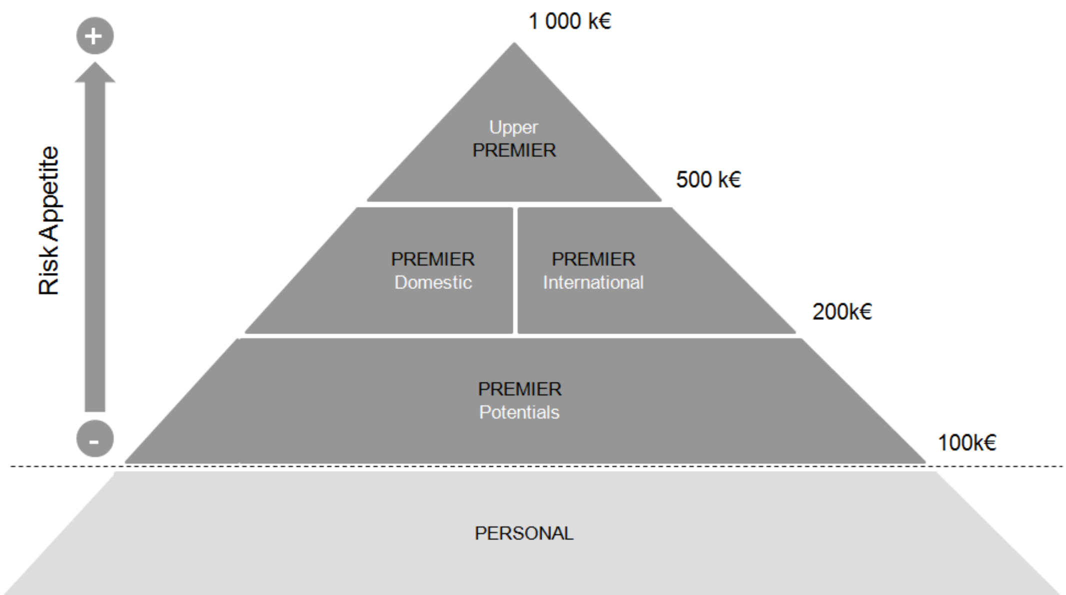


Figure 8: Marketing segmentation of Barclays' Retail Customers regarding their annual wages.

Regarding the Barclays internal strategy, Customers are categorized regarding their annual income and/or assets as presented Figure 8 and are regrouped in two main categories: “*Personal*” Customers and “*Premier*” Customers. The first category represents the most important part of retail bank population but contributes less to Barclays’ profitability than the second main category. Then the “*Premier*” category is divided into 4 segments which represent their potential and / or their wealth origins. Lastly it is important to note that the wealthiest customers are not represented in this segmentation because they are eligible to “family office” services.

In early 2009, Barclays France created a new business unit dedicated to special investment sales to fulfil the objectives defined by the Premier Strategy in terms of investment products. This division called “*Centre of Excellence on Structured Products*” (CoE) aims to create new investment supports fitting the customer behaviour (i.e. risk appetite) and local investment practices (i.e. favourite investment support). At this time, the supplied investment support can either be a structured note (issued by external counterpart), deposit (issued by Barclays with a BMTN as Liquidity support), or a term account (opened in Barclays’ book) format (see Figure 9)

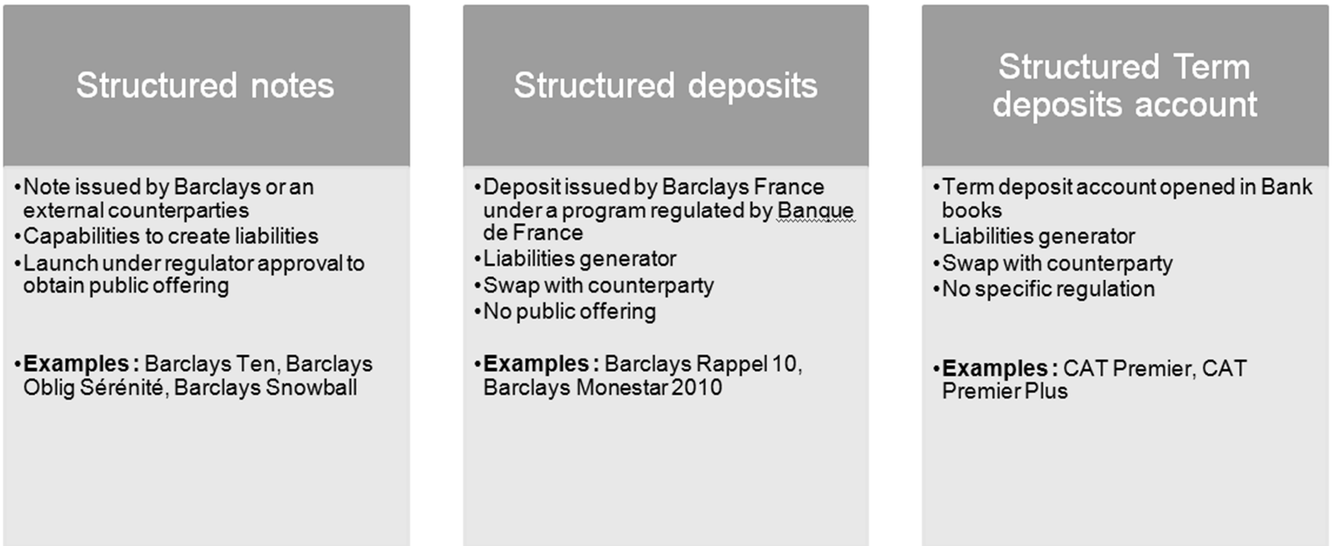


Figure 9: Types of investment products supplied by the CoE

The CoE is a trans-national division because it organizes and coordinates structured products sales across Retail Distribution Networks of the Western Europe area (WE), i.e. France, Spain, Italy and Portugal. Hence CoE is organized between Local Investment Teams and CoE Central Organization (see Figure 10):

1. Local Teams are in charge to define and calibrate new investment strategies regarding the local customer’s habits and risk appetite, the adequacy with local regulatory rules and customer protection rules. Moreover local teams are in charge to assess underlying risk because they are risk owners (i.e. they will have to endorse potential losses). And at last they are in charge to get the “*New Product Approval*” committee validation to distribute new products (see Figure 13 for more details),
2. CoE Central Organization is located both in London (organization and governance line) and Paris (Front Office and Risk Management). It is in charge to organize and coordinate the distribution of structured investments. This integrates several functions such as:
    a. The interaction with local investment teams to structure accurate strategies regarding collected needs,
    b. The negotiation of each financial sub-component with external market counterpart after receiving the final go from local investment team. This negotiation is driven by a compromise between underlying risks and product’s profitability at issuance during primary market phase.
    c. The management of secondary market of sold structured products after the initialization date (i.e. strike date),
    d. The report in term of P&L and Risk Management,
    e. The interaction with Barclays Group in term of development and objectives,
    f. And lastly (but not least) an overall risk unit to assess, valuate and manage risks generated by the business unit.

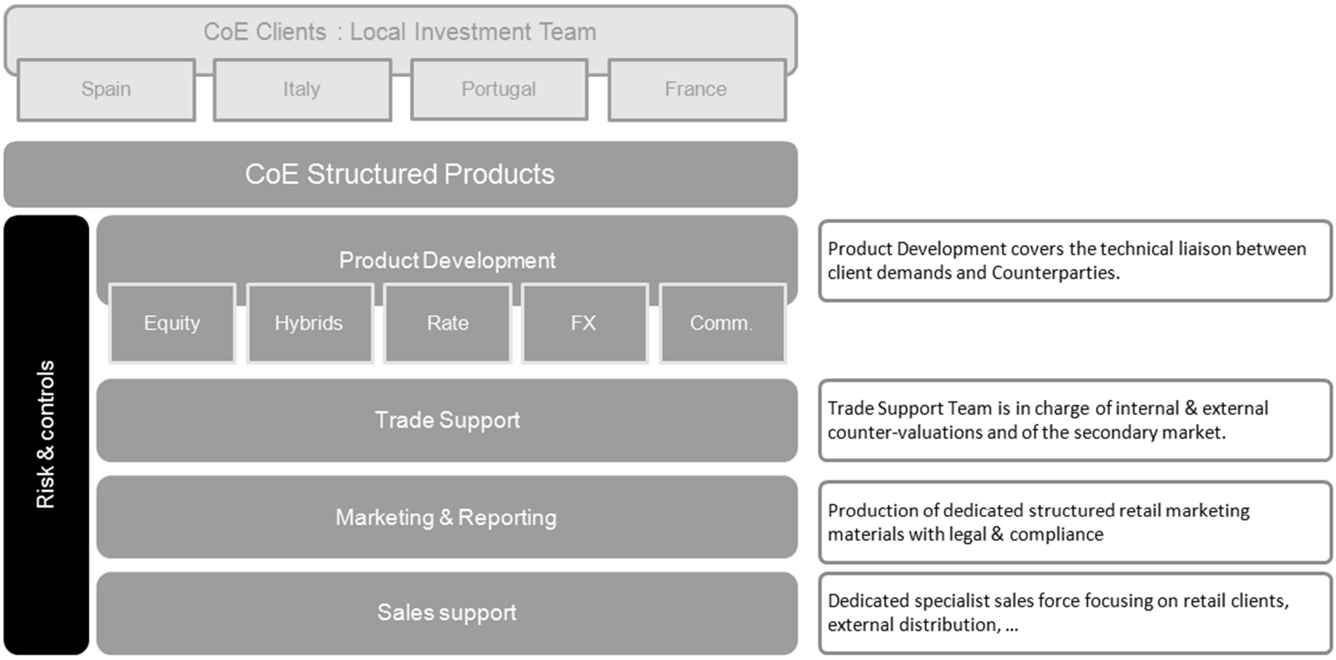


Figure 10: Organization Chart of CoE both for Western Europe and Local Divisions.

Figure 11 summarizes the key figures generated by this business unit and gives several feedbacks:

1. A strong interest from our customers for this category of investments,
2. A significant difference risk appetite regarding the countries in terms of
    a. Capital protection,
    b. Asset natures.

Key figures

	Volume			Income by country			Country Margin		
	2009	2010	TOTAL	2009	2010	TOTAL	2009	2010	TOTAL
France	649.1	410.3	1 059.4	22.37	20.55	42.92	3.4%	5.0%	4.1%
Spain	685.9	334.8	1 020.6	21.42	13.72	35.15	3.1%	4.1%	3.4%
Portugal	93.0	427.6	520.6	4.05	15.81	19.86	4.3%	3.7%	3.8%
Italy	38.9	34.0	72.9	1.39	0.79	2.18	3.6%	2.3%	3.0%
TOTAL	1 466.8	1 206.7	2 673.5	49.23	50.87	100.10	3.36%	4.22%	3.74%

Product characteristics

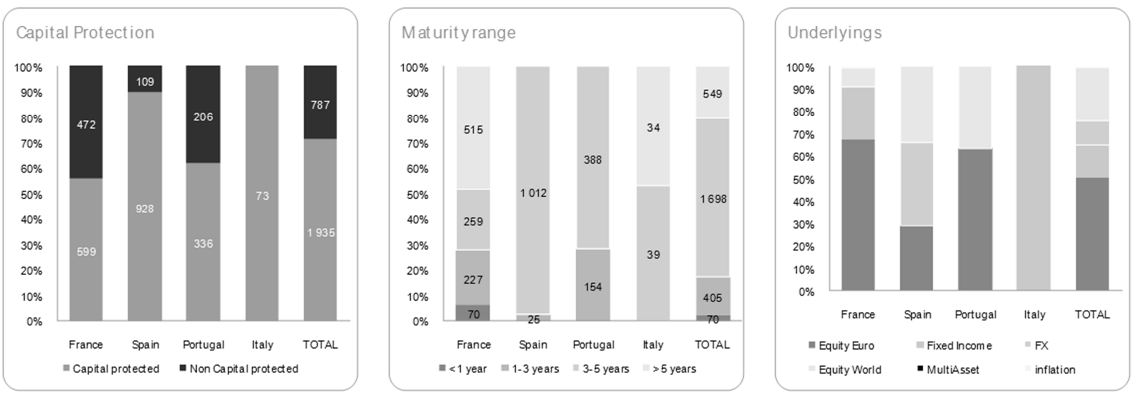


Figure 11: CoE key figures for exercise periods 2009 and 2010

These elements constitute also a proof of concept for Premier Strategy regarding the development of dedicated investment solutions. However this activity requires a strong and efficient risk management either to protect customers or to preserve business' profitability. This risk management must cover every kind of risk and requires an industrialized process with accurate checks. And this state of risk can be reached through a deep knowledge of underlying activity cycle and hazardous event triggering.

3.3.3 Business cycle

To illustrate the process in place in Barclays, we will present it regarding the life cycle of a structured product. This cycle follows the key steps (see Figure 12):

1. Product Conception: This step concerns all marketing actions concerning customer need appreciation and product design under constraints of risk adjustment.
2. Product Certification: every product distributed through Barclays retail networks must be assessed and certified by the "*New Product Approval*" (NPA) Committee.
3. Sales: The certified product is sold by retail networks for a period between 3 to 6 months. Please note that most structured products with options are bought in "forward starting" mode, i.e. the reference value of underlying will be set in the future. Moreover every structured product is closed to subscriptions at the end of Sales Period.
4. Living Products Management: this step concerns the management of closed structured products, i.e. collect repurchase orders and dispatch retrieved cash to final customers. This step will focus essentially on valuation controls and customers reporting.

Figure 12: Life steps of Structured Product Issued by CoE.

In Figure 13 we present a detailed view of new product approval process which assesses and qualifies every new product distributed by Barclays Plc. The key steps of this overall assessment are:

1. "*Ideas Generation*" step corresponds to the product design phase where CoE conceives and plans new products production. These draft products are presented to key functions (Compliance, Risk ...) to be reviewed, challenged and if necessary aligned to become eligible to retail distribution.
2. "*Clients Need identification*" step corresponds to assess the adequacy between customers' needs and products offer,
3. "*NPA agreement*" corresponds of the full certification process by all function stakeholders (IT, Human Resources, Finance ...). This process establishes that risk factors are identified and under control either from operational point of view or from internal policies.
4. "*Trade, issuance & settlement*" step is a key step for Market Risk. The new product is eligible to retail distribution and so BPLC has to constitute a stock, this position is borrowed on BPLC's balance sheet for a given period (a.k.a. "*sales period*") and any residual position has to be sold at the end of this period. Thus BPLC borrows a potential market risk by trading for given specific market conditions which may induce potential losses if a residual position remains and if market conditions worsen. Hence every new trade must be assessed and authorized before dealing by a dedicated process. This risk is called "*Pipeline Risk*" and will be presented in §3.4.
5. "*Pre-launch*" step corresponds to the generation of official product documentation and its review by regulatory authorities (e.g. AMF for France). But also to the preparation of all marketing materials to prepare the product selling campaign.
6. "*Selling Period*" step corresponds to the product selling period which is finely monitored either in term of risk management or profitability.

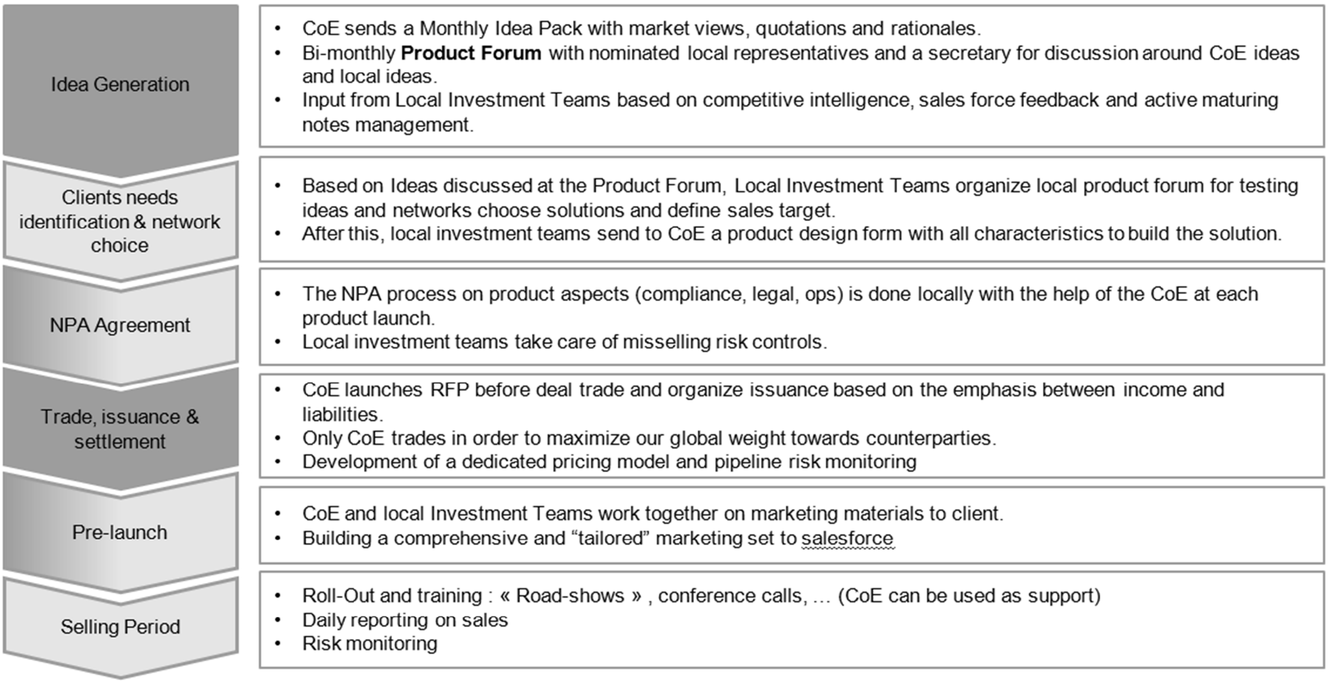


Figure 13: Stepping process followed for each new product launch

As said in previous paragraph, CoE is in charge of all activities and so must integrate all essential functions to manage secondary market (see Figure 14):

1. “*Repurchases Management*” function: like for Mutual Funds, customers have the right to repurchase their investments at every moment for a given living product. The counterpart is customers are exposed to market variations and their consequences on product’s price. Hence CoE organizes this activity and act as financial intermediate between end customers and product’s issuer. By the way, this activity generates a residual market risk due to issuers’ constraints (minimal amount) and intraday market variations (i.e. elapsed time between customers repurchase order and selling to the issuer). This risk will be reviewed in §3.4.
2. “*Price controls*” function: Structured products may have complex valuation process due to option or/and credit stacked structure and a high sensitivity to market conditions variations. Moreover most of structured products are OTC and quotes are produced only by issuers. All these facts together give an important issue regarding the price evaluation of these products. Hence regulatory authorities ask for financial intermediates to counter valuate this kind of product to assess the price fairness. CoE establishes a pricing process based on two items: an internal pricing process with a dedicated team and a valuation with help of external pricing providers such as Euro VL, SunGard or Pricing Partners.
3. “*Risk Management*” function assesses risks generated by CoE activities, put in place a control plan to limit operational risk, and reports to Barclays Group the risk statement.

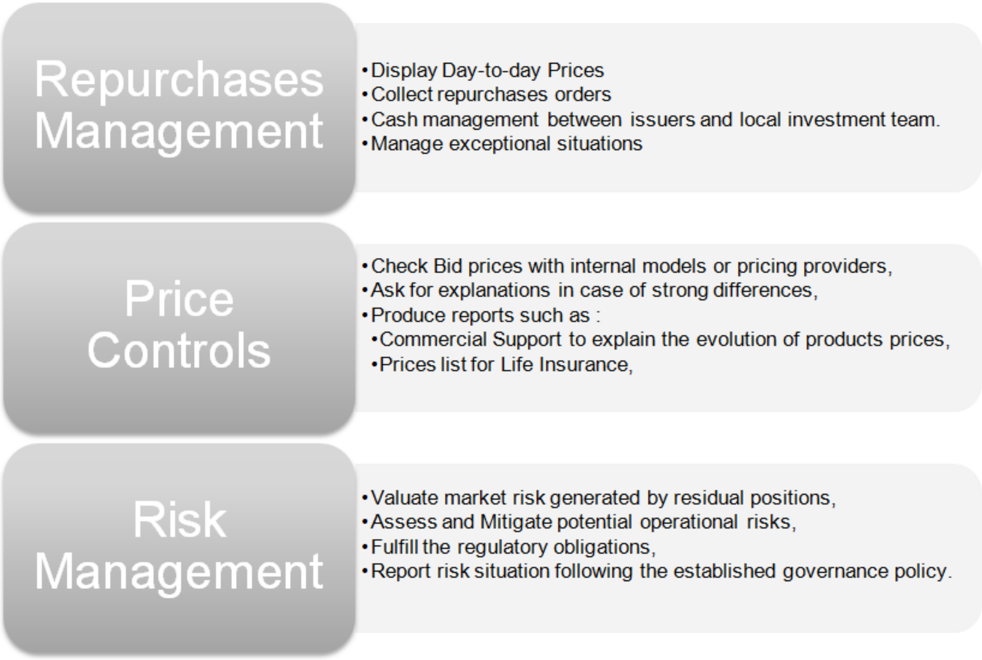


Figure 14: Key functions concerning living products.

3.4 RISK EXPOSURE GENERATED BY IS ACTIVITIES

3.4.1 Risk Analysis of the generated Risk Exposure

In previous paragraph we started with a brief description of potential risks generated by residual positions either at product issuance (primary market) or during its life after closing customers' subscriptions (secondary market). In this section we will investigate further the nature of risk exposures generated by these activities to produce a detailed analysis. Thereafter its conclusions will be helpful to define the underlying risk nature and its dynamics. This will drive our theoretical researches to define all the necessary components to produce a risk framework such as pricing model, assumptions, limits and calibration process.

At first let's start our analysis by a short description of underlying risks and we will use following figures to illustrate the risk induced by a residual position at a given time (B) and the variation of the market value (A), either for the primary market (Figure 15) or secondary market (Figure 16).

Figure 15 represents the potential risk generated on primary market and we can describe it as follows:

1. CoE starts the stock constitution at time $t = a$ after getting the NPA certification and the authorization to trade. This induces a market position for a given price and market conditions which can either match the whole target commercial objective (100 in figure) or a part of it (symbolized by line "Bank Position"),
2. Sales Period will finish at time $t = b$ where all residual position must be sold whatever the price. The reason of such decision found its sources in Barclays internal risk rules and IFRS constraints: retail activities are not allowed to keep products with Market Values on On-Sheet Balance. The exception is the sales period where commitments on this category of products are listed on Off-Sheet Balance. Moreover structured products are initialized at $t = b$ to issue the note with underlying credit product and to define reference level(s) (i.e. "*the strike level*") used by future conditions.

3. During the interval $]a,b[$, the risk exposure is materialized by the difference $\text{Risk exposure} = \sum \text{authorized deals} - \sum \text{sales}$. Thus the risk exposure is essentially driven by customers' behaviour and we can define this hazardous as the risk trigger event (see §3.2.2).
4. The cost/profit linked to this risk exposure is driven by price variations regarding market conditions, materialized by the difference $\Delta\text{Price} = \text{Price}_t - \text{Price}_a$ with $t \in]a,b[$ and stock constituted only at $t = a$. Hence we can define the price variations as the risk realization event.
5. Thus the compounding of both events gives the primary pipeline risk, i.e. the risk of potential losses induced by the residual expositions.

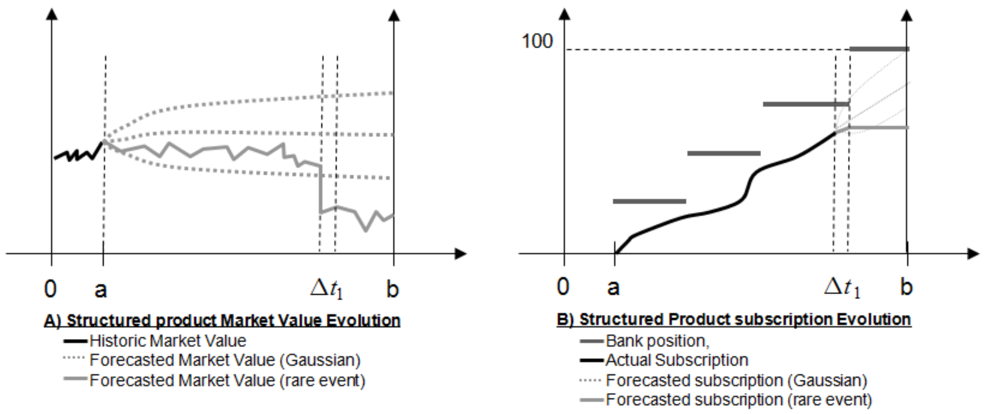


Figure 15: Representation of potential on primary market.

Figure 16 summarizes the potential risk generated by residual positions in secondary market. Please note that the underlying principles are the same as those presented in Figure 15 but with different conditions:

1. At time $t = b$, the subscriptions are closed and only outflows can proceed. Moreover all reference conditions are initialized.
2. The product will live until $t = c$ which corresponds to product's contractual maturity where the final redemption may proceed regarding the structure (presence of early redemption conditions, final conditions on final payments ...).
3. During the interval $]b,c[$ the customers may proceed to repurchases, i.e. sell the bought product for a given reference price to CoE.
4. The CoE resells the product to primary issuer either instantaneously or after a delay Δt_2. The presence of a potential delay will depend on issuer's constraints on secondary market (generally a minimal amount for treating the repurchases). Hence the customer behaviour will have a strong incidence on risk exposure generation regarding issuer's constraints and so we can describe it as risk trigger event.
5. According to delay Δt_2, market conditions may evolve a lot with the implicit consequences on product's price. The greater the delay Δt_2 is, the more the cost may be. Thus we can design this item as the risk materialization event.
6. As previously the compounding of both events will give the secondary pipeline risk.

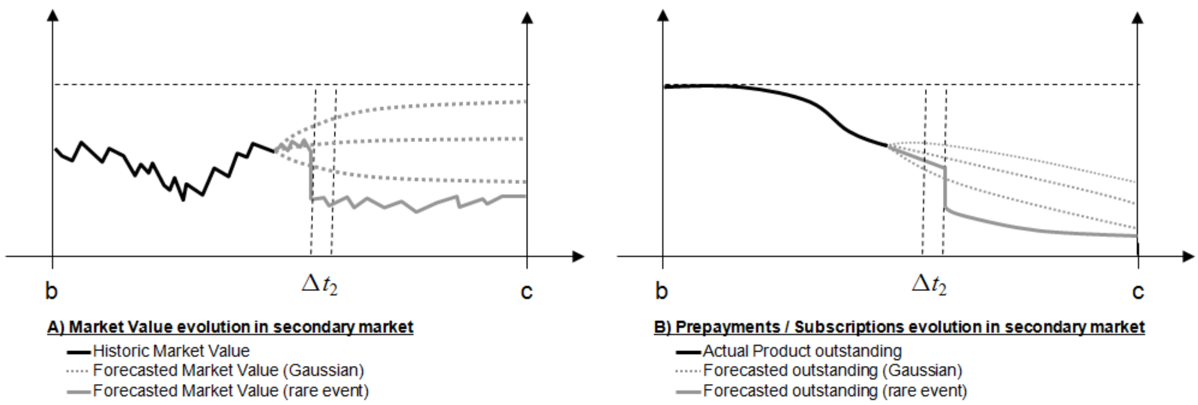


Figure 16: Representation of potential risk on secondary market

Please note that we represented only a long selling position which is the only one authorized by Barclays authorities and so will constitute one of the core assumption in current dissertation.

These two linear descriptions give us a taste of what are the constituents of pipeline risk and how they influence the risk dynamic. Hence the pipeline risk structure is compounded by two constituents: a risk exposure driven by customers' behaviour (conditioned by its personal situation and the interactions with sales people) and a potential loss driven by underlying market conditions. However they produce only a slight insight of the underlying dynamic and its factors.

To go further, we have to solve pending questions to appreciate fully the dynamic of pipeline risk and so produce an accurate risk management framework. These unsolved questions can be categorized in three sets: the first one will concern the customer behaviour to determine its main drivers, the second one will focus on the underlying pricing model of structured products and the last one will concern the potential dependency links either among these two random events or with external events.

The first set of questions will be related to the customer behaviour as a trigger event:

1. *What are the underlying drivers which influence the customer behaviour*? This question will focus our attention on how to model a future customer behaviour and hence its impact on final risk exposure. As we will see in next chapter, there are several ways to appreciate and estimate future customer behaviour and each approach has its own specificities and difficulties.
2. *Will a customer have the same behaviour on primary or secondary market*? The two situations are different because on primary market the customer estimates the potential yield generated by the structured products and on secondary market, the customer estimates the stopping time of its investments. Hence there is a significant difference regarding the arbitrage position which may be influenced by several factors which may be specific to a customer or due to a contextual situation.

The second set of questions will focus on the realization event (i.e. the structured product's price):

1. *What are the factors which influence the price of such products*? Many factors may influence a product's price according to its structure. Some are common (such as interest rates, credit quality or remaining time to maturity), others are specific (e.g. an equity linked note will be sensitive to underlying factor level unlike a credit linked note without option).
2. *What kind of model is eligible to a risk framework*? To estimate a structured product requires the use a pricing model. Since (Black, et al., 1973) and (Merton, 1976), pricing models extended a lot regarding the payoff structure, the underlying composition or the way to integrate hazardous market. Moreover the markets development generated a richer and richer environment either in term of data or calculation capacities and influenced at last the pricing methods used by financial intermediates. Lastly the past three decades produced a lot of financial crisis with more or less impacts and helped to extend existing or create new pricing model integrating lessons from the past.
3. *How the market type may influence the price*? There is a fundamental difference between the market types: on primary market, all structured products are bought in forward while they are living in secondary market. This difference will have consequences on relevant sensitivity factors and the pricing method to use.

And the last set of questions will deal with the relationship(s) between these two variables:

1. *How market conditions may influence a customer decision*? The past thirty years, information became more and more available with a significant increase due to the use of internet. Hence customers as economic agents are more and more aware of market conditions nearly instantly. But how it can influence its behaviour and its decisions? Moreover financial cycles are punctuated by crises which generate an increase of anxiogenic information. To illustrate this point, we use Google Trends, a tool which supplies historical search data concerning the presence of keywords in Google queries and news online. Figure 17 represents these data for the "*Lehman Brothers*" keywords and we can notice that historical data are punctuated by peaks closely from trigger events (flagged by letters). Hence we have to question ourselves if customer behaviour remain the same whatever the market conditions. If not, what will be the differences between normal or distressed market conditions? Lastly a structured product will integrate market conditions in its price and so how can its variations contribute to a cascading effect on price?

2. *How the commercial offer composition may generate an interest from customers*? This question highlights several points concerning a commercial offer. The first concern is the products composition and the underlying investment options proposed. Thus, the more adjusted the offer is, the more a customer may be interested. So a dedicated analysis has to be conducted to optimize the commercial offer. The second concern is how to drive the interaction between customers and the commercial taskforce. This is a very important point because it corresponds to the time point which establishes the contractual commitments between Barclays and its customers. It is also the reason why most of regulators focus their attention on how sales are conducted (see (AMF, 2010) and (ACP, 2010)).

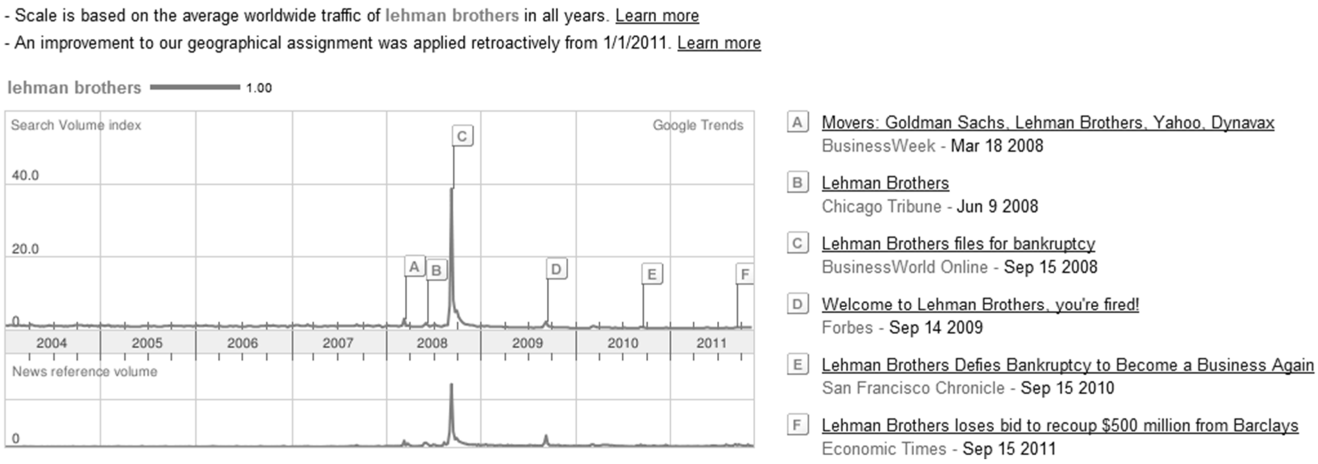


Figure 17: Google queries analysis on keywords "*Lehman brothers*" (left graphic), where query numbers are scaled based on the relative average search traffic (i.e. "1.0"). On Right, Google Trends linked referred web pages with events regarding the query numbers. For instance, "C" referred to the Lehman's bankruptcy day where query numbers were forty higher than the average search traffic.

To get a clearer picture, we try to represent the dynamic structure and its underlying factors in Figure 18. To do this, we started from the simple equation of pipeline risk to define the random variables which will influence it and we add the dynamic factors highlighted with previous set of questions.

Regarding the dynamic factors, we classify them in the following three categories:

- Exogenous factors: all items produced by the environment at a given period and which have an influence on random variables. We define also two sub categories according to the frequency of such factors (structural or punctual).
- Endogenous factors: these are all personal factors specific to a customer situation. As previously we distinguished these factors according to their frequency.
- Commercial factors: these represent the dynamic generated by a commercial taskforce and the product packaging.

Next, we add the dynamic impulses generated by these dynamic factors on random variables to highlight the integration process and the potential correlations between factors, random variables or both. Looking at the whole picture we can distinguish several cases according to their complexities. The simplest case is represented by the interaction between the "*price estimation*" random variable and the exogenous factors which can be collected simply and directly to be used for price integration. The opposite case is represented by the influence of exogenous factors on the "*customer behaviour*" random variable where we can define three levels of integration: all of these are indirect and may influence a customer on several ways with more or less measurable aspects:

- The less measurable is the customer sensitivity to global anxiogenic information provided by Medias. This will essentially depend on two factors: the customer sensitivity to risk (i.e. its risk appetite) and its capacity to be informed. And Figure 17 is a good representation of investors' capacity to be informed in real time with use of internet.
- The intermediate aspect will be related to the influence of exogenous factors on customer's investment capacity represented by endogenous factors. For instance, a crisis may significantly reduce a customer investment capacity either due to unemployment or income's drop through its variable constituent(s). However these impacts will spread more or less quickly and will create a delay between the market event and the realized impacts.

- And the most measurable aspect will be the estimated prices of customer's investments: these are updated on a regular basis (daily or weekly) and integrate exogenous phenomena.

The final touch integrates the commercial capacity of a retail network to interact with its customers and help them to invest on appropriate investment supports. Nonetheless the composition of commercial offer is dependent on data provided willingly by customers regarding their personal information and so their investment capacity and appetite. Hence it is indirectly connected to exogenous factors and their impacts on a customer's situation as represented in Figure 18. Please note that updating the data can suffer a material delay which may have important commercial impact by creating an inaccurate offer.

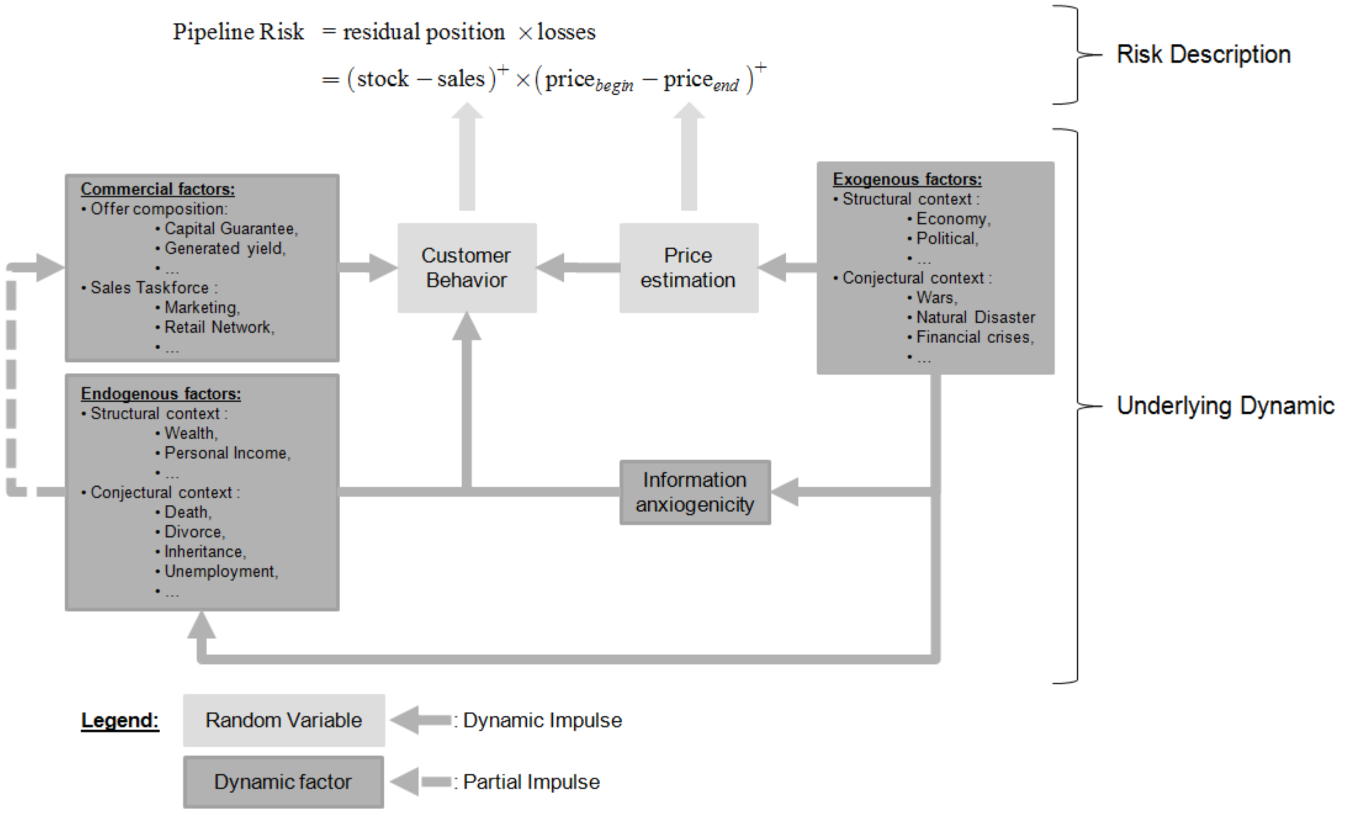


Figure 18: Structural representation of Pipeline Risk's underlying dynamic.

At this stage, we performed a static risk analysis which gives a picture of intrinsic value drivers of pipeline risk. The next step of the process modelling will be to integrate time value so to estimate the potential risk linked to option structure either on structured product or customer. However this may be more or less complex depending on the integrated factors and correlations as we described it in previous static analysis.

To give a flavour of the potential model complexity, we represented several model cases according to the market type, the used random generator and the potential relationship(s) between the two random variables. We summarized the key points of this intuition regarding the items in panel on the next page:

<table>
<tr><th>Market type</th><th>Random item</th><th>Comments</th></tr>
<tr><td rowspan="3">Primary
(Figure 15)</td><td>Market Price</td><td>We figure out the forecasted values regarding two random generators:
1. Either a Gaussian process to give an average picture of price dispersion and hence the potential risk of losses.
2. A Jump process to give a more realistic picture of potential price dispersion. In current figure, we represented only one path with a one rare negative event (see Δt_1).
From a mathematical point of view, both processes described belong to the Lévy Process Family and may be decomposed following the Lévy-Ito theorem. The difference between the two processes is the lack of discontinuous component in first one, which has lots of consequences regarding the notion of market completeness and the pricing process.</td></tr>
<tr><td>Customer
Behavior</td><td>We figure out the influence of endogenous and commercial factors on customer's decision to invest or not, which is related to a financial arbitrage with no implication but to miss an opportunity. This sum of decisions will have a direct consequence on sales level and on the final residual risk exposure.
According to the number of customers, this phenomenon tends to be normal following the central limit theorem and so we can estimate the risk exposure dispersion for a given confidence level. Some abnormalities may appear due to extreme life events like decease or inappropriate commercial offer and may downsize significantly the sales level if it deals with a relevant number of customers.
In current figure, we limited the customer behavior impact by drawing a normal dispersion of sales level.</td></tr>
<tr><td>Correlation(s)</td><td>We described in Figure 18 the potential impulses generated by exogenous factors and how they may influence a customer behavior. We represented in Figure 15 this phenomenon with two cases:
1. The first is the absence of correlation(s) as described in Figure 18 and so the customer acts "normally". Hence we can have an estimation of risk exposure dispersion at maturity and define a worst case according a confidence level.
2. The second case is 100% correlation factor and we established the second case out with help of the rare event represented by Δt_1 delay and how it may influence customers (B, bold gray plain line). Thus the most extreme case will be a sales stop which definitively defines the residual risk exposure.
These two cases highlight several issues: the first one is the presence or not of correlations and the second one is how to integrate a correlation structure in terms of dynamic and weighting.</td></tr>
<tr><td rowspan="3">Secondary
(Figure 16)</td><td>Market Price</td><td>Same as previously</td></tr>
<tr><td>Customer
Behavior</td><td>We represented the customer arbitrage following the principles described for primary market case. However the measured behavior has a significant difference on psychological point of view because it measures the arbitrage on an underlying risk. Hence a customer may be more sensitive and acts in a more drastically manner which may promote abnormalities appearance.</td></tr>
<tr><td>Correlation(s)</td><td>This is represented as previously but it integrates the difference of measured behavior. A customer may act in a faster way with a delay $\Delta t_2 \leq \Delta t_1$</td></tr>
</table>

In this section, we analysed the pipeline risk by defining a set of questions and trying to fill the empty lines with the help of our preliminary investigations. We presented at first the phenomenon according to the market type; next step we described the structure of underlying factors and their potential links; and lastly the propagation of random events through such a structure according to the random type. From an ideal perspective, all items must be integrated to manage fully the pipeline risk and that's what we try to demonstrate in current dissertation.

3.4.2 Defining a Risk Appetite Measure in Complex Environment

All important elements concerning the pipeline risk can be divided into several constraint categories:

- First we presented a quick overview of risk definition and how it has been integrated by main actors of financial markets (financial institutions, institutional investors, regulators ...). This will help to define general context and constraints that we will call "*The Set of General Constraints*".
- Second we set up the industrial context in terms of business strategy, commercial objectives and also in terms of internal risk rules. All these elements will therefore constitute the "*The Set of Local Constraints*".
- And lastly we presented a macro risk analysis to highlight the essential technical elements which must be analysed and integrated to get a complete risk valuation and to define a risk hedging strategy. Thereby these elements will constitute "*The Set of Technical Constraints*".

We have so far presented, analysed and discussed these elements from a theoretical point of view. Now we have to define the implementation strategy to fulfil our main goal: defining a risk measure which is in line with most of reviewed principles and rules presented in previous sections.

In this purpose, we will review them to determine those which are integrated or not according to the original set of constraints:

3.4.2.1 <u>General constraints:</u>

These are the most important because they are followed by every actor. Most of them are dictated by financial regulators and establish the core rules followed by each financial actor. In Barclays' situation, Paris Branch is under three regulatory authorities: FSA (the UK financial regulator), AMF (French financial regulator) and ACP (French Bank and Insurance regulator).

The first one imposes that every bank must follow Basel 2 rules which define a set of risk measures. The most important and most used is the Value-at-Risk measure and hence this will be our first choice to be in adequacy with FSA principles.

The two French regulators define a set of rules to protect customers' best interest. Recently they published public opinions (i.e. new constraints) on complex investment supports. This aims to canalize the inner complexity of such products and to define general rules on sales practices compliant with MIF rules. These rules are more stringent than those enacted by FSA, symbolized by the Acronym KYC (for "*Know Your Clients*") because it defines new additional constraints in term of customer information, audit trail, required guarantees and pricing fairness.

Hence these rules must belong to the core constraints to be integrated. In Barclays' situation, all principles governed by FSA are under Group Risk's perimeter and followed the "*Identify, assess, control, report, manage and challenge*" principle. And customers' protection and sales practices are assessed and controlled by Group Compliance. Thus these essential constraints are already integrated in Barclays' core rules defined in its General Risk Framework.

3.4.2.2 <u>Local Constraints:</u>

From business point of view, Investment Solutions represent one of the cornerstones of Premier Strategy due to its importance in expected income. However the marketed products are very volatile and carry a more important part of risk comparing to classical investment supports.

This issue is apprehended in the General Risk Framework provided by Barclays' Group Risk which set a risk segregation and valuation of each risk component. Each business is in charge to adapt and implement this risk framework and periodic audits practiced by Risk Group insure that the implemented risk framework encompasses and aggregates all generated risks.

Regarding the pipeline risk, we can identify three important risk categories:

- Operation risk: this risk encompasses from operational errors to frauds, the last one being banks' main concern since the infamous rogue trading performed by Jérome Kerviel. In case of structured products, it requires to hire dedicated professionals with the granted power to perform market operations. This implies to define a limited action perimeter in terms of authorized products and trading limits, but also to define a set of controls and deals reviews to assess their conformity. This dictates also a clear segregation of duties between front office (i.e. the market operators and first level controllers), middle office (i.e. the second level of controllers) and back office (i.e. settlements and cash flows managers).

- Credit Risk: this risk includes all issues related to a structured product's default event (i.e. a non-respect of contractual payments). From a structural point of view, a credit event may rise at several levels: the first level deals with the product's issuer (i.e. the counterpart), the second level is related to the issuer's external financial intermediates which provide one or more financial components integrated into the product's structure (e.g. liquidity support, derivatives, insurance, guarantee(s) ...) and the third level corresponds to a legal distortion on contractual documentation.

- Market Risk: this risk includes all events which can influence negatively the value of a financial product when a risk exposure is reported. We explained in §3.4 that risk exposure is reported either when a traded structured product is not entirely sold (primary market) or when a customer repurchase has not been sold back to the product's issuer (secondary market). Hence these two categories of risk exposures are sensitive to the product's price variations and so may generate consequent losses. This is generated by components participating into the product's structure, i.e. the liquidity support and derivatives. The nature of introduced market risks essentially depend on the component nature. For instance, a structured product integrating a bond as liquidity support and a structured swap using an equity underlying will introduce several market risks such as sensitivities to interest rates, credit rating, and equity sensitivities. Regarding derivatives, their pricings will be driven by the fact that their inner strike values are defined (secondary market) or not (primary market).

Both Operation and Credit Risks are well integrated by Barclays Group and defined into its General Risk Framework. Thereby existing policies define mandatory rules, constraints and reports that every business must implement: dedicated IT reporting systems collect evaluate and aggregate all contracted risk; periodical reviews assess and challenge the implementations validity; exceptional business situations are reviewed by governance instances to mitigate them either by temporary dispensations, waivers or unwinds. Frauds are prevented or managed by regular audits provided by Barclays Internal Audit.

Regarding Market Risk, Barclays Group defined two situations: a standard risk framework for usual risks similar to those defined for Operation and Credit Risks, and general policies for more unusual products. This last case gives more freedom in risk definition and measure because they are locally defined by risk teams. However followed principles and implemented models are reviewed and challenged by Group Market Risk in order to be listed and integrated into existing risk framework. This bottom-up process is common into Anglo-Saxon financial institutions to help and support business development. The principle settles on the fact that businesses are more aware of underlying risks generated by their most unusual activities and it allows the businesses to achieve their objectives by letting them defining their risk measures. Hence the risk integration doesn't stop the business development but canalizes it by adapting limits, according to the risk management experiences and the granted risk appetite limit.

The Investment Solutions are led by this process according to the unusual business nature. And the purpose of current dissertation is to describe the construction of an appropriate market risk measure following the technical constraints we will select.

3.4.2.3 Technical constraints

At this stage, we will proceed to a merging operation to produce an accurate risk measure of pipeline risk by selecting appropriate components regarding the available knowledge and resources. We will use a pragmatic approach due to this last (but not least) constraint.

First we will select the VaR methodology presented in §3.2.3 as the backbone of the future risk framework. This method has shown its strengths since its first presentation by JPMorgan, especially in results due to its simple and structured calculation process. However we will adapt the steps of VaR methodology to Pipeline Risk Measurement as follows:

1. Mark the Structured Products positions exposed to market movement,
2. Measure their price sensitivities for a given set of underlying risk factors,
3. Estimate the risk factors dispersion for a given confidence level,
4. And finally evaluate the potential losses.

Now let's analyse each step to estimate the technical requirements and constraints.

3.4.2.3.1 Mark the Structured Products Risk Exposure

Regarding the Pipeline analysis presented in §3.4, a residual exposition is clearly driven by customers' appetite for a given structured product. A preliminary literature review shows that the customer behaviour toward risk has been deeply analysed by two different schools regarding the perception of underlying returns/risks by the investor. They are demonstrating:

1. <u>Either a linear relationship between real and perceived returns</u>: this field was studied at first by John Von Neumann and Oskar Morgenstern in their book "*Theory of Games and Economic Behaviour*" (Von Neumann, et al., 1944) and set up several principles such as the existence of an expected utility function used by investors to drive their investments. These principles set up the fundamentals of modern monetary principles spread by the economical school of Chicago. The most well-known application is the modern portfolio theory set up by Harry Markowitz (Markovitz, 1952) who defined a risk limit regarding the investor risk perception. To do this, Markowitz defined a utility function based on the following constraints: maximize the expected return of the investment portfolio while minimizing its underlying risks.

2. <u>Or A nonlinear relationship between the real return and the investor perception</u>: this point was raised by Maurice Allais (Allais, 1953) when he reviewed the cases presented by Von Neumann. He structured them in a way to show a distortion of investor behaviour which can't respect the core principles of Von Neumann's theory. These cases are known as the "Allais paradoxes" and were exploited further by Daniel Kahneman and Amos Tversky to define the "*Prospect Theory*" (Kahneman, et al., 1979) (Tversky, et al., 1992) and the "*Framing Decision*" (Tversky, et al., 1981). These two principles show a nonlinear relationship between the real underlying risks and their perception by investors. They were applied in finance field to give birth to the "behavioural finance" either to estimate the risk appetite of investors for a given investment or to price more efficiently options' prices regarding the investors' interactions (Zanotti, et al., 2010).

Integrating a customer behaviour model (whatever the selected methodology) can be achieved by following the next steps:

1. Collect customers' data and analyse them to categorize customers and to define their essential drivers (e.g. capital protection, investment capacity, risk profile and so on ...),
2. Define a relevant customer behaviour model by determining or not the presence of a linear perception and adapt it to current structure,
3. Fitting the data to the selected model and checking the forecasting capacity,
4. Integrate the model process into an industrial process to deliver the required performance.

Regarding the Barclays' context, the integration of such model raises several issues:

- <u>Sample size</u>: most of applied examples show that such models require a lot of refined data to get efficient results. But Barclays France has a tiny customer population (near 150k customers) comparing to other banks. Preliminary studies showed the presence of a customer effect on SP activity however results are subject to biases regarding the population size. Hence a sophisticated model may see its added value spoiled by random effects due to a lack of convergence and so may generate model errors with significant financial consequences.

- <u>Industrialization process</u>: Adapt the defined model will inevitably require the definition of a project structure. Indeed it will require the collaboration of several actors (CoE, Marketing, Risk, IT, HR, Compliance and so on) and so we have to define the goals and responsibilities of each actor. Moreover the project advancement must be monitored regularly to determine and solve the potential issues / dead ends / bottlenecks to achieve the final objectives. Such management structure will be highly time consuming and may be subject to important delays.

- Resource allocation: defining a project structure implies to allocate an identified budget in terms of money, equipment or human resources. This will be challenged by Barclays Group by comparing 1) the estimated costs and the potential incomes generated by this project and 2) its priority versus current projects plan. Nowadays preliminary studies proved only the allocation of insufficient budget to achieve complete model integration.

- Confidentiality: customers represent the core business component of Barclays France and the Premier Strategy details can't be publicly communicated without the approvals of Compliance Direction and Executive Committee Members. Thereby only a simplified version of applied model can be communicated without restriction.

Due to these issues, we won't integrate at this stage dynamic customer behaviours into Pipeline Risk Framework and will assume only deterministic risk exposure profiles all along this dissertation. However we will investigate the impact of customer behaviour based on "*what-if*" scenarios in future developments.

3.4.2.3.2 Measure their price sensitivities for a given set of underlying risk factors

The original Black-Scholes-Merton (BSM) pricing model defined a way to estimate the price of a European option but also its sensitivities with respect of underlying factors variations by estimating the appropriate partial derivatives. These sensitivity factors are nowadays used widely by financial investors to get a proxy result of their potential losses and became a de facto cornerstone for VaR valuation.

However the complexity of derivative products increased a lot since these old days and BSM can't encompass such complexity due to its inner limitations. Nowadays the finance industry increased its knowledge of underlying random processes and developed several numerical processes to improve its pricing capacity with more or less successes and limitations.

Specifications of future pricing component must take into account the following constraints:
- To encompass most of payoff structures from the simplest to the most complex,
- To deliver stable price and sensitivity factors,
- To integrate any random generators of the Levy Family to get a better picture of potential risk,
- To be efficient and with a low maintenance cost.

There is a lot of available literature and we will present later in current dissertation the selected approach and how we implemented it regarding standard methodologies.

3.4.2.3.3 Estimate the risk factors dispersion for a given confidence level

We explored previously the importance of sensitivity factors to estimate future potential losses but it is linked also to the way we forecast the future variations of underlying factors.

An overview of current technical knowledge show several relevant points:
- A predominance of Monte Carlo based approach to estimate the dispersion at risk regarding a confidence level,
- The generated paths are generated either with help of
  - Market models (Libor model for interest rates, GMB for stocks, SABR for highly volatile assets ...),
  - Econometrical models (AR, MA, ARMA family, GARCH and so on),
  - Or Markovian models (Regime switching process)
- The model can be univariate or multivariate. In last case, a Variance-Covariance matrix has to be integrated (with or without correlation) with static/dynamic correlation structure.

Hence we will select the most appropriate model according to the modelled factor to get the most robust result. This selection and its application will be presented later in current dissertation (see §4and §5).

3.4.2.3.4 Evaluate the potential losses

This section will be divided in two steps:

1. The way to combine the implied factors to estimate potential losses and to produce the Risk Measure (i.e. VaR).
2. The definition of the methodology, used to calibrate model parameters and to integrate market data.

To do this requires selecting relevant tools:

- To collect necessary market data,
- To produce the required calculations
- To report risk levels and their underlying constituents.

We will present the way we solved these issues in §5.

4. THEORITICAL REVIEW OF OPTION PRICING PRINCIPLES

In this section, we will review most of theoretical pricing principles to develop a pricing strategy in line with the constraints presented in §3.4.2.

In first part, we will present most important pricing theories, methods and paradigm changes since the well-known "*Black, Scholes and Merton*" Model (see (Black, et al., 1973) and (Merton, 1973)). This presentation will focus mainly on equity markets specificities for the sake of clarity and we will describe only essential differences among the various assets. In the second part, we will review the Fourier Transform principles, properties and uses in statistics and probability application fields. The third part will focus on the first application of Fourier Transform in finance field by presenting the essential points of Carr and Madan paper on European Option Pricing by FT (see (Carr, et al., 1999)) with its pros and cons. And at last, the fourth part will be dedicated to the presentation of the Fourier Space Time-stepping methodology and its uses in Option Pricing.

4.1 PRICING MODELS AT GLANCE

We start this section by a short review of Black-Scholes-Merton (BSM) pricing model which established the core principles of most of model pricing. By defining a theoretical portfolio composed of a variable asset, a risk-free asset and a number of option contract to hedge the variable asset, BSM defined the theoretical framework employed in Modern Mathematical Finance.

But it also defined the basic structure of every pricing model by defining:

1. A payoff structure ($\varphi(S,K)=\max(S-K,0)$) where S represents the variable asset , K the strike and $\varphi(S,K)$ the payoff function regarding S and K ,
2. A Random Generator (e.g. the asset follows a Geometric Brownian Motion stochastic process known as GBM),
3. A numerical method to solve the pricing structure (e.g. use of heat equation solutions to get the close formula of the PDE defined by BSM),
4. And an implicit calibration process to fit the model to market data (introducing financial parameters).

We don't investigate further the mathematical demonstration because it is well documented and reviewed in most of mathematical finance courses. However we will replace this model into the historical timeline to highlight the function of this model as baseline of modern pricing models. And we will base our comments on the Figure 19 to give the intuition of most of developments since the BSM model.

The first field of investigation is the extension of BSM Model with the development of more and more complex option structure either by developing mono payoff structure (American, Asian, Digital and more Exotic options), or by using one or several underlying assets to trigger the payoff (introduction of correlation matrix) (see (Hull, 2005) to investigate further). These developments induced the development of more and more complex models based on the BSM scheme. However these developments showed the limits of BSM paradigm and required an investigation in other research areas.

The second field of investigation is related to the random process and its embedded random generator. Indeed the GBM follows a log-normal stochastic process whose properties and developments are well known since Ito's works. However it shows several limitations especially comparing to the nature of replicated assets. The GBM process assumes that the underlying asset follows a continuous process. Nowadays several studies showed that this assumption is false. For instance, the distribution of equity indices is leptokurtic and showed lots of discontinuities (especially in period of distressed markets).

Merton (Merton, 1976) produced the first study on alternative random generator with the development of GBM process with oriented jumps. The second important investigation was done by Heston (Heston, 1993) who defined GBM model with an embedded stochastic volatility. This model aimed to integrate the particular nature of equity assets' volatility. Please note that this work corresponds to the first investigation of pricing model with help of Fourier Transform and so the use of $\mathbb{C}$ space. More recently, modern finance researchers discovered the properties of the Lévy Process Family and since the end of 90's, lots of new stochastic processes were developed and integrated in modern pricing models (see (Kou, 2002), (Kou, et al., 2004), (Barndorff-Nielsen, 1998) and Stochastic processes collected in Panel 5 to get a flavour).

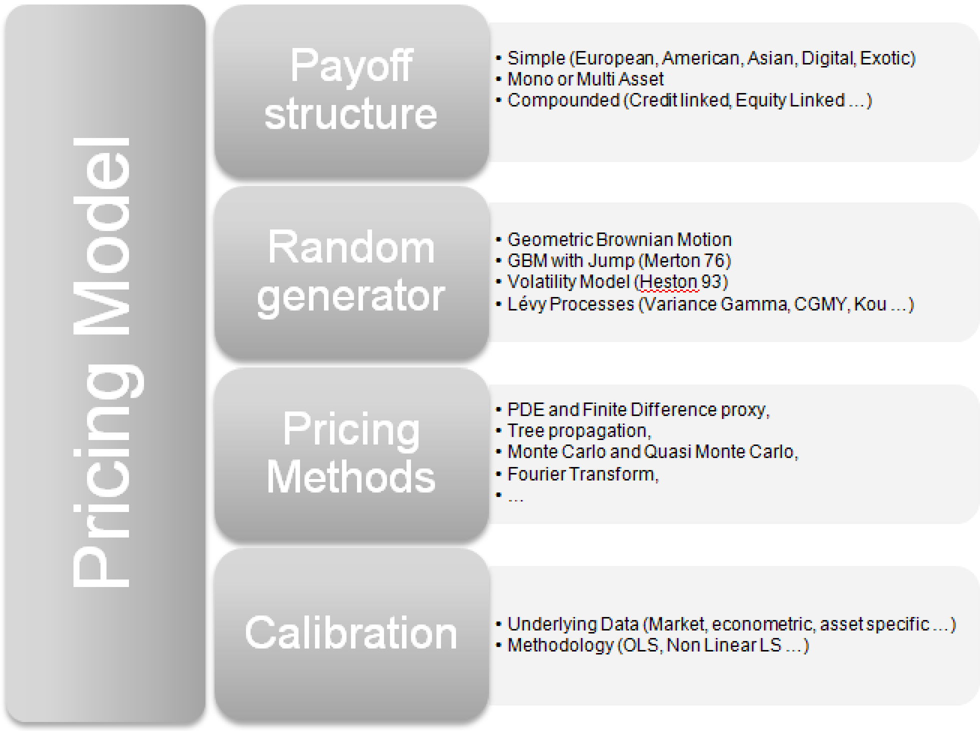


Figure 19: Pricing Model Structure and related methods

The third field of investigation deals with numerical methodologies, i.e. the counterpart of theoretical demonstrations. Indeed most of papers compile lots of equations however their implementations into real algorithms may be very challenging. For instance, the formula of a Matrix $\mathbf{A}$ can be simply written as $\mathbf{A}^{-1}$ however the implementation of an accurate numerical solution will require lots of skills and craftiness. In our examples, the matrix form will impact deeply the choice of the employed algorithm. Thereby we won't use the same algorithm regarding the matrix form (more or less symmetric, flattened or narrowed). Moreover the choice of an algorithm will also depend on its performance in terms of pricing convergence and time consumption. Some solutions require lots of IT resources to get accurate results and so can be very expensive. This can be the source of important dilemma but as market professionals say, "*If you give peanuts, you get monkeys*".

Let's go back to the developments in modern financial model and review numerical solutions developed since the BSM model. The first approach was the use of Partial Differential Equations (PDE) to define a theoretical solution of pricing issue as did Black and Scholes. With smartness, some of these PDE can be solved with closed-form solutions; however it is near impossible to find closed form solutions on complex PDEs. Hence most of papers use the extension of "*Finite Difference Method*", adapted for solving Stochastic Differential Equations (SDE). This is referred in stochastic area as "*Kolmogorov equations*" which defines a solving process either by backward or forward induction. Hence we defined a grid to represent the asset diffusion and by extension the value of payoff.

The development of exotic options so-called "*path dependent*" promoted the development of lattice / grid solutions which mimics the diffusion process of the underlying asset with help of a tree representation (see (Cox, et al., 1979) and (Hull, 2005) for further investigations). But the most used technique remains the Monte Carlo Method (MCM) due to its simplicity of understanding and its statistical properties in terms of convergence (see (Glasserman, 2004)). However this technique presents some lacks when used to estimate the sensitivity factors of complex option structure and lots of studies aimed to improve the MCM efficiency (see (Fries, 2005), (Giles, et al., 2006) and (Giles, 2007)).

Following the work produced by Heston, new developments investigated the use of Fourier Transform and the most representative is the study produce by Carr and Madan (Carr, et al., 1999) who demonstrated how to price a European Option with help of Fourier Transform. Since this paper, new approaches were developed to extend this study as we will see with the developments published by (Surkov, 2009).

Process	Formula
Geometric Brownian Motion (GBM)	$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$
GBM with Jump **(Merton, 1976)**	$\frac{dS_t}{S_t} = \mu dt + \sigma dW(t) + dJ(t)$
GBM with Stochastic Volatility **(Heston, 1993)**	$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{\upsilon(t)}dW_S(t)$ $d\upsilon(t) = \kappa(\theta - \upsilon(t))dt + \sigma\sqrt{\upsilon(t)}dW_\upsilon(t)$ Where $dW_S(t)dW_\upsilon(t) = \rho dt$
Exponential Levy model	$S(t) = S(0)e^{X(t)}$
Variance Gamma **(Madan, et al., 1990)** **(Madan, et al., 1998)**	$\frac{dS(t)}{S(t)} = \mu g_{dt} + \sigma dW(g_{dt}) = \mu g + \sigma\sqrt{g_{dt}}dZ$ Where $g_{dt} \to \Gamma(\lambda dt, \nu dt)$ or $g_{dt} = \gamma(t+dt;\tilde{\mu},\tilde{\sigma}) - \gamma(t;\tilde{\mu},\tilde{\sigma})$ We may define the g_{dt} PDF such as: $f_{dt}(g) = \left(\frac{\tilde{\mu}}{\tilde{\sigma}}\right)^{\frac{\tilde{\mu}^2 dt}{\tilde{\sigma}}} \frac{g^{\frac{\tilde{\mu}^2 dt}{\tilde{\sigma}}} e^{-\frac{\tilde{\mu}}{\tilde{\sigma}}g}}{\Gamma\left(\frac{\tilde{\mu}^2 dt}{\tilde{\sigma}}\right)}$

Panel 5: Main Stochastic Processes

And the last field of study deals with the calibration of pricing models with market data. This is the meeting point of financial models, numerical solution and market realities. Indeed a good pricing model is widely adopted by practitioners because it combines simplicity of use, speediness and efficient price estimation.

These are the most important reasons of BSM model adoption by market practitioners. The practical use of BSM by traders developed the famous "*implied volatility*", i.e. the volatility level corresponding to the BSM inversion according to fixed market conditions. However its overuse showed also its lacks especially during the 1987 equity crash which illustrated that equity indices were not continuous processes.

This major event led to investigate further existing models to improve their accuracies and their fits to market data. This contributed to the development of a new risk life cycle as illustrated in Panel 3. This issue was integrated as major constraints of famous pricing models such as the BGM Interest Rate Model (Brace, et al., 1997) whose declinations gave the famous LIBOR Interest Rate model, or the Hull-White Interest Rate model (one and two factors) (Hull, et al., 1990) (Hull, et al., 1996).

Regarding the specifications developed in §3.4.2.3, we will now investigate further the use of Fourier Transform in Option Pricing and how it can help us to deliver accurate evaluations for Structured Products.

4.2 FOURIER TRANSFORM (FT) REVIEW

This part aims to present most important FT properties to help readers without prior knowledge on this mathematical tool. Thus we will review the FT definition, properties and constraints. We will also review its extension in statistical and probability fields with the presentation of characteristic function.

Please see (Matsuda, 2004) for a more detailed presentation.

4.2.1 Definition

The Fourier Transform is a mathematical operation that "*transforms*" one complex-valued function of a real variable into another. Most significant uses are found in physic study, wave motion or optics field due to its capacity as integrand operator. FT may be continuous or discrete: both operations have the same properties but the second is more useful for its ease and integration in algorithms. However it has a sampling issue raised from the discretization process (see §4.2.4).

In following panel we present several Continuous FT (CFT) formulas with the first formula as the general FT formula and others as specialized formulas depending on values of parameters (a,b). Please note that a and b are selected on a conventional basis regarding the field of application.

Application Field	Fourier Transform and Inverse Fourier Transform
General formula	$\mathcal{G}(\omega) \equiv \mathcal{F}\left[g(t)\right](\omega) \equiv \sqrt{\frac{\lvert b \rvert}{(2\pi)^{1-a}}} \int_{-\infty}^{+\infty} e^{ib\omega t} g(t)\,dt$ $g(t) \equiv \mathcal{F}^{-1}\left[\mathcal{G}(\omega)\right](t) \equiv \sqrt{\frac{\lvert b \rvert}{(2\pi)^{1+a}}} \int_{-\infty}^{+\infty} e^{-ib\omega t} \mathcal{G}(\omega)\,d\omega$
Characteristic function formula $(a,b)=(1,1)$	$\mathcal{G}(\omega) \equiv \mathcal{F}\left[g(t)\right](\omega) \equiv \int_{-\infty}^{+\infty} e^{i\omega t} g(t)\,dt$ $g(t) \equiv \mathcal{F}^{-1}\left[\mathcal{G}(\omega)\right](t) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \mathcal{G}(\omega)\,d\omega$
Mathematical formula $(a,b)=(1,-1)$	$\mathcal{G}(\omega) \equiv \mathcal{F}\left[g(t)\right](\omega) \equiv \int_{-\infty}^{+\infty} e^{-i\omega t} g(t)\,dt$ $g(t) \equiv \mathcal{F}^{-1}\left[\mathcal{G}(\omega)\right](t) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \mathcal{G}(\omega)\,d\omega$
Physical formula $(a,b)=(0,1)$	$\mathcal{G}(\omega) \equiv \mathcal{F}\left[g(t)\right](\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} g(t)\,dt$ $g(t) \equiv \mathcal{F}^{-1}\left[\mathcal{G}(\omega)\right](t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} \mathcal{G}(\omega)\,d\omega$
Signal processing formula $(a,b)=(0,-2\pi)$ and $\omega = 2\pi f$	$\mathcal{G}(f) \equiv \mathcal{F}\left[g(t)\right](f) \equiv \int_{-\infty}^{+\infty} e^{-2\pi i f t} g(t)\,dt$ $g(t) \equiv \mathcal{F}^{-1}\left[\mathcal{G}(f)\right](t) \equiv \int_{-\infty}^{+\infty} e^{i\pi i f t} \mathcal{G}(\omega)\,d\omega$

Panel 6: Main FT and Inverse FT Definitions regarding the Application Field

4.2.2 Condition of Existence

The use of Fourier Transform requires assessing the integrability capacity to prove the existence of the FT pair $(g(t), \mathcal{G}(\omega))$. A sufficient condition (but not necessary) is to prove that $g(t)$ is a convergent function such as $\int_{-\infty}^{+\infty} |g(t)| dt < \infty$. This condition is very important and will raise an issue on FFT uses in option pricing (Carr, et al., 1999).

4.2.3 Properties

Fourier Transform defines a set of useful properties which are listed in panel below. In current dissertation, we will use essentially the "*differentiation*" property in sensitivity factors pricing.

Properties	Time Domain Function $y(t)$	Fourier Transform $F[y(t)](\omega)$
Linearity	$af(t)+bg(t)$	$a\mathcal{F}(\omega)+b\mathcal{G}(\omega)$
Even Function	$f(t)$ is even	$\mathcal{F}(\omega)\in\mathbb{R}$
Odd Function	$f(t)$ is odd	$\mathcal{F}(\omega)\in\mathbb{I}$
Symmetry	$\mathcal{F}(t)$	$2\pi f(-\omega)$
Differentiation	$\frac{df(t)}{dt}$	$-i\omega\mathcal{F}(\omega)$
Time Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}\mathcal{F}\left(\frac{\omega}{a}\right)$
Time Shifting	$f(t-t_0)$	$e^{i\omega t_0}\mathcal{F}(\omega)$
Convolution	$f*g\equiv\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$	$\mathcal{F}(\omega)\mathcal{G}(\omega)$
Multiplication	$f(t)g(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(\varpi)\mathcal{G}(\omega-\varpi)d\varpi$
Modulation (Frequency Shifting)	$e^{-i\omega_0} f(t)$	$\mathcal{F}(\omega-\omega_0)$

Panel 7: Fourier Transform's main properties

4.2.4 Discrete Fourier Transform and Discretization issue

Discrete FT (DFT) is a special case of continuous FT and hence both share the same properties. The purpose of DFT is to approximate FT as close as possible by sampling a finite number of points N of a continuous time domain function $g(t)$ with time domain sampling interval Δt. Similarly in FT space, we aim to approximate a continuous FT $\mathcal{G}(\omega)$ with angular frequency sampling interval $\Delta\omega$.

We define:

$$\Delta t = \frac{T}{N} \text{ and } t_n = n.\Delta t$$

$$\Delta\omega = \frac{2\pi}{N\Delta t} \text{ and } \omega_k = k.\Delta\omega = k\frac{2\pi}{N\Delta t}$$

Hence

$$\begin{aligned}
\mathcal{G}\left(\omega_k = k\frac{2\pi}{N\Delta t}\right) &= \sum_{n=0}^{N-1} g(t_n = n\Delta t)e^{i\omega_k t_n} \\
&= \sum_{n=0}^{N-1} g(n\Delta t)e^{ikn\frac{2\pi}{N}n} \\
&\simeq \mathcal{G}(\omega) \\
g(t_n = n\Delta t) &= \frac{1}{N}\sum_{n=0}^{N-1} \mathcal{G}\left(\omega_k = k\frac{2\pi}{N\Delta t}\right)e^{-i\omega_k t_n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \mathcal{G}\left(k\frac{2\pi}{N\Delta t}\right)e^{-ikn\frac{2\pi}{N}n} \\
&\simeq g(t)
\end{aligned}$$

Thus we replace the integration step for FT and inverse FT by a sum of N points. This proxy is very useful in applied sciences fields and permits the development of the well-known “*Fast Fourier Transform*” (FFT).

However the time and periodic samplings raise a question on the most relevant size for both samplings. If a domain is sampled at an insufficient high rate, it implies a poor approximation of FT function and generates information losses

Below you’ll find an illustration of the issue in signal field called “*aliasing*” during a signal reconstruction:

1. An original signal is available for studies (see “*High Frequency Signal*”) and we have to do treatments with help of Fourier Transform,
2. Therefore a sample is extracted from the original signal to use FFT but it contains not enough information,
3. After intermediate treatments, we reconstructed the signal with help of Inverse Fourier Transform. However this operation will produce a lower frequency signal due to the irrelevant signal.

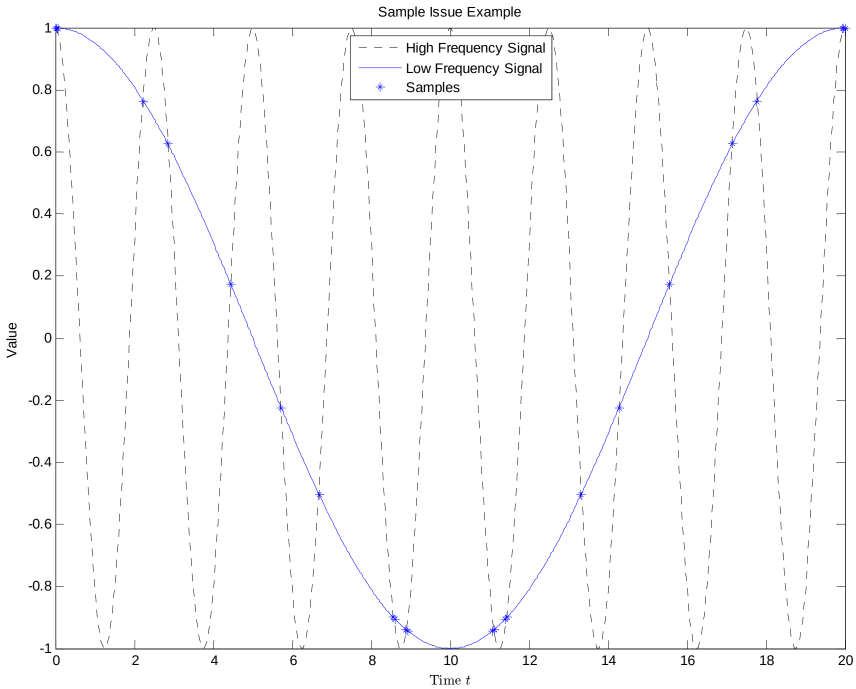


Figure 20: Sampling issue illustration where the reconstruction of a high frequency signal from a low size sample gives a low frequency signal.

This issue has been solved by Nyquist-Shannon sampling theorem which states that only half of N points are reliable. So $k=\frac{N}{2}$ and the implication on ω sampling is $\Delta\omega=\frac{\omega_{\max}}{k}=\frac{2\omega_{\max}}{N}$ where $\omega_{\max}$ is the Nyquist critical frequency. $\omega_{\max}$ is defined as $\omega_{\max}=\frac{1}{2\Delta t}$. This point will be key implementation point of option pricing with FFT as we will see in §5.1.

4.2.5 Characteristic Function

Right now we will review the use of FT and Inverse FT in probability field under the "*characteristic function*" theory which provides an alternative way to analyse random variables other than by their Probability Density Functions (PDF) or Cumulative Density Functions (CDF). The cautious readers can find a more rigorous presentation in (Grimmet, et al., 2001).

4.2.5.1 Definition

Let X be a random variable with its probability density function $\mathbb{P}(x)$. A characteristic function $\phi_X(\omega)$ with $\omega\in\mathbb{R}$ is defined as the Fourier Transform of the PDF $\mathbb{P}(x)$ using parameters $(a,b)=(1,1)$ applied to general formula. Hence:

$$\phi_X(\omega)\equiv\mathcal{F}[\mathbb{P}(x)]\equiv\int_{-\infty}^{+\infty}e^{i\omega x}\mathbb{P}(x)\,dx=\mathbb{E}\left[e^{i\omega x}\right]$$

A characteristic function $\phi(\omega)$ may be expressed by using Euler's as:

$$\phi_X(\omega) = \mathbb{E}\left[e^{i\omega x}\right]$$
$$= \mathbb{E}[\cos(\omega x)] + i\mathbb{E}[\sin(\omega x)]$$

Also we may rewrite a characteristic function by using Taylor series expansion about a point $x = 0$ as:

$$\phi_X(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x}\mathbb{P}(x)\,dx$$
$$= \int_{-\infty}^{+\infty} \left(1 + i\omega x + \frac{1}{2!}(i\omega x)^2 + \frac{1}{3!}(i\omega x)^3 + \frac{1}{4!}(i\omega x)^4 + \ldots\right)\mathbb{P}(x)\,dx$$
$$= \int_{-\infty}^{+\infty} \mathbb{P}(x)\,dx + i\omega\int_{-\infty}^{+\infty} x\mathbb{P}(x)\,dx + \frac{1}{2!}(i\omega x)^2\int_{-\infty}^{+\infty} x^2\mathbb{P}(x)\,dx + \frac{1}{3!}(i\omega x)^3\int_{-\infty}^{+\infty} x^3\mathbb{P}(x)\,dx + \ldots$$
$$= r_0 + i\omega r_1 + \frac{1}{2!}(i\omega x)^2 r_2 + \frac{1}{3!}(i\omega x)^3 r_3 + \ldots$$
$$= \sum_{n=0}^{\infty}\frac{(i\omega)^n}{n!}r_n$$

Where r_n is the n-th moment around 0 (called raw moment).

4.2.5.2 Properties

Most important properties of a characteristic function $\phi_X(\omega)$ are:

1. $\phi_X(\omega) \leq 1$
2. $\phi_X(0) = 1$
3. $\phi_X(\omega)$ is uniformly continuous on $\mathbb{R}$
4. $\phi_{X_1+X_2+\ldots+X_n}(\omega) = \prod_{k=1}^{\infty}\phi_{X_k}(\omega)$ where $\{X_1, X_2, \ldots, X_n\}$ are independent random variables.
5. A random variable X has a symmetric PDF $\mathbb{P}(x)$ if and only if $\phi_X(\omega) \in \mathbb{R}$ for $\omega \in \mathbb{R}$

4.2.5.3 Calculation of statistical moments

Characteristic functions are very useful to estimate statistical moment with help of characteristic exponent $\Psi(\omega)$ defined as $\Psi(\omega) = \log(\phi(\omega))$ and we define the n-th-cumulant such as:

$$C_n = \frac{1}{i^n}\frac{\partial^n\Psi_X(\omega)}{\partial\omega^n}\bigg|_{\omega=0}$$

This approach may be compared with the Moment Generating Function approach defined as:

$$M(\omega) = \int_{-\infty}^{+\infty} e^{\omega x}\mathbb{P}(x)dx$$
.

And n-th raw moment

$$r_n = \frac{\partial^n M(\omega)}{\partial \omega^n}\bigg|_{\omega=0}$$

Panel below summarizes the common statistical moment's calculations with help either of characteristic function or moment generating function.

Calculation	Moment	nth cumulant	nth raw moment
Mean	$\mathbb{E}[X]$	C_1	r_1
Variance	$\mathbb{E}\left[\left(X-\mathbb{E}[X]^2\right)\right]$	C_2	$r_2 - r_1^2$
Skew	$\frac{\mathbb{E}\left[\left(X-\mathbb{E}[X]^3\right)\right]}{\sqrt{\left(\mathbb{E}\left[\left(X-\mathbb{E}[X]^2\right)\right]\right)^3}}$	$\frac{C_3}{(C_2)^{3/2}}$	$\frac{2r_1^3+3r_1r_2+r_3}{\left(r_2-r_1^2\right)^{3/2}}$
Excess Kurtosis	$\frac{\mathbb{E}\left[\left(X-\mathbb{E}[X]^4\right)\right]}{\sqrt{\left(\mathbb{E}\left[\left(X-\mathbb{E}[X]^2\right)\right]\right)^4}}-3$	$\frac{C_4}{(C_2)^2}$	$\frac{-6r_1^4+12r_1^2r_2-3r_2^2-4}{\left(r_2-r_1^2\right)^2}$

Panel 8: Statistical Moments calculation comparison between Moment Generating Function and Characteristic Function.

4.2.6 Fourier Transform and Stochastic Processes

In this part we will review firstly some important definitions concerning stochastic processes, secondly we will investigate the notion of distribution divisibility with help of Lévy-Itô Decomposition theorem, thirdly we will formulate the Lévy-Kintchine representation and the expression of most important Levy processes.

Most of theorems and propositions are presented without proofs and a curious reader may found a rigorous treatment of these points in (Cont, et al., 2004).

4.2.6.1 <u>Definitions</u>

A right-continuous with left limits ("*cadlag*" in French) or adapted (non-anticipating) stochastic process $\{X_t; 0 \le t < \infty\}$ on space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ is said to be a Lévy process if it satisfies the following conditions:

1. Its increments are independent of past values. Thus $X_{t+1} - X_t$ is said to be independent of the filtration $\mathcal{F}_t$ if $P(X_{t+1} - X_t | \mathcal{F}_t) = P(X_{t+1} - X_t)$.
2. Its increments are stationary, i.e. distribution of increments is independent of t. Thus $X_{t+h} - X_t$ follows the same probability distribution of X_h.
3. $X_0 = 0$.
4. X_t is continuous in probability, i.e. $\forall \varepsilon > 0, \lim_{h \to 0} P(|X_{t+h} - X_t| \ge \varepsilon) = 0$.

Hence:

- A process which fulfils the first three conditions is said to be a process with stationary independent increments.
- If a process fulfils also the fourth condition, then the probability of jumps at time t is zero however they may occur in the future. Hence the fourth condition doesn't imply a continuous sample path.

A stochastic process $\{X_t; 0 \leq t < \infty\}$ is said to be "cadlag" if it satisfies the following conditions:

1. Left limit of the process $X_{t-} = \lim_{s \to t, s<t} X_s$ exists,
2. Right limit of the process $X_{t+} = \lim_{s \to t, s>t} X_s$ exists,
3. And $X_t = X_{t+}$.

We will finish the review of basic definitions with the notion of distribution divisibility:

- A random variable is said divisible if it can be represented as the sum of two independent random variables with identical distributions. Example: $Y = Y_1 + Y_2$.
- A random variable is said infinitely divisible if it can be represented as the sum of n independent random variables with identical distributions and $n > 2$. Example: $Y = Y_1 + Y_2 + \ldots + Y_n$

Regarding the characteristic function of an infinitely divisible process Y, we may highlight the relationships between $\phi(\omega)$ and $\phi_n(\omega)$:

- $\phi(\omega) = (\phi_n(\omega))^n$
- $\phi_n(\omega) = (\phi(\omega))^{\frac{1}{n}}$

Now we will formulate the following lemma which is very important: indeed it means infinitely distributions can generate Lévy Process and conversely.

$$\text{Lévy Process } \{X_t; t \geq 0\} \Leftrightarrow P(X_{t+h} - X_t) \text{ is an infinitely divisible distribution}$$

But what does this notion of distribution divisibility mean and what is its implication in stochastic process inner capacities?

4.2.6.2 Levy-Itô decomposition theorem

The use of Lévy-Itô decomposition theorem allows a further investigation by stating that a Lévy process may be represented as the sum of two independent processes:

- On one hand you have a continuous process constituted by a Brownian motion with drift,
- On the other hand, you have a discontinuous jump process, i.e. a sum of centred independent jumps.

Let $\{X_t; t \geq 0\}$ be a Lévy process on $\mathbb{R}$ with a Lévy measure ν which measures the number of jumps per unit of time whose sizes belong to any positive measurable set A.

The Lévy-Itô Decomposition theorem states in formal way that:

1. A Lévy measure must satisfies two conditions:
   $\int_{-\infty}^{+\infty} \mathbb{1}_{|x| \geq 1} \cdot \nu(dx) < \infty$ and $\int_{\mathbb{R}-\{0\}} x^2 \cdot \mathbb{1}_{|x|<1} \cdot \nu(dx) < \infty$
   a. The first condition implies that a Lévy process must have an almost-surely finite number of large jumps (i.e. jumps with absolute values greater or equal to 1) by time unit,
   b. The second one implies that a Lévy measure must be square-integrable around the origin.
2. There exists a drift μ and a Brownian motion with diffusion coefficient σ

Hence the Lévy process $\{X_t; t \geq 0\}$ is characterized by its characteristic triplet (μ, σ, ν) and can be defined such as:

$$\begin{aligned} X_t &= \mu t + \sigma_t W_t + X_t^L + \lim_{s \to 0} X_t^s \\ &= \mu t + \sigma_t W_t + X_t^L + \tilde{X}_t^s \end{aligned}$$

Where

- μ : Drift
- σ_t: Diffusion coefficient
- W_t: Wiener process where $W_t \sim> N(0,t)$
- X_t^L : Sum of finite number of large jumps during the interval $[0,t]$,
- $\tilde{X}_t^s$: Sum of number of small jumps during the interval $[0,t]$. Please note the number of small jumps may be infinite in the limit $s \to 0$.

Hence $\mu t + \sigma_t W_t$ represents the continuous part and $X_t^L + \lim_{s \to 0} \tilde{X}_t^s$ the jump part which are represented by compensated (i.e. centred around 0) Poisson processes according to their definition range.

With help of this theorem, it is possible to characterize all Lévy processes by looking at their characteristic functions (i.e. the Fourier Transform of the Lévy processes) and this will supply two important properties.

The first important property concerns the Time Divisibility of Lévy processes. Let define $\{X_t; t \geq 0\}$ an infinitely divisible Lévy process with a triplet (μ, σ, ν) and a measure satisfying $\nu(\{0\}) = 0$. Thus the time divisibility can be defined such as:

$$\phi_{X_t}(\omega) = \mathbb{E}\left[e^{i\omega X_t}\right] = \mathbb{E}\left[e^{i\omega t X_1}\right] = \left(\phi_{X_1}(\omega)\right)^t = \phi_{X_1}(t\omega).$$

Sketch of proof:

$$\begin{aligned} \phi_{X_{t+s}}(\omega) &= \mathbb{E}\left[e^{i\omega X_{t+s}}\right] \\ &= \mathbb{E}\left[e^{i\omega(X_{t+s}-X_s)} e^{i\omega X_s}\right] \\ &= \mathbb{E}\left[e^{i\omega(X_{t+s}-X_s)}\right]\mathbb{E}\left[e^{i\omega X_s}\right] \\ &= \phi_{X_t}(\omega)\phi_{X_s}(\omega) \end{aligned}$$

The second property is that the general characterization of random variables distribution underlyings the Lévy jump-diffusion defined previously by the Lévy-Kintchine formula. Let define $\{X_t; t \geq 0\}$ an infinitely divisible Lévy process with a triplet (μ, σ, ν) and a measure satisfying $\nu(\{0\}) = 0$. Thus a law P of a X_t may be defined by the Lévy Kintchine formula such as:

$$\phi_{X_t}(\omega) = \mathbb{E}\left[e^{i\omega X_t}\right] = e^{t\left(i\omega\mu - \frac{\omega^2\sigma}{2} + \int_{\mathbb{R}}\left(e^{i\omega x} - 1 - i\omega x \mathbb{1}_{\{|x|<1\}}\right)\nu(dx)\right)} = e^{t\Psi(\omega)}$$

Where $\Psi(\omega)$ defines the characteristic exponent of Lévy process X_t.

Sketch of Proof:

$$\begin{aligned}\phi_{X_t}(\omega) &= \mathbb{E}\left[e^{i\omega X_t}\right] \\ &= \mathbb{E}\left[e^{i\omega t X_1}\right] \\ &= \mathbb{E}\left[e^{i\omega t\left(\mu+\sigma W_1+X_1^L+\tilde{X}_1^s\right)}\right] \\ &= \mathbb{E}\left[e^{i\omega t(\mu)}\right]\cdot\mathbb{E}\left[e^{i\omega t(\sigma W_1)}\right]\cdot\mathbb{E}\left[e^{i\omega t\left(X_1^L\right)}\right]\cdot\mathbb{E}\left[e^{i\omega t\left(\tilde{X}_1^s\right)}\right] \\ &= \left(\phi_\mu(\omega)\cdot\phi_{\sigma W}(\omega)\cdot\phi_{X_1^L}(\omega)\cdot\phi_{\tilde{X}_1^s}(\omega)\right)^t\end{aligned}$$

With

$$\phi_\mu(\omega) = \mathbb{E}\left[e^{i\omega\mu}\right] = e^{i\omega\mu t}$$

$$\phi_{\sigma W_t}(\omega) = \mathbb{E}\left[e^{i\omega W_t}\right] = e^{-\frac{\omega^2\sigma^2 t}{2}}$$

$$\phi_{X_1^L}(\omega) = \mathbb{E}\left[e^{i\omega X_1^L}\right] = e^{\int_{\{|x|\geq 1\}}\left(e^{i\omega x}-1\right)\nu(dx)}$$

$$\phi_{\tilde{X}_1^s}(\omega) = \mathbb{E}\left[e^{i\omega\tilde{X}_1^s}\right] = e^{\int_{\{|x|<1\}}\left(e^{i\omega x}-1-i\omega x\right)\nu(dx)}$$

Thus

$$\begin{aligned}\phi_{X_t}(\omega) &= \left(e^{i\omega\mu t}\cdot e^{-\frac{\omega^2\sigma^2 t}{2}}\cdot e^{\int_{\{|x|\geq 1\}}\left(e^{i\omega x}-1\right)\nu(dx)}\cdot e^{\int_{\{|x|<1\}}\left(e^{i\omega x}-1-i\omega x\right)\nu(dx)}\right)^t \\ &= \left(e^{i\omega\mu t-\frac{\omega^2\sigma^2 t}{2}+\int_{\{|x|\geq 1\}}\left(e^{i\omega x}-1\right)\nu(dx)+\int_{\{|x|<1\}}\left(e^{i\omega x}-1-i\omega x\right)\nu(dx)}\right)^t \\ &= \left(e^{i\omega\mu t-\frac{\omega^2\sigma^2 t}{2}+\int_{\mathbb{R}}\left(e^{i\omega x}-1-i\omega x\mathbb{1}_{\{|x|<1\}}\right)\nu(dx)}\right)^t \\ &= e^{t\left(i\omega\mu t-\frac{\omega^2\sigma^2 t}{2}+\int_{\mathbb{R}}\left(e^{i\omega x}-1-i\omega x\mathbb{1}_{\{|x|<1\}}\right)\nu(dx)\right)}\end{aligned}$$

We collect in Panel 9 examples of Lévy densities and characteristic exponents for most used Lévy processes (single mode).

Process	Lévy Density $\nu(d\nu)$	Characteristic Exponent $\Psi(\omega)$
Geometric Brownian Motion	0	$i\gamma\omega-\frac{\sigma^2\omega^2}{2}$ where $\gamma=r-q-\frac{\sigma^2}{2}$
Merton Jump-Diffusion **(Merton, 1976)**	$\frac{\lambda}{\sqrt{2\pi\tilde{\sigma}^2}}e^{-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2}}$	$i\gamma\omega-\frac{\sigma^2\omega^2}{2}+\lambda\left(e^{i\tilde{\mu}\omega-\frac{\tilde{\sigma}^2\omega^2}{2}}-1\right)$ where $\gamma=r-q-\frac{\sigma^2}{2}-\lambda\left(e^{\tilde{\mu}-\frac{\tilde{\sigma}^2}{2}}-1\right)$
Kou Jump-Diffusion **(Kou, 2002)**	$\lambda\left(\frac{\eta_p}{\eta_+}e^{-\frac{y}{\eta_+}}\mathbb{1}_{\{y>0\}}+\frac{1-\eta_p}{\eta_-}e^{-\frac{\lvert y\rvert}{\eta_-}}\mathbb{1}_{\{y<0\}}\right)$	$i\gamma\omega-\frac{\sigma^2\omega^2}{2}+\lambda\left(\frac{\eta_p}{1-i\omega\eta_+}+\frac{1-\eta_p}{1+i\omega\eta_-}-1\right)$
Variance Gamma **(Madan, et al., 1990)** **(Madan, et al., 1998)**	$\frac{1}{\mu\lvert y\rvert}e^{C_1y-C_2\lvert y\rvert}$	$-\frac{1}{\mu}\ln\left(1-i\gamma\mu\omega+\frac{\sigma^2\mu\omega^2}{2}\right)$ where $\gamma=r-q-\frac{\sigma^2}{2}$
Normal Inverse Gaussian **(Barndorff-Nielsen, 1998)**	$\frac{C_3}{\lvert y\rvert}e^{C_1y}K_1\left(C_4\lvert y\rvert\right)$	$-\frac{1}{\mu}\left(\sqrt{1-2\imath\gamma\mu\omega+\sigma^2\mu\omega^2}-1\right)$
Carr-Geman-Madan-Yor **(Carr, et al., 2002)**	$\frac{C}{\lvert y\rvert^{1+Y}}\left(e^{-G\lvert y\rvert}\mathbb{1}_{\{y<0\}}+e^{-My}\mathbb{1}_{\{y>0\}}\right)$	$C\Gamma(-Y)\left[(M-i\omega)^Y-M^Y+(G+i\omega)^Y-G^Y\right]$

Panel 9: Levy densities and Characteristic Exponents of main stochastic processes.

Where r is the risk-free rate, q the dividend yield, γ the risk-neutral drift and σ is the volatility of the driving Brownian Motion, $C_1=\frac{\gamma}{\sigma^2}$, $C_2=\frac{\sqrt{\gamma+2\sigma^2/\mu}}{\sigma^2}$, $C_3=\frac{\sqrt{\gamma+2\sigma^2/\mu}}{\pi\sigma^2\sqrt{\mu}}$, $C_4=\frac{\sqrt{\gamma+\sigma^2/\mu}}{\sigma^2}$ and $K_p(x)$ is the modified Bessel function of the second kind.

4.3 OPTION PRICING WITH FOURIER TRANSFORM

Now we will present briefly the method used by (Carr, et al., 1999) which defines a stable way of inverting the characteristic function to get the price of a European call option.

4.3.1 Payoff function definition

The first step will focus on the definition of an accurate payoff function compliant with Fourier Transform. At first glance, the payoff function of a European call can be expressed as $C_T(K) = (S_t - K)^+$ or $C_T(k) = \left(e^x - e^k\right)^+$ in the exponential form.

However $C_T(k)$ tends to S_0 as $k \to -\infty$ and hence the payoff function is not square integrable, which is a mandatory condition to use Fourier Transform as seen in §4.2.2. This issue may be escaped by using a damping factor to modify the call price function and to obtain a square integrable function. Thus a new payoff function $c_T(k)$ is defined by $c_T(k) = e^{\alpha k} C_T(k)$ where $\alpha > 0$.

4.3.2 Fourier Transform of a European Call function

From the $c_T(k)$ definition, we can express the Fourier Transform and its Inverse as follows:

$$\begin{aligned}\mathcal{F}[c_T(k)](\omega) &= \phi_{c_T}(\omega) \\ &= \int_{-\infty}^{+\infty} e^{i\omega k} c_T(k)\,dk\end{aligned}$$

$$\begin{aligned}c_T(k) &= \mathcal{F}^{-1}\left[\phi_{c_T}(\omega)\right](k) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega k}\phi_{c_T}(\omega)\,d\omega \\ &= \frac{1}{\pi}\int_{0}^{+\infty} e^{-i\omega k}\phi_{c_T}(\omega)\,d\omega\end{aligned}$$

And by consequence $C_T(k)$ can be written such as:

$$\begin{aligned}C_T(k) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega k}\phi_{c_T}(\omega)\,dk \\ &= \frac{e^{-\alpha k}}{\pi}\int_{0}^{+\infty} e^{-i\omega k}\phi_{c_T}(\omega)\,dk\end{aligned}$$

The next step followed by Carr and Madan is to find the closed form expression of $\phi_{c_T}(\omega)$ depending on $\phi_T(\omega)$, the characteristic function of underlying diffusion process, following the solving process below:

$$\begin{aligned}\phi_{c_T}(\omega) &= \int_{-\infty}^{+\infty} e^{i\omega k} \int_{k}^{+\infty} e^{\alpha k} e^{-rT}\left(e^{x}-e^{k}\right) P(x)\cdot dx\cdot dk \\ &= \int_{-\infty}^{+\infty} e^{-rT} P(x) \int_{-\infty}^{x} e^{i\omega k}\left(e^{x+\alpha k}-e^{(1+\alpha)k}\right)\cdot dk\cdot dx \\ &= \int_{-\infty}^{+\infty} e^{-rT} P(x)\left[\frac{e^{(\alpha+1+i\omega)x}}{\alpha+i\omega}-\frac{e^{(\alpha+1+i\omega)x}}{\alpha+1+i\omega}\right]\cdot dx \\ &= \frac{e^{-rT}\phi_T(\omega-(1+\alpha)i)}{\alpha^2+\alpha-\omega^2+i(1+2\alpha)\omega}\end{aligned}$$

4.3.3 FFT implementation of Carr-Madan Scheme

After obtaining the characteristic function of $\phi_T(\omega)$, we reintroduce it into the closed form solution of $\phi_{c_T}(\omega)$ to solve the integral defined by $C_T(k)$. In this purpose, Carr and Madam showed how to use the FFT algorithm to solve $C_T(k)$ with help of trapezoidal rule. The numerical scheme is outlined as follows:

$$\begin{aligned}c_T(k) &= \frac{1}{\pi}\int_{0}^{+\infty} \underbrace{e^{-i\omega k}\phi_{c_T}(\omega)}_{g(\omega)} d\omega \\ &\approx \int_{0}^{M} g(\omega) d\omega \\ &\approx \frac{\Delta\omega}{2}\left[g(\omega_1)+2\sum_{n=2}^{N} g(\omega_n)+g(\omega_N)\right] \\ &\approx \Delta\omega\left[\sum_{n=1}^{N} g(\omega_n)-\frac{1}{2}[g(\omega_1)+g(\omega_N)]\right]\end{aligned}$$

With

$$\begin{aligned}M &= N\Delta u \\ u_n &= (n-1)\Delta u\,.\end{aligned}$$

Then Carr and Madan discretized the log-strike axis to create a full Fourier Matrix. The discretization proceeds such as $k_m = -b + (m-1)\Delta k$, with $m = 1,\ldots,N$ and $b \in \mathbb{R}$ a constant assumed as the lower boundary point of the log-strike axis. By inserting the discretized k_m into previous equation we will obtain the following result:

$$
\begin{aligned}
c_T(k_m) &= \frac{1}{\pi}\int_0^{+\infty} \underbrace{e^{-i\omega k_m}\phi_{c_T}(\omega)}_{g(\omega)}d\omega \\
&\approx \Delta\omega\left[\sum_{n=1}^{N}\underbrace{e^{-i\Delta k\Delta u(n-1)(m-1)}}_{\xi_N^{(n-1)(m-1)}}e^{i(n-1)b\Delta u}\phi_{c_T}(\omega_n)-\frac{1}{2}[g(\omega_1)+g(\omega_N)]\right] \\
&\approx \Delta\omega\left[\sum_{n=1}^{N}\xi_N^{(n-1)(m-1)}\left[e^{i(n-1)b\Delta u}\phi_{c_T}(\omega_n)\right]-\frac{1}{2}[g(\omega_1)+g(\omega_N)]\right]
\end{aligned}
$$

Thus we can rewrite the pricing process in matrix form such as:

$$
\begin{aligned}
\begin{pmatrix} c_T(k_1) \\ c_T(k_2) \\ \vdots \\ c_T(k_N) \end{pmatrix} &\approx \Delta u \begin{pmatrix} e^{-iu_1k_1} & e^{-iu_2k_1} & \cdots & e^{-iu_Nk_1} \\ e^{-iu_1k_2} & e^{-iu_2k_2} & \cdots & e^{-iu_Nk_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-iu_1k_N} & e^{-iu_2k_N} & \cdots & e^{-iu_Nk_N} \end{pmatrix} \times \begin{pmatrix} \phi_{c_T}(\omega_1) \\ \phi_{c_T}(\omega_2) \\ \vdots \\ \phi_{c_T}(\omega_N) \end{pmatrix} - \frac{1}{2}\begin{pmatrix} g(u_1) \\ 0 \\ \vdots \\ g(u_N) \end{pmatrix} \\
&\approx \Delta u \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-i\Delta u\Delta k} & \cdots & e^{-i(N-1)\Delta u\Delta k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i\Delta u(N-1)\Delta k} & \cdots & e^{-i(N-1)\Delta u(N-1)\Delta k} \end{pmatrix}}_{\text{Matrix A}} \times \begin{pmatrix} e^{iu_1b}\phi_{c_T}(\omega_1) \\ e^{iu_2b}\phi_{c_T}(\omega_2) \\ \vdots \\ e^{iu_Nb}\phi_{c_T}(\omega_N) \end{pmatrix} - \frac{1}{2}\begin{pmatrix} g(u_1) \\ 0 \\ \vdots \\ g(u_N) \end{pmatrix}
\end{aligned}
$$

And Matrix A is solved efficiently by the FFT algorithm.

4.4 <u>FOURIER SPACE TIME-STEPPING METHOD</u>

4.4.1 Introduction

In the previous section, we reviewed the approach developed by Carr and Madan concerning the pricing of a European Option with help of Fourier Transform. The followed process requires:

1. Defining a square integrable payoff,
2. Calculating its characteristic counterpart
3. And defining the discretization process in order to use the Fast Fourier Transform.

This is a first step which presents lots of limitations if applied to more complex options, especially with the two first steps of the process which can be limitative enough.

A new methodology was developed by Vladimir Surkov in his Thesis (Surkov, 2009) to outreach these limitations but also to fulfil some essential functions to develop modern pricing software such as:

1. The necessity to get convergent results with precision, high speed,
2. To handle efficiently path-independent and path-dependent derivatives,
3. To handle mono and multi-assets derivatives,
4. To define a generic handling of random process generators,
5. And to be naturally adaptable whatever the IT architecture complexity.

During the next sections, we will refer to the several papers written by the author during his thesis (see (Jackson, et al., 2008), (Jaimungal, et al., 2009), (Jaimungal, et al., 2010) and (Davison, et al., 2010). In §4.4.2, we will recall the mathematical context of a pricing model using Lévy processes. The section §4.4.3 will introduce the PIDE context and how to solve it and lastly in §4.4.4, we will present how to use it to estimate sensitivity factors

4.4.2 Spot Price Model

In current section, we will recall the mathematical definitions used by the author to develop the FST pricing model.

At first, the author defines a "*Spot Price process*" such as $\mathbf{S}(t)=\mathbf{S}(0)e^{\mathbf{X}(t)}$ and $\mathbf{X}(t)$ a Lévy process with characteristic triplet $(\boldsymbol{\gamma},\boldsymbol{\Sigma},\boldsymbol{\nu})$ where $\boldsymbol{\gamma}$ represents the vector of unadjusted drift, $\boldsymbol{\Sigma}$ the variance-covariance matrix and $\boldsymbol{\nu}$ the multi-dimensional Lévy density. So the process $\mathbf{X}(t)$ is eligible to the Levy-Ito Decomposition Theorem and can be represented with its diffusion and jump components as follows:

$$\mathbf{X}(t)=\boldsymbol{\gamma}t+\mathbf{W}(t)+\mathbf{J}^{l}(t)+\lim_{\varepsilon\to 0}\mathbf{J}^{\varepsilon}(t)$$

$$\mathbf{J}^{l}(t)=\int_0^t\int_{|y|\geq 1}\mathbf{y}\,\tilde{\upsilon}(d\mathbf{y}\times ds),$$

$$\mathbf{J}^{\varepsilon}(t)=\int_0^t\int_{\epsilon\leq|y|<1}\mathbf{y}\left[\tilde{\upsilon}(d\mathbf{y}\times ds)-\upsilon(d\mathbf{y}\times ds)\right]$$

Where

$\mathbf{W}(t)$ is a standard Brownian motion,

$\tilde{\upsilon}(d\mathbf{y}\times ds)$ is a Poisson random measure counting the number of jumps of size $\mathbf{y}$ occurring at time s,

And $\upsilon(d\mathbf{y}\times ds)=\upsilon(d\mathbf{y})ds$ is its compensator.

$\mathbf{J}^{l}(t)$ and $\mathbf{J}^{\varepsilon}(t)$ are interpreted as presence tokens of large and small jumps which can have some incidence from a theoretical point of view. Thus the presence or not of an infinite number of small jumps defines if the process is with finite or infinite activity. In the last case, the small jumps integral must be centred to obtain the convergence.

Secondly the author enforced the risk-neutral condition to determine a unique drift which is uniquely when the volatility and Lévy density are specified. Thus γ can be chosen such as:

$$\mathbb{E}_0\left[e^{\mathbf{X}_j(1)}\right]=e^{r}\quad\Rightarrow\quad\Psi\left(-i\mathbf{1}_j\right)=r$$

Where $\Psi(\boldsymbol{\omega})$ is the characteristic exponent of d-dimensional Lévy process, defined with help of the Lévy-Khintchine formula:

$$\Psi(\boldsymbol{\omega})=i\boldsymbol{\gamma}\cdot\boldsymbol{\omega}-\frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{\omega}\cdot\boldsymbol{\omega}+\int_{\mathbb{R}^n}\left(e^{i\boldsymbol{\omega}\mathbf{y}}-1-i\mathbb{1}_{\{|\mathbf{y}|\leq 1\}}\boldsymbol{\omega}\cdot\mathbf{y}\right)\nu(d\mathbf{y})\tag{1}$$

These elements define a pricing framework where the BSM model is integrated by simply setting the Lévy density to zero, as presented by Cont and Tankov in their book (Cont, et al., 2004).

4.4.3 Partial Integral Differential Equation (PIDE) Solution

Next the author uses the fundamental theorem of asset pricing to recall that a discounted log-transformed price process $v\left(t,\mathbf{X}(t)\right)\triangleq e^{r(T-t)}V\left(t,\mathbf{S}(0)e^{\mathbf{X}(t)}\right)$ is a martingale under the measure $\mathbb{Q}$. This implies that the associated drift of underlying stochastic process equals zero. Thus the author defined the PIDE by coupling the application of previous condition on $v(t,\mathbf{x})$ and its boundary condition at maturity which gives the following system:

$$\begin{cases}(\partial_t+\mathcal{L})v(t,\mathbf{x}) & =0\\ V(T,\mathbf{x}) & =\varphi\left(\mathbf{S}(0)e^{\mathbf{X}}\right)\end{cases} \tag{2}$$

Where $\mathcal{L}$ represents the infinitesimal generator of the multi-dimensional Lévy process and acts as a twice differentiable function $g(\mathbf{x})$ as follows:

$$\mathcal{L}g(\mathbf{x})=\left(\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}}+\frac{1}{2}\boldsymbol{\Sigma}\partial_{\mathbf{x}}\cdot\partial_{\mathbf{x}}\right)g(\mathbf{x})+\int_{\mathbb{R}^n\setminus\{0\}}\left(g(\mathbf{x}+\mathbf{y})-g(\mathbf{x})+\mathbb{1}_{\{|\mathbf{y}|<1\}}\mathbf{y}\cdot\partial_{\mathbf{x}}g(\mathbf{x})\right)\nu(d\mathbf{y}) \tag{3}$$

As the author reminds, the Fourier and Laplace Transforms have been used widely to solve PDEs, and the next stage aims to develop a methodology to solve PIDEs as those presented in equation (2). The idea beneath such approach is avoiding the limitations highlighted in Carr and Madan's work.

To do this, the author reminds the definitions of Continuous Fourier Transform (CTF), its inverse function (ICFT) and the CFT property that transforms partial derivative $\partial_{\mathbf{x}}^n$ into a linear operator according to spatial (see §4.2):

Definition	Equation
Continuous FT	$\mathcal{F}[g](\boldsymbol{\omega})\triangleq\int_{-\infty}^{+\infty}g(\mathbf{x})e^{-i\boldsymbol{\omega}'\mathbf{x}}d\mathbf{x}$
Inversed Continuous FT	$\mathcal{F}^{-1}[\hat{g}](\mathbf{x})\triangleq\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{g}(\boldsymbol{\omega})e^{i\boldsymbol{\omega}'\mathbf{x}}d\boldsymbol{\omega}$
Property on partial derivatives	$\mathcal{F}\left[\partial_{\mathbf{x}}^n g\right](\boldsymbol{\omega})=i\boldsymbol{\omega}\mathcal{F}\left[\partial_{\mathbf{x}}^{n-1}g\right](\boldsymbol{\omega})=\cdots=(i\boldsymbol{\omega})^n\mathcal{F}[g](\boldsymbol{\omega})$ (4)

Next he applies the CFT to the infinitesimal generator $\mathcal{L}$ of $\mathbf{X}(t)$ defined in equation (3) which allows the definition of a solution by factorizing out the characteristic exponent of $\mathbf{X}(t)$:

$$\begin{aligned}\mathcal{F}[\mathcal{L}v](t,\boldsymbol{\omega})&=\left\{i\boldsymbol{\gamma}\cdot\boldsymbol{\omega}-\frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{\omega}\cdot\boldsymbol{\omega}+\int_{\mathbb{R}^n}\left(e^{i\boldsymbol{\omega}\mathbf{y}}-1-i\mathbb{1}_{\{|\mathbf{y}|\le1\}}\boldsymbol{\omega}\cdot\mathbf{y}\right)\nu(d\mathbf{y})\right\}\mathcal{F}[v](t,\boldsymbol{\omega})\\&=\boldsymbol{\Psi}(\boldsymbol{\omega})\mathcal{F}[v](t,\boldsymbol{\omega})\end{aligned} \tag{5}$$

Applying Fourier Transform to both part of the PIDE (2) and replacing $\mathcal{F}[\mathcal{L}v](t,\boldsymbol{\omega})$ by (5) give the following system:

$$\begin{cases}\partial_t\mathcal{F}[v](t,\boldsymbol{\omega})+\boldsymbol{\Psi}(\boldsymbol{\omega})\mathcal{F}[v](t,\boldsymbol{\omega})=0\\ \mathcal{F}[V](T,\boldsymbol{\omega}) =\mathcal{F}[\varphi](\boldsymbol{\omega})=\mathcal{F}\left[\varphi\left(S(T)\right)\right](\boldsymbol{\omega})\end{cases} \tag{6}$$

Hence the PIDE is transformed into a multidimensional ODE system, parameterized by ω. Given the value of $\mathcal{F}[v](t_2,\boldsymbol{\omega})$ at time $t_2 \leq T$, the system can be solved to find the value at time $t_1 < t_2$:

$$\mathcal{F}[v](t_1,\boldsymbol{\omega}) = \mathcal{F}[v](t_2,\boldsymbol{\omega})e^{\Psi(\boldsymbol{\omega})(t_2-t_1)} \text{ with } t_1 < t_2 \leq T \tag{7}$$

Taking the inverse transform of (7) leads to the final result:

$$v(t_1,\mathbf{x}) = \mathcal{F}^{-1}\left[\mathcal{F}[v](t_2,\boldsymbol{\omega})e^{\Psi(\boldsymbol{\omega})(t_2-t_1)}\right](\mathbf{x}) \tag{8}$$

Applying previous results to payoff valuation gives the following result:

$$\begin{aligned}
v(t,\mathbf{x}) &= V(T,\mathbf{x})e^{-r(T-t)} \\
&= \mathcal{F}^{-1}\left[\mathcal{F}[V](T,\boldsymbol{\omega})e^{-r(T-t)}\right](t,\mathbf{x}) \\
&= \mathcal{F}^{-1}\left[\mathcal{F}[\varphi](\boldsymbol{\omega})e^{-r(T-t)}\right](t,\mathbf{x}) \\
&= \mathcal{F}^{-1}\left[\mathcal{F}\left[\varphi(S(T))\right](\boldsymbol{\omega})e^{-r(T-t)}\right](t,\mathbf{x}) \\
&= \mathcal{F}^{-1}\left[\mathcal{F}\left[\varphi\left(S(0)e^{\Psi_{LK}(\boldsymbol{\omega})(T-t)}\right)\right](\boldsymbol{\omega})e^{-r(T-t)}\right](t,\mathbf{x}) \\
&= \mathcal{F}^{-1}\left[\mathcal{F}\left[\varphi\left(S(0)e^{(\Psi_{LK}(\boldsymbol{\omega})-r)(T-t)}\right)\right](\boldsymbol{\omega})\right](t,\mathbf{x}) \\
&= \mathcal{F}^{-1}\left[\mathcal{F}[v](T,\boldsymbol{\omega})e^{\Psi_{FST}(\boldsymbol{\omega})(T-t)}\right](t,\mathbf{x})
\end{aligned} \tag{9}$$

Where

- $\boldsymbol{\Psi}_{FST}(\boldsymbol{\omega}) = (\boldsymbol{\Psi}_{LK}(\boldsymbol{\omega}) - r)$
- $\boldsymbol{\Psi}_{LK}(\boldsymbol{\omega})$: the characteristic exponent defined by the Lévy Khintchine formula in equation (1),
- $\boldsymbol{\Psi}_{FST}(\boldsymbol{\omega})$: the transformed characteristic exponent employed by FST methodology. For the remainder of the dissertation, we will design $\boldsymbol{\Psi}(\boldsymbol{\omega}) = \boldsymbol{\Psi}_{FST}(\boldsymbol{\omega})$ by convention.

The consequences of this demonstration are:

1. No parameter is introduced to define a square integrable process,
2. No payoff transformation is required to proceed to the evaluation. You just have to define a vector of intrinsic value according to the payoff structure and proceed to the FT. Then you will add the time value with help equation (7) and applying the IFT gives a vector of actualized values.
3. The infinitesimal generator is just a modular component of the evaluation process. Hence it can be changed easily only if the characteristic exponent is defined, which is the case for most of stochastic processes (see Panel 9 for examples).

4.4.4 Computing Sensitivity Factors

Previous section was dedicated to the definition of the pricing process and now we will see how to estimate the sensitivity factors for a given option structure.

We will distinguish the sensitivity factors into two categories:

1. Sensitivity factors depending on state variables (i.e. variables integrated into the payoff vector $v(t,\mathbf{x})$),
2. And those depending on model parameters (i.e. variables integrated into $\boldsymbol{\Psi}(\boldsymbol{\omega})$)

Please note we will assume that no parameter belongs into both categories for the sake of simplicity.

4.4.4.1 Sensitivity Factors versus State Variables

The first thing to define is the so-called "*Scaling Principle*", i.e. how to establish a relationship between $\partial_{\mathbf{x}_k} v(t,\mathbf{x})$ and $\partial_{\mathbf{S}_k} v(t,\mathbf{x})$ according to the relationship established between the two variables by $\mathbf{S}_k = \mathbf{S}_k(0)e^{\mathbf{x}_k}$. From this formula, we can obtain its inverse defined as $\mathbf{x}_k = \ln\left(\frac{\mathbf{S}_k}{\mathbf{S}_k(0)}\right)$. Let's now introduce this relationship into partial derivatives versus $\mathbf{x}_k$ with help of the Product Rule.

The results of these calculations are summarized into Panel 10.

Order	Impact of scaling on PDE
1st	$\partial_{\mathbf{x}_k} v(t,\mathbf{x}) = \frac{\partial v(t,\mathbf{x})}{\partial \mathbf{x}_k} = \frac{\partial v(t,\mathbf{x})}{\partial \mathbf{S}_k}\frac{\partial \mathbf{S}_k}{\partial \mathbf{x}_k} = \partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k(0)e^{\mathbf{x}_k} = \partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k$ $\Rightarrow \partial_{\mathbf{S}_k} v(t,\mathbf{x}) = \frac{\partial_{\mathbf{x}_k} v(t,\mathbf{x})}{\mathbf{S}_k(0)e^{\mathbf{x}_k}}$
2nd (mono asset)t	$\partial^2_{\mathbf{x}_k} v(t,\mathbf{x}) = \frac{\partial\left[\partial_{\mathbf{x}_k} v(t,\mathbf{x})\right]}{\partial \mathbf{x}_k} = \frac{\partial\left[\partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k(0)e^{\mathbf{x}_k}\right]}{\partial \mathbf{S}_k}\frac{\partial \mathbf{S}_k}{\partial \mathbf{x}_k} = \frac{\partial\left[\partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k\right]}{\partial \mathbf{S}_k}\frac{\partial \mathbf{S}_k}{\partial \mathbf{x}_k}$ $= \left[\frac{\partial\left[\partial_{\mathbf{S}_k} v(t,\mathbf{x})\right]}{\partial \mathbf{S}_k} \cdot \mathbf{S}_k + \partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \frac{\partial[\mathbf{S}_k]}{\partial \mathbf{S}_k}\right] \cdot \mathbf{S}_k(0)e^{\mathbf{x}_k}$ $= \left[\partial^2_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k + \partial_{\mathbf{S}_k} v(t,\mathbf{x})\right] \cdot \mathbf{S}_k = \partial^2_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k^2 + \partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k$ $= \partial^2_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k^2 + \partial_{\mathbf{x}_k} v(t,\mathbf{x})$ $\Rightarrow \partial^2_{\mathbf{S}_k} v(t,\mathbf{x}) = \frac{\left(\partial^2_{\mathbf{x}_k} v(t,\mathbf{x}) - \partial_{\mathbf{x}_k} v(t,\mathbf{x})\right)}{\left(\mathbf{S}_k(0)e^{\mathbf{x}_k}\right)^2}$
2nd (multi assets)	$\partial^2_{\mathbf{x}_k\mathbf{x}_l} v(t,\mathbf{x}) = \frac{\partial\left[\partial_{\mathbf{x}_k} v(t,\mathbf{x})\right]}{\partial \mathbf{x}_l} = \frac{\partial\left[\partial_{\mathbf{S}_k} v(t,\mathbf{x}) \cdot \mathbf{S}_k\right]}{\partial \mathbf{S}_l}\frac{\partial \mathbf{S}_l}{\partial \mathbf{x}_l} = \partial^2_{\mathbf{S}_k\mathbf{S}_l} v(t,\mathbf{x}) \cdot \mathbf{S}_k \cdot \mathbf{S}_l$ $\Rightarrow \partial^2_{\mathbf{S}_k\mathbf{S}_l} v(t,\mathbf{x}) = \partial^2_{\mathbf{x}_k\mathbf{x}_l} v(t,\mathbf{x}) / \left(\mathbf{S}_k(0)e^{\mathbf{x}_k} \cdot \mathbf{S}_l(0)e^{\mathbf{x}_l}\right)$

Panel 10: Scaling principles adapted to first order and second order partial derivatives.

So Greeks based on state variables can be calculated according to the PIDE solution expressed in equation (8), the property on partial derivatives (see equation (4)) while in Fourier Space and the scaling principle defined previously in Panel 10 regarding the derived variable ($\mathbf{x}_k$ or $\mathbf{S}_k$).

The results of these calculations are summarized in Panel 11.

Sensitivity Factor	Versus $\mathbf{x}_k$	Versus $\mathbf{S}_k$
Delta	Continuous: $\partial_{\mathbf{x}_k} v(t,\mathbf{x}) = \mathcal{F}^{-1}\left[i\boldsymbol{\omega}_k \cdot \mathcal{F}[v](t,\boldsymbol{\omega})\right](t,\mathbf{x})$ Discrete: $\boldsymbol{\Delta}_{\mathbf{x}_k,m-1} = \mathrm{FFT}^{-1}\left[i\boldsymbol{\omega}_k \cdot \hat{\mathbf{v}}_{m-1}\right]$ Where $\hat{\mathbf{v}}_{m-1} = \mathrm{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}$	Continuous: $\partial_{\mathbf{S}_k} v(t,\mathbf{x}) = \frac{\partial_{\mathbf{x}_k} v(t,\mathbf{x})}{\mathbf{S}_k(0)e^{\mathbf{x}_k}} = \frac{\mathcal{F}^{-1}\left[i\boldsymbol{\omega}_k \cdot \mathcal{F}[v](t,\boldsymbol{\omega})\right](t,\mathbf{x})}{\mathbf{S}_k(0)e^{\mathbf{x}_k}}$ Discrete: $\boldsymbol{\Delta}_{\mathbf{S}_k,m-1} = \frac{\mathrm{FFT}^{-1}\left[i\boldsymbol{\omega}_k \cdot \hat{\mathbf{v}}_{m-1}\right]}{\mathbf{S}_k(0)e^{\mathbf{x}_k}}$ Where $\hat{\mathbf{v}}_{m-1} = \mathrm{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}$
Gamma	Continuous: $\partial^2_{\mathbf{x}_k} v(t,\mathbf{x}) = \mathcal{F}^{-1}\left[-\boldsymbol{\omega}_k^2 \cdot \mathcal{F}[v](t,\boldsymbol{\omega})\right](t,\mathbf{x})$ Discrete: $\boldsymbol{\Gamma}_{\mathbf{x}_k,m-1} = \mathrm{FFT}^{-1}\left[-\boldsymbol{\omega}_k^2 \cdot \hat{\mathbf{v}}_{m-1}\right]$	Continuous: $\partial^2_{\mathbf{S}_k} v(t,\mathbf{x}) = \frac{\left(-\partial_{\mathbf{x}_k} v(t,\mathbf{x}) + \partial^2_{\mathbf{x}_k} v(t,\mathbf{x})\right)}{\left(\mathbf{S}_k(0)e^{\mathbf{x}_k}\right)^2} = \frac{\mathcal{F}^{-1}\left[\left(-i\boldsymbol{\omega}_k - \boldsymbol{\omega}_k^2\right) \cdot \mathcal{F}[v](t,\boldsymbol{\omega})\right](\mathbf{x})}{\left(\mathbf{S}_k(0)e^{\mathbf{x}_k}\right)^2}$ Discrete: $\boldsymbol{\Gamma}_{\mathbf{S}_k,m-1} = \frac{\mathrm{FFT}^{-1}\left[\left(-i\boldsymbol{\omega}_k - \boldsymbol{\omega}_k^2\right) \cdot \hat{\mathbf{v}}_{m-1}\right]}{\left(\mathbf{S}_k(0)e^{\mathbf{x}_k}\right)^2}$

Panel 11: Sensitivity Factors of State Variables.

4.4.4.2 Sensitivity Factors versus Model Parameters

In current section, we will use the following sign convention to distinguish two model parameters such as:

1. Model parameter #1: ★
2. Model parameter #2: ◇

Now let's start the derivation of equation (8) according to the Fourier Transform property with partial derivatives.

The results are presented below:

1st order partial derivative:

We present the demonstration of how to calculate 1st order partial derivative with help of Continuous Fourier Transform:

$$\begin{aligned}\partial_\star \hat{v}(t,\boldsymbol{\omega}) &= \partial_\star\left(\hat{v}(T,\boldsymbol{\omega})\cdot e^{\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)}\right)\\ &=\hat{v}(T,\boldsymbol{\omega})\cdot\partial_\star\left(e^{\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)}\right)\\ &=\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(T,\boldsymbol{\omega})\cdot e^{\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)}\\ &=\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})\end{aligned}$$

We assume that $\partial_\star\hat{v}(T,\boldsymbol{\omega})=0$, hence:

$$\partial_\star v(t,\mathbf{x})=\mathscr{F}^{-1}\left[\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})\right](\mathbf{x}) \tag{10}$$

The equivalent DFT form is:

$$\nabla_{\star,m-1}=\mathrm{FFT}^{-1}\left[\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\Delta t_m\cdot\hat{\mathbf{v}}_{m-1}\right] \tag{11}$$

<u>2nd order partial derivative (mono assets):</u>

As previously, we provide the demonstration of the calculation of 2nd order partial derivative with respect to a single asset with help of CFT:

$$\begin{aligned}\partial_\star^2\hat{v}(t,\boldsymbol{\omega}) &= \partial_\star\left[\partial_\star\hat{v}(t,\boldsymbol{\omega})\right]\\ &=\partial_\star\left[\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})\right]\\ &=\partial_\star^2\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})+\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\partial_\star\hat{v}(t,\boldsymbol{\omega})\\ &=\partial_\star^2\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})+\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})\\ &=\left[\partial_\star^2\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)+\left(\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\right)^2\right]\cdot\hat{v}(t,\boldsymbol{\omega})\end{aligned}$$

With $\partial_\star\hat{v}(T,\boldsymbol{\omega})=0$

Thereby,

$$\partial_\star^2 v(t,\mathbf{x})=\mathscr{F}^{-1}\left[\left(\partial_\star^2\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)+\left[\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\right]^2\right)\cdot\hat{v}(t,\boldsymbol{\omega})\right](\mathbf{x}) \tag{12}$$

The equivalent DFT form is:

$$\nabla_{\star,m-1}^2=\mathrm{FFT}^{-1}\left[\left(\partial_\star^2\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\Delta t_m+\left[\partial_\star\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\Delta t_m\right]^2\right)\cdot\hat{\mathbf{v}}_{m-1}\right] \tag{13}$$

<u>2nd order partial derivative (multi assets):</u>

Lastly we provide the demonstration of the calculation of 2nd order partial derivatives with respect to two assets (based on CFT):

$$\begin{aligned}\partial^2_{\star\diamond}\hat{v}(t,\boldsymbol{\omega}) &= \partial_{\diamond}\left[\partial_{\star}\hat{v}(t,\boldsymbol{\omega})\right] \\ &= \partial_{\diamond}\left[\partial_{\star}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega})\right] \\ &= \partial^2_{\star\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega}) + \partial_{\star}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\partial_{\diamond}\hat{v}(t,\boldsymbol{\omega}) \\ &= \partial^2_{\star\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega}) + \partial_{\star}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\partial_{\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)\cdot\hat{v}(t,\boldsymbol{\omega}) \\ &= \left[\partial^2_{\star\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t) + \partial_{\star}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\partial_{\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)^2\right]\cdot\hat{v}(t,\boldsymbol{\omega})\end{aligned}$$

By assuming that $\partial_{\star}\hat{v}(T,\boldsymbol{\omega}) = 0$ and $\partial_{\diamond}\hat{v}(T,\boldsymbol{\omega}) = 0$, we get:

$$\partial^2_{\star\diamond}v(t,\mathbf{x}) = \mathscr{F}^{-1}\left[\left(\partial^2_{\star\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t) + \partial_{\star}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\partial_{\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot(T-t)^2\right)\cdot\hat{v}(t,\boldsymbol{\omega})\right](\mathbf{x}) \tag{14}$$

Below you will find the discrete version:

$$\nabla^2_{\star\diamond,m-1} = \mathrm{FFT}^{-1}\left[\left(\partial^2_{\star\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\Delta t_m + \partial_{\star}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\partial_{\diamond}\boldsymbol{\Psi}(\boldsymbol{\omega})\cdot\Delta t_m^2\right)\cdot\hat{\mathbf{v}}_{m-1}\right] \tag{15}$$

4.4.4.3 Partial Derivatives of the Characteristic Exponent: An example with the GBM process

Now let's apply the valuation of the sensitivity factors based on Model parameters according to the Geometric Brownian Motion process. This implies to derivate its characteristic exponent factor Ψ (see Panel 9) versus each model parameters (i.e. r, t and σ).

The results are listed in panel below:

Model	Parameter	Formula
BSM	Interest Rate r	$\partial_r \Psi(\omega) = \partial_r \Psi_{FST}(\omega) = \partial_r \left(\left(\Psi_{LK}(\omega) - r \right) \cdot (T-t) \right) = \left(\partial_r \Psi_{LK}(\omega) - 1 \right) \cdot (T-t) = (i\omega - 1) \cdot (T-t)$ with $\partial_r \Psi_{LK}(\omega) = \partial_r \left(i\omega \left(r - q - \frac{\sigma^2}{2} \right) - \frac{\sigma^2 \omega^2}{2} \right) = i\omega$
	Time t	$\partial_t \Psi(\omega) = \partial_t \Psi_{FST}(\omega) = \partial_t \left(\left(\Psi_{LK}(\omega) - r \right) \cdot (T-t) \right) = -\left(\Psi_{LK}(\omega) - r \right) + \partial_t \Psi_{LK}(\omega) = -\left(\Psi_{LK}(\omega) - r \right) = -\Psi(\omega)$ with $\partial_t \Psi_{LK}(\omega) = \partial_t \left(i\omega \left(r - q - \frac{\sigma^2}{2} \right) - \frac{\sigma^2 \omega^2}{2} \right) = 0$
	Diffusion Volatility σ	$\partial_\sigma \Psi(\omega) = \partial_\sigma \Psi_{FST}(\omega) = \partial_\sigma \left(\left(\Psi_{LK}(\omega) - r \right) \cdot (T-t) \right) = \partial_\sigma \left(\left(\Psi_{LK}(\omega) - r \right) \cdot (T-t) \right) = -\left(i\omega + \omega^2 \right) \cdot \sigma \cdot (T-t)$ with $\partial_\sigma \Psi_{LK}(\omega) = \partial_\sigma \left(i\omega \left(r - q - \frac{\sigma^2}{2} \right) - \frac{\sigma^2 \omega^2}{2} \right) = -\left(i\omega + \omega^2 \right) \cdot \sigma$

Panel 12: Partial Derivatives of the GBM characteristic exponent factor versus a model parameter.

Hence these partial derivatives can be reintroduced in equations presented in §4.4.4 to get the value of Rho, Theta and Vega.

In this situation we use only equation (11), however these results can be extended regarding the desired Sensitivity Factors (see §7.3 for examples of uncommon sensitivity factors) with help of equations (13) and (15).

5. IMPLEMENTATION OF PIPELINE RISK FRAMEWORK

This section will present how to use the FST Method to evaluate structured products and how we will integrate it into the future Pipeline Risk Framework:

- The first stage will focus on the discretization principles to follow and how to produce pricing algorithms with help of the iterative integration process.
- The second stage will present a pricing benchmark to compare the FST efficiency versus well-known pricing models. This benchmark will be based on a set of eight classical options and three typical option structures.
- The third stage will be dedicated to the implementation aspects of Pipeline Risk Framework. We will see how to forecast risk factors, how to calibrate the FST parameters regarding the option structure and the market data. Lastly we will present the Excel Calculator used to monitor and manage risk exposure generated by Structured Products.

5.1 DISCRETIZATION PRINCIPLES OF FST METHOD

This section will focus on two important concepts to be applied during the discretization of FST method: the mapping process of $\mathbb{R}$ and $\mathbb{C}$, and the Iterative Integration process. These two concepts are essential and will be used all along this dissertation.

5.1.1 Mapping Process

In section §4.2.4, we presented the "*aliasing*" issue raised by the discretization of Fourier Transform and how to solve it with help of the Optimal frequency of Nyquist.

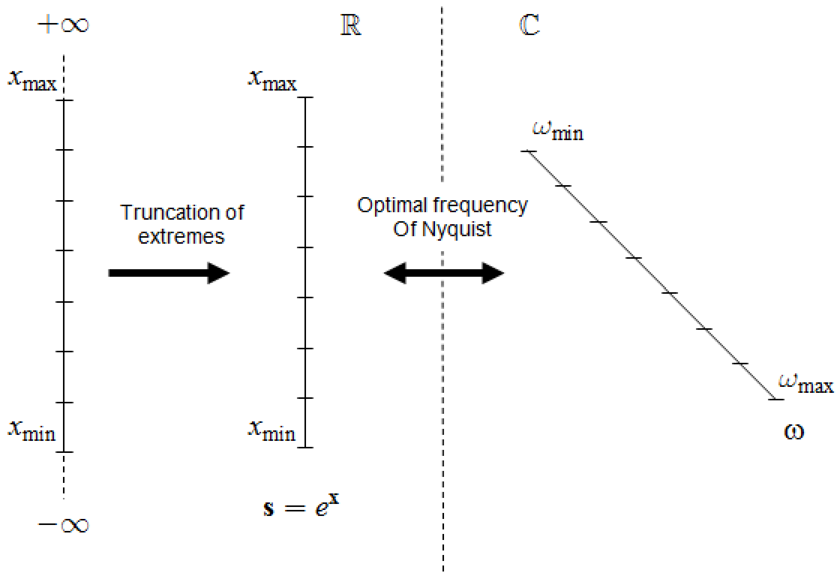


Figure 21: Mapping process between vectors x, s (Time Space) and ω (Fourier Space)

Hence we are able to map $\mathbb{C}$ to $\mathbb{R}$ following the mapping process presented in Figure 21:

1. Definition of Fourier Parameters in $\mathbb{R}$: Let's define $x_{\min}$ and $x_{\max}$ (these two values will act as proxy values of $-\infty$ and $+\infty$) and N, the number of points used to discretize $[x_{\min}, x_{\max}]$ and produce vector $\mathbf{x}$.

2. Vector Creation: Next we define the vector $\mathbf{x}$ which contains $N+1$ points with $x_{\min}$ and $x_{\max}$ as extrema. This vector will help to define the $\mathbf{s}$ vector such as $\mathbf{s}=e^{\mathbf{x}}$ and will represent the potential values of the option underlying.
3. Mapping between $\mathbb{R}$ and $\mathbb{C}$ spaces: the final stage will be the definition of the $\boldsymbol{\omega}$ vector, the counterpart of $\mathbf{x}$ in $\mathbb{C}$ space with help of Nyquist's Optimal Frequency.

In the current disseration, we will apply a linear discretization (i.e. a constant Δx discretization step) as presented in (Surkov, 2009), (Jaimungal, et al., 2009) and (Jaimungal, et al., 2010).

5.1.2 Iterative Integration Process

Following the mapping process presented in previous section, we will describe the integration process of an option value (see Figure 22 for a graphical representation).

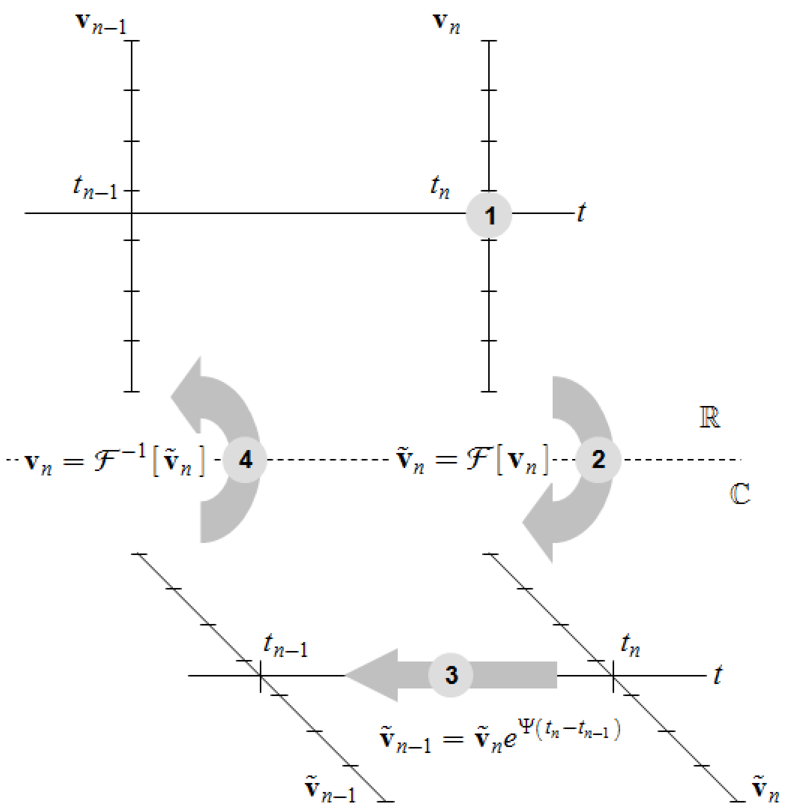


Figure 22: FST Iterative Integration Process. 1) Evaluate the option intrinsic value at t(n), 2) Apply Fourier Transform, 3) Add Time value with help of PIDE solution and 4) Back to Real Space to get option's present values at t(n-1).

This process follows the next steps:

1. Adding the Intrinsic Value: we define the vector $\mathbf{v}_n$ which represents the vector of intrinsic value of an option such as $\mathbf{v}_n=\varphi(\mathbf{s},K)=\varphi\left(e^{\mathbf{x}},K\right)$.
2. Going to the FT Space: With help of FFT, we convert $\mathbf{v}_n$ into $\tilde{\mathbf{v}}_n$, its $\mathbb{C}$ counterpart. Due to the mapping process, $\tilde{\mathbf{v}}_n=\varphi\left(e^{\boldsymbol{\omega}},K\right)$.

3. Adding the Time Value: we add the time value of an option such as $\tilde{\mathbf{v}}_{n-1} = \tilde{\mathbf{v}}_n e^{\Psi(t_n - t_{n-1})}$. Please note that Ψ corresponds to the characteristic exponent factor and so is specific to each random generator process.
4. Back to the Real Space: this last step is done with help of Inversed FFT and gives the vector $\mathbf{v}_{n-1}$

We just described the integration process of one-step option. However this process can be reiterated for n-steps option. As example, the $\mathbf{v}_{n-1}$ calculated by the integration process may be reagregated into a final $\mathbf{v}_{n-1}$ which will be rebalanced into a new integration step. This case will be presented more explicitely from sections §5.2.2 to §5.2.3.

5.2 PRICING BENCHMARK

In this section, we present the benchmark pricing results of several options according to multiple valuation approaches. The idea of this benchmark is to start with simple options to explore the way to use FST method and compare its pricing capacity by comparing with well-known methods. The next step of the benchmark will be to confront FST method with more complex options which belong to structured products. Lastly we will compare the results of structured products valuation. An issue will rise shortly from the second part of the benchmark: the decrease of comparison method only to the Monte Carlo approach. Indeed these structures can't be priced with closed-formula and PDE approaches are not efficient enough to price such option structures. Thus the only reference method which will remain will be the Monte Carlo approach. It's why the first part will be important to prove the accuracy of FST approach.

In first stages, we will use INTLAB (Rump, 1999), a Matlab implementation of "*Automatic Differentiation*" (AD) Algorithm. This method consists of a set of techniques based on the mechanical application of the chain rule to obtain derivatives of a function given as a computer program. AD exploits the fact that every computer program, no matter how complicated, executes a sequence of elementary arithmetic operations such as additions or elementary functions such as exponential. By applying the chain rule of derivative calculus repeatedly to these operations, derivatives of arbitrary order can be computed automatically, and accurate to working precision. This approach is highly reliable when a closed-formula is defined however it seems limited when you have to integrate one or more conditional terms such as $\mathbb{1}_{\{x<p\}}$ for instance due to the difficulty to define its elementary operation. This approach is currently studied to improve Monte Carlo efficiency (see (Giles, et al., 2006), (Giles, 2007) and (Giles, 2008)). For more details on this numerical approach, please see www.autodiff.org.

The pricing models used in this section are referred as follows:

1. BSM: Black Scholes Closed Formula,
2. FST: the FST method,
3. ADF: the Automatic Differentiation method implemented in INTLAB,
4. MCM: Monte Carlo Method,
5. CRR: the Cox Ross Rubinstein method (Cox, et al., 1979)
6. FiD: the Finite Difference Method,
7. LSM: The Longstaff and Schwartz Method (Longstaff, et al., 2001).

For the sake of simplicity and dissertation clarity, we will use only the Geometric Brownian Motion Stochastic Process as Random Generator Process. However the results presented can be extended with more complex processes but this would require to define the partial derivative of their characteristic exponent factor Ψ to proceed the valuation, as we did in §4.4.4.3.

This benchmark will price options based on theoretical parameters and will focus on market data calibration in section §5.3.3. The value of each parameter will be summarized in the results panel. All calculations are done with help of a Mac Book Pro with a 2.80GHz Intel Core(TM) 2 Duo CPU with 4.00 Go of RAM Memory and Matlab 2010b.

5.2.1 Path independent Options

5.2.1.1 European Option

We start the pricing benchmark with the "guinea pig" of every option pricing model: the European Option. In this section, we will compare the pricing results of four pricing models: BSM, FST, ADF and MCM.

In panel below, we present the FST Iterative Integration algorithm (see §5.1.2) adapted to European option. Comparing the initial algorithm, we detailed the last stage of option valuation: determining the price and greeks value at $t = 0$. This step proceed by interpolating the dedicated real vector according to the S_0 value with an accurate interpolation method (we will use cubic polynomial interpolation).

Please note that this calculation step will be applied all along this dissertation and so won't be detailed in next stages.

t	$\mathbb{R}$	$\mathbb{C}$
n	1) $\mathbf{v}_n = \varphi(\mathbf{S}, K) = \max(\mathbf{S} - K, 0)$ →	2) $\hat{\mathbf{v}}_n = \mathrm{FT}[\mathbf{v}_n]$ ↓
0	4) Price $v_0 = \mathrm{interpolate}\left(\mathrm{FT}^{-1}[\hat{\mathbf{v}}_0], \mathbf{S}, S_0\right)$ 5) Greeks $\delta = \mathrm{interpolate}\left(\frac{\mathrm{FFT}^{-1}[i\boldsymbol{\omega}_k \cdot \hat{\mathbf{v}}_0]}{\mathbf{S}_k(0)e^{\mathbf{x}_k}}, \mathbf{S}, S_0\right)$ $\gamma = \mathrm{interpolate}\left(\frac{\mathrm{FFT}^{-1}\left[\left(-i\boldsymbol{\omega}_k - \boldsymbol{\omega}_k^2\right) \cdot \hat{\mathbf{v}}_0\right]}{\left(\mathbf{S}_k(0)e^{\mathbf{x}_k}\right)^2}, \mathbf{S}, S_0\right)$ $\rho = \mathrm{interpolate}\left(\mathrm{FFT}^{-1}\left[\partial_{\mathrm{rho}} \boldsymbol{\Psi}(\boldsymbol{\omega}) \cdot t_n \cdot \hat{\mathbf{v}}_0\right], \mathbf{S}, S_0\right)$ $\tau = \mathrm{interpolate}\left(\mathrm{FFT}^{-1}\left[-\boldsymbol{\Psi}(\boldsymbol{\omega}) \cdot \hat{\mathbf{v}}_{m-1}\right], \mathbf{S}, S_0\right)$ $\nu = \mathrm{interpolate}\left(\mathrm{FFT}^{-1}\left[\partial_{\mathrm{vega}} \boldsymbol{\Psi}(\boldsymbol{\omega}) \cdot t_n \cdot \hat{\mathbf{v}}_0\right], \mathbf{S}, S_0\right)$	← 3) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_n e^{\Psi(n-0)}$

Panel 13: FST Algorithm used to evaluate a European Option.

Now we present the pricing results and the used parameter values in Figure 23. As we can see, most results are in line with the pricing results produced by the BSM closed formulae (see §7.3). The ADF method will estimate price and greeks by using the call / put closed formula used to estimate the price. Regarding the performance, the FST is ranked third with a time consumption near instant with pricing errors below the bips (i.e. 0.01%) value. We will study later in §5.3.3 the pricing errors between BSM and FST according to the variations of financial parameters.

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	3.00
basis	1.00

Method	Parameters	Values
MCM	Nb Sims	1 000 000.00
FST	N	8 192.00
	X_min	- 10.00
	X_max	10.00

Option	Payoff	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
European	Call	BSM	370.92	0.60	0.00	3 361.51	- 100.71	1 676.61	0.01
		FST	370.92	0.60	0.00	3 361.51	- 100.71	1 676.61	0.03
		ADF	370.92	0.60	0.00	3 361.51	- 100.71	1 676.61	0.02
		MCM	372.34	0.57	- 0.00	3 319.90	- 354.27	1 587.57	1.81
	Put	BSM	309.95	- 0.40	0.00	- 3 955.58	- 3.15	1 676.61	0.01
		FST	309.95	- 0.40	0.00	- 3 955.58	- 3.15	1 676.61	0.02
		ADF	309.95	- 0.40	0.00	- 3 955.58	- 3.15	1 676.61	0.01
		MCM	309.73	- 0.40	0.00	- 2 444.10	- 110.68	1 340.36	1.83

Figure 23: Example of pricing results of a European Option (Call and Put).

Let's have a deeper look at the MCM results which present few differences due to the stochastic nature of its valuation process.

Here we recall the two statistical theorems underlying to the MCM valuation process:

1) The Law of the Large Numbers:
If we define $(X_i, i \geq 1)$ a serie of real random numbers which are independent and equally distributed such as $\mathbb{E}(X_1) < +\infty$, then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n \to +\infty]{} \mathbb{E}[X_1]$ almost surely.

2) The Central Limit Theorem:
We define $(X_i, i \geq 1)$ a serie of real random numbers which are independent and equally distributed such as $\mathbb{E}(X_1) < +\infty$. We note σ^2 the variance of X_1 and $N(0,1)$ a centered and scaled gaussian distribution.
Then $\sqrt{n}\dfrac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X_1]}{\sigma} \xrightarrow[n \to +\infty]{d} N(0,1)$

Thus the first theorem demonstrates that Monte Carlo Simulations will converge to the mean results of Price and Greeks. And the second theorem demonstrates that we can produce a confidence interval for all results produced by MCM such as $X_1 < \mathbb{E}[X_1] \pm \frac{\sigma}{\sqrt{n}} t_{\alpha/2}$. Most of the time σ^2 is unknown but it can be estimated with its unbiaised estimator $s^2 = \frac{1}{N-1}\sum_{i=1}^{n}(X_i - \mathbb{E}(X_1))^2$.

Hence there is a stability issue due to the statistical construction of MCM which is well known (see (Glasserman, 2004)) and is highlighted while estimating sensitivity factors. To converge toward stable estimation, we have to reduce the width of the confidence interval. So several variance reduction methods were developed to decrease the results' volatitliy and the most employed consists to increase the number of simulations.

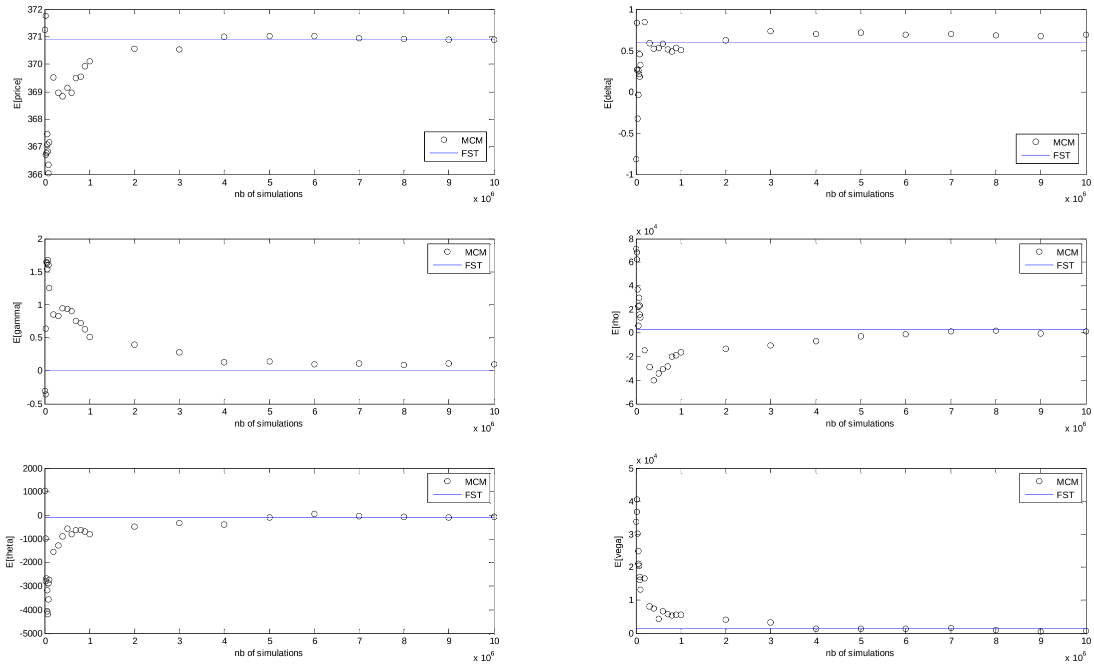


Figure 24: .Study of MCM convergence on mean pricing values (Price and Greeks) for a European Call Option.

In Figure 24 we compare the MCM pricing results according to the simulations number and as expected we note the convergence effect with the increase of simulations. Please note that we estimate the greeks nature by using the difference finite as derivative proxy. This method produces convergent results on Price evaluation however the derivative results remain volatile enough as we can see in Figure 25.

Regarding the sensitivity factors estimation by MCM, Glasserman summarized several methods to solve this issue such as chain-rule application on $\mathbb{E}\left[\frac{\partial V}{\partial x}\right]$ however most of them have limited impacts on more complex option structures. Unfortunately, proxy evaluation of sensitivity factors is also highly employed on trading floors and belongs as core constituent of prices broadcasted through information networks such as Reuters, Bloomberg or Fininfo. Hence we will use only this proxy approach in current disseration to highlight this phenomena and so the stability of FST method.

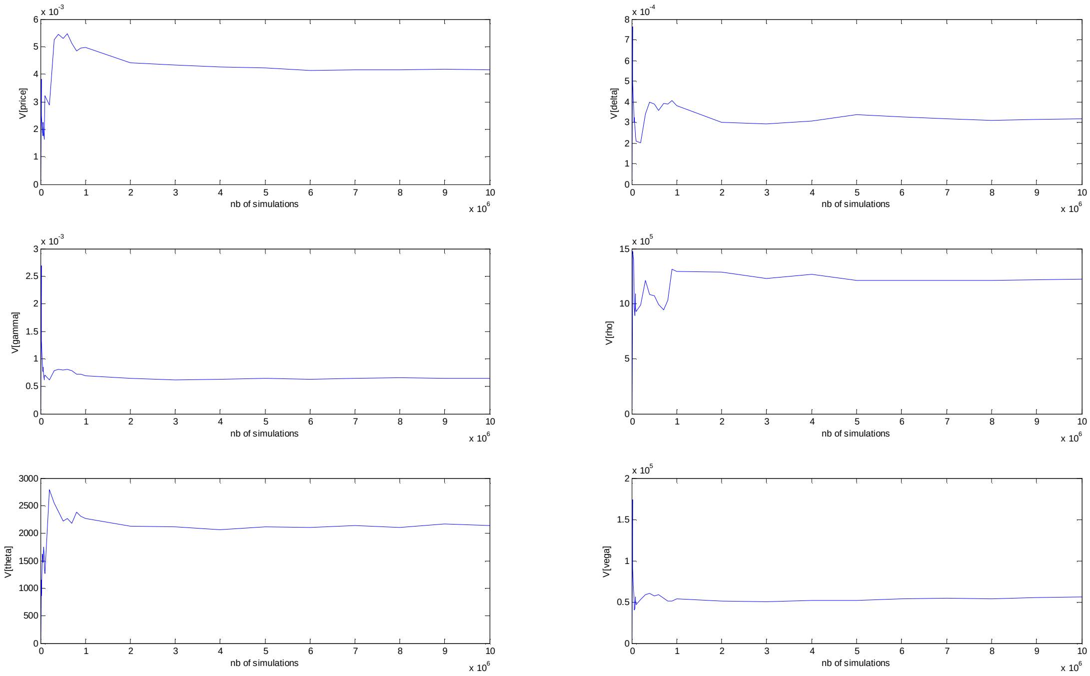


Figure 25: Variance estimation of MCM Pricing (price and Greeks) for a European Call Option.

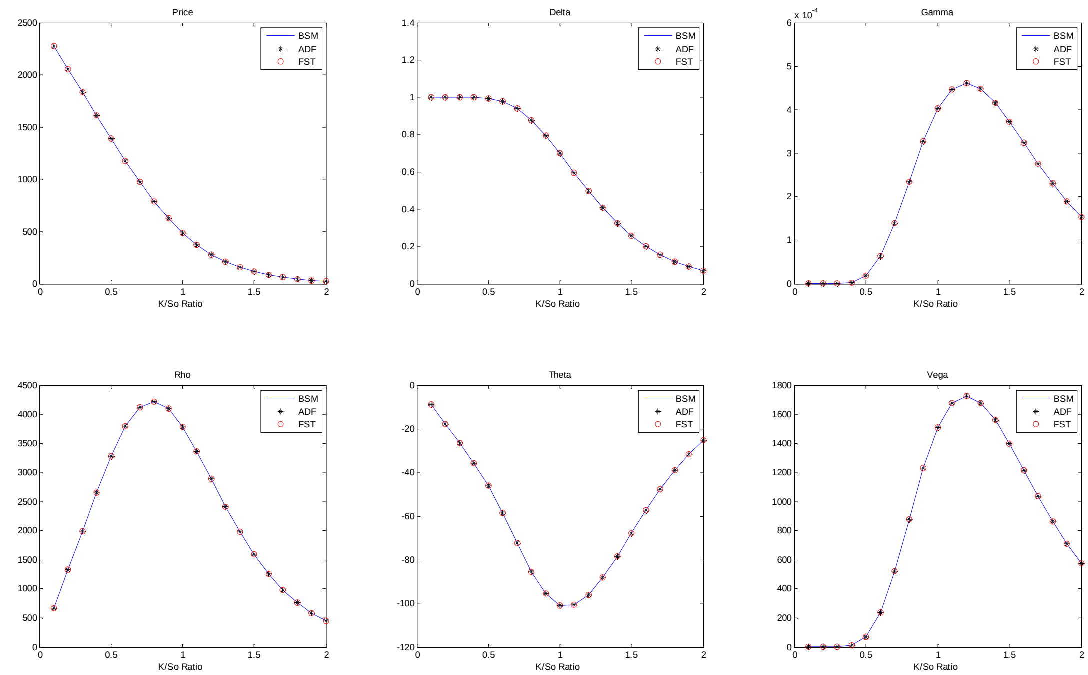


Figure 26: Pricing results of a European Call according to the Pricing method and the Strike level.

Finally we assessed the pricing capacity of FST method by producing a pricing comparison between BSM, FST and ADF methods according to the strike level. These results are summarized in Figure 26 where we draw the priced items and we can notice few differences between the three methods.

5.2.1.2 Digital Options

Now let's study another "*guinea pig*" option with the digital option. Previously we highlighted the volatile issue on MCM evaluations, especially while concerning the Greeks estimations. In case of Digital Option, this phenomena is highlighted by the presence of an indicator function (i.e. $\mathbb{1}_{\{x>p\}}$) in the payoff structure.

In panel below, we presented the FST algorithm employed in digital option evaluation (whatever the type) and please note that the only difference with the European one presented in Panel 13 is the definition of payoff at $t=n$.

t	$\mathbb{R}$		$\mathbb{C}$	
n	1) $\mathbf{v}_n = \varphi(\mathbf{S},K) = \text{asset} \times \mathbb{1}_{(\mathbf{S}-K)^+}$ where asset $\in \{cash, S_n\}$	→	2) $\hat{\mathbf{v}}_n = \text{FT}[\mathbf{v}_n]$	↓
0	4) Price & Greeks See algorithm on §5.2.1.1	←	3) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_n e^{\Psi(n-0)}$	

Panel 14: FST Algorithm used to evaluate a Digital Call Option (Cash or Asset).

5.2.1.2.1 *"Cash or Nothing" Digital Option (DCON)*

We start the digital benchmark with the most likely indicator function known as "*Cash or Nothing*" Option in market jargon. Its payoff structure is simple enough and can be expressed as $\varphi(S,K) = \text{cash} \times \mathbb{1}_{(S-K)^+}$ (call case), i.e. the counterpart gives a cash amount when the index is above the fixed strike.

The Figure 27 presents the pricing results produced by the four following methods: BSM, FST, ADF and MCM. Most of priced items are strongly convergent with little time consumption (i.e. a few milliseconds). However some MCM results show some relevant differences especially on sensitivity factors based on rate, time and volatility (around $\pm$ 50 – 100% difference versus the BSM results).

Parameters	Values
So	2 500.00
K	2 750.00
Cash	100.00
r	4.00%
Vol	20.00%
T	3.00
basis	1.00

Method	Parameters	Values
MCM	Nb Sims	1 000 000.00
FST	N	8 192.00
	X_min	- 10.00
	X_max	10.00

Option	Payoff	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
Digital "Cash or Nothing"	Call	BSM	40.75	0.04	- 0.00	182.60	- 1.00	- 43.03	0.00
		FST	40.75	0.04	- 0.00	182.58	- 1.00	- 43.07	0.03
		ADF	40.75	0.04	- 0.00	182.60	- 1.00	- 43.03	0.00
		MCM	40.77	0.04	- 0.00	286.14	- 2.97	- 32.86	1.88
	Put	BSM	47.95	- 0.04	0.00	- 448.68	4.55	43.03	0.00
		FST	47.94	- 0.04	0.00	- 448.66	4.55	43.07	0.03
		ADF	47.95	- 0.04	0.00	- 448.68	4.55	43.03	0.00
		MCM	48.02	- 0.04	- 0.00	- 307.32	9.52	19.93	1.84

Figure 27: Example of pricing results of a DCON Option (Call and Put).

These volatile results are the perfect illustration of derivative estimation of an indicator function. In Figure 28 we plotted the convergence speed regarding the simulation number for each priced items. We can notice that convergence speeds can be divided into two categories: fast convergence and slow convergence. This second category applies especially to Rho and Theta measures which show important differences with FST measures even for high simulations number.

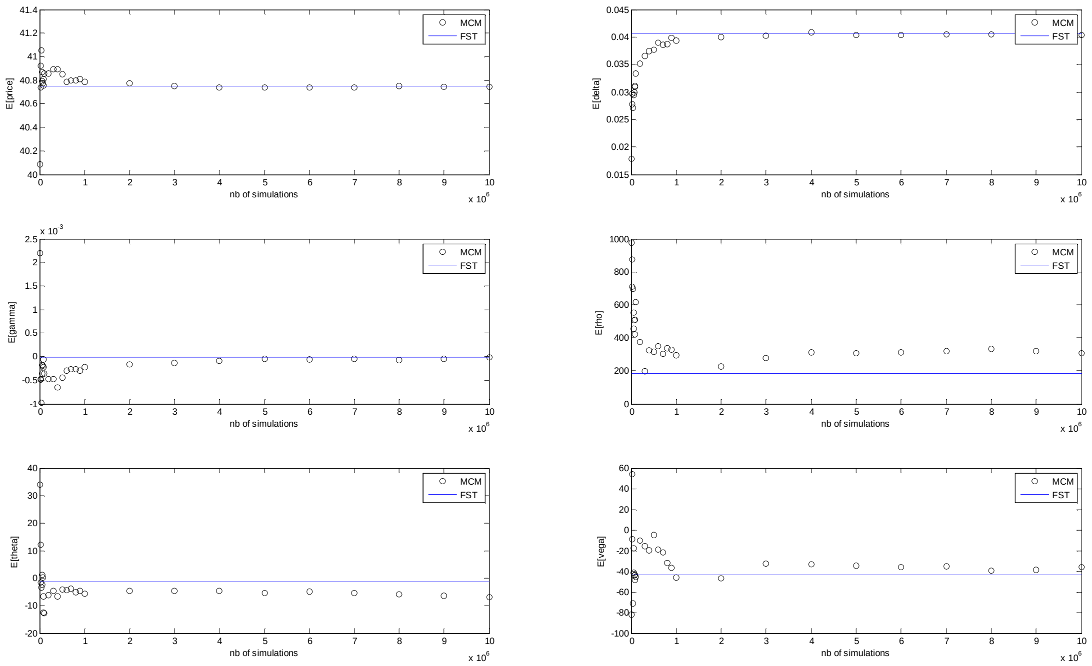


Figure 28: MCM convergence speed on mean pricing values (Price and Greeks) for a DCON Call.

This fact is correlated with the Variance estimations plotted in Figure 29 where variances remain stable whatever the simulations number. This phenomenon represents the influence of the presence of indicator functions into a payoff structure in terms of lack o f convergence.

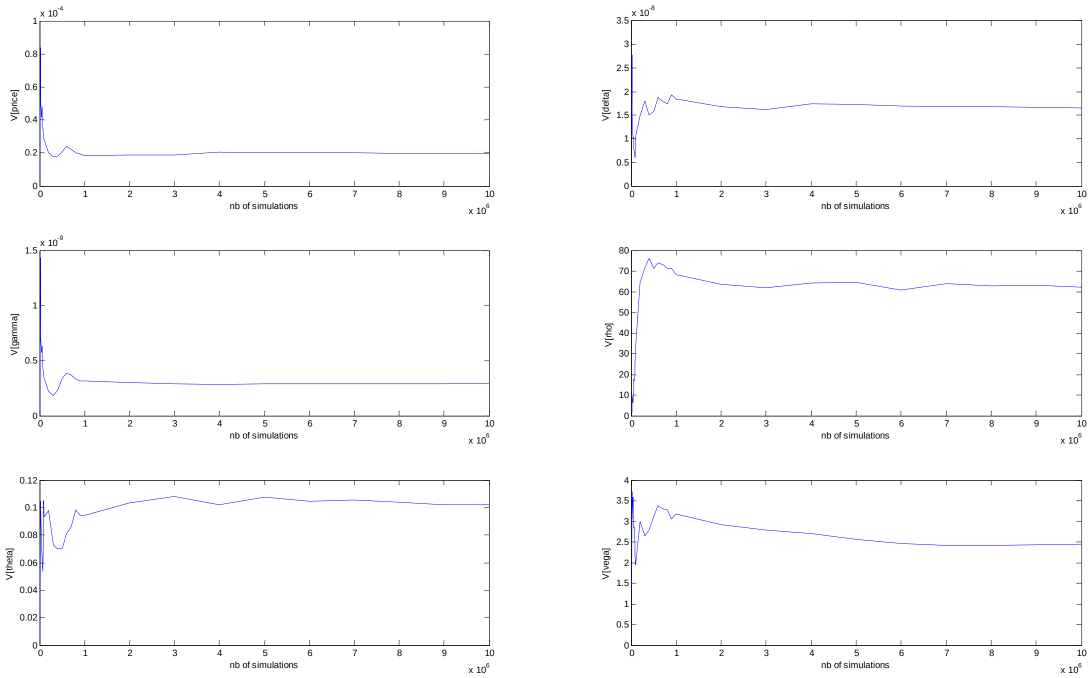


Figure 29: Variance estimation of MCM Pricing (price and Greeks) for a DCON Call.

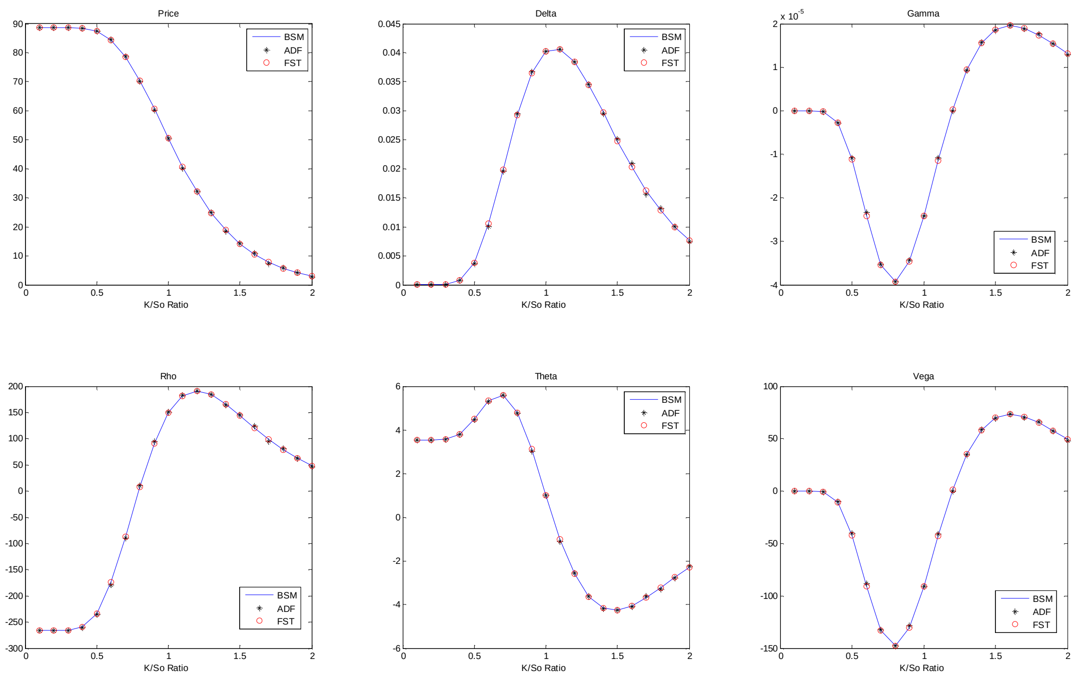


Figure 30: Pricing results of a DCON Call according to the Methodology type and the Strike level.

We assess the FST pricing capacity by comparing the priced items according to the strike level with BSM and ADF methods. We can notice in Figure 30 that there are few differences among the three methods and results seem coherent $\left(\lim_{K \to 0} \text{Price} = \text{Cash} \right)$

5.2.1.2.2 Digital "Asset or Nothing" Option (DAON)

Now we continue the digital benchmark with the "*Asset or Nothing*" Option which has the a payoff structure defined as $\varphi(S,K) = S \times \mathbb{1}_{(S-K)^+}$ (call case). The difference with the previous digital option is in the delivered asset, i.e. the underlying index/stock instead of a cash amount.

We presented in Figure 31 the pricing results produced by BSM, ADF, FST and MCM methods. Thus we can see that BSM, ADF and FST produced convergent pricings with few time consumption. Regarding the MCM results, we see strong divergences essentially located in greeks estimation. Thus the most striking point is the calculation of non zero gamma which is in contraction with the closed formula calculation. Moreover we can see high volatile results on other sensitivity factors estimations, especially with Vega which is ten times higher than values calculated by other methods.

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	3.00
basis	1.00

Method	Parameters		Values
MCM	Nb Sims		1 000 000.00
FST	N		8 192.00
	X_min	-	10.00
	X_max		10.00

Option	Payoff	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
Digital "Asset or Nothing"	Call	BSM	1 491.42	1.71	0.00	8 383.05	- 128.22	493.34	0.00
		FST	1 491.66	1.71	0.00	8 382.57	- 128.17	492.14	0.03
		ADF	1 491.42	1.71	0.00	8 383.05	- 128.22	493.34	0.00
		MCM	1 491.66	1.18	- 1.10	6 064.80	- 238.58	5 909.40	1.89
	Put	BSM	1 008.58	- 0.71	- 0.00	- 8 383.05	128.22	- 493.34	0.00
		FST	1 008.34	- 0.71	- 0.00	- 8 382.57	128.17	- 492.14	0.03
		ADF	1 008.58	- 0.71	- 0.00	- 8 383.05	128.22	- 493.34	0.00
		MCM	1 009.13	- 1.22	1.09	- 7 405.06	152.56	- 6 702.98	1.88

Figure 31: Example of pricing results of a DAON Option (Call and Put).

We continue the investigation by analyzing the convergence speed evolution (see Figure 32) by comparing FST and MCM results. The striking fact while looking at the results is the lack of convergence capacity for most of measures. Some differences are negligible (e.g. delta and gamma), others are important (e.g. Theta and Vega).

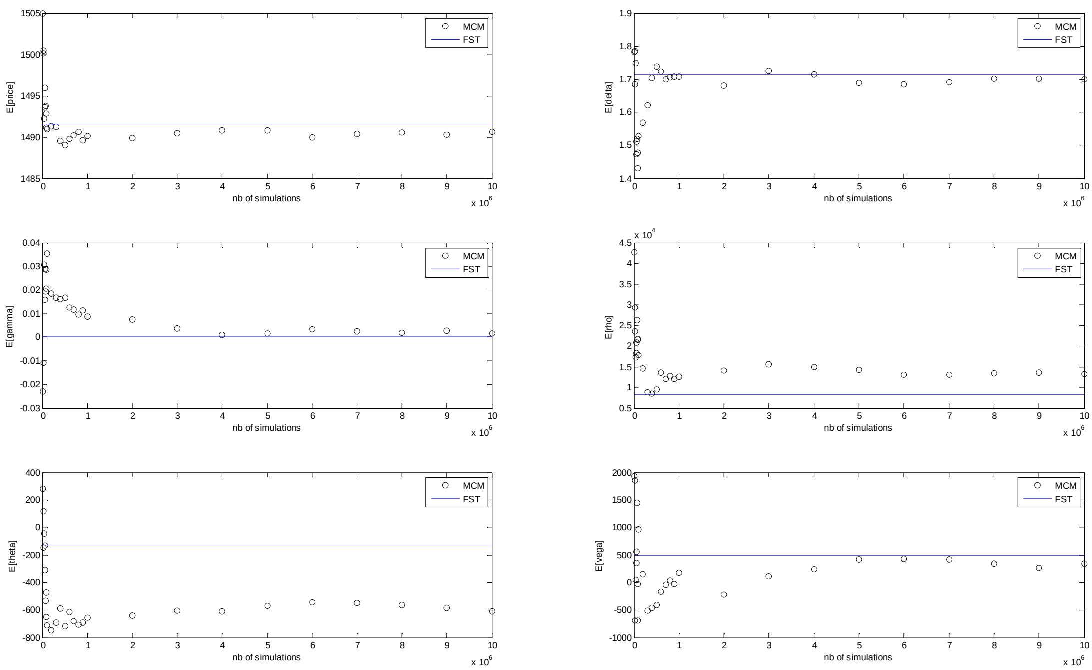


Figure 32 MCM convergence speed on mean pricing values (Price and Greeks) for DAON Call.

The next question is “*Is this phenomenon raised from a high variance?*” The analysis of variance measures (see Figure 33) shows that most of them are stable and but some of them remain at high variance level. Hence the lack of convergence could find its origin in the presence of high and stable variance, in the presence of the function indicator into the payoff structure, but also in the delivery of underlying Index / Stock.

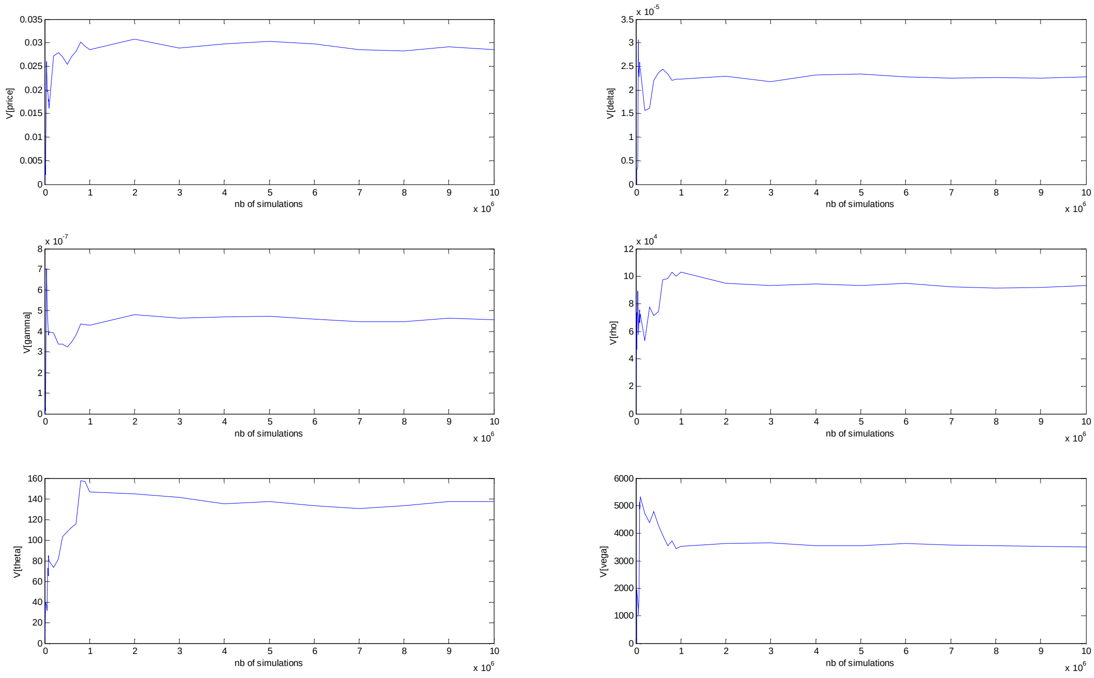


Figure 33: Variance estimation of MCM Pricing (price and Greeks) for a DAON Call.

The last part of the analysis is to compare FST pricing results with other methods (only BSM and ADF). We followed the same approach seen in previous section and results are produced in Figure 34. As we can see, there are few differences and results seem coherent $\left(\lim_{K \to 0} \text{Price} = S_0 \right)$.

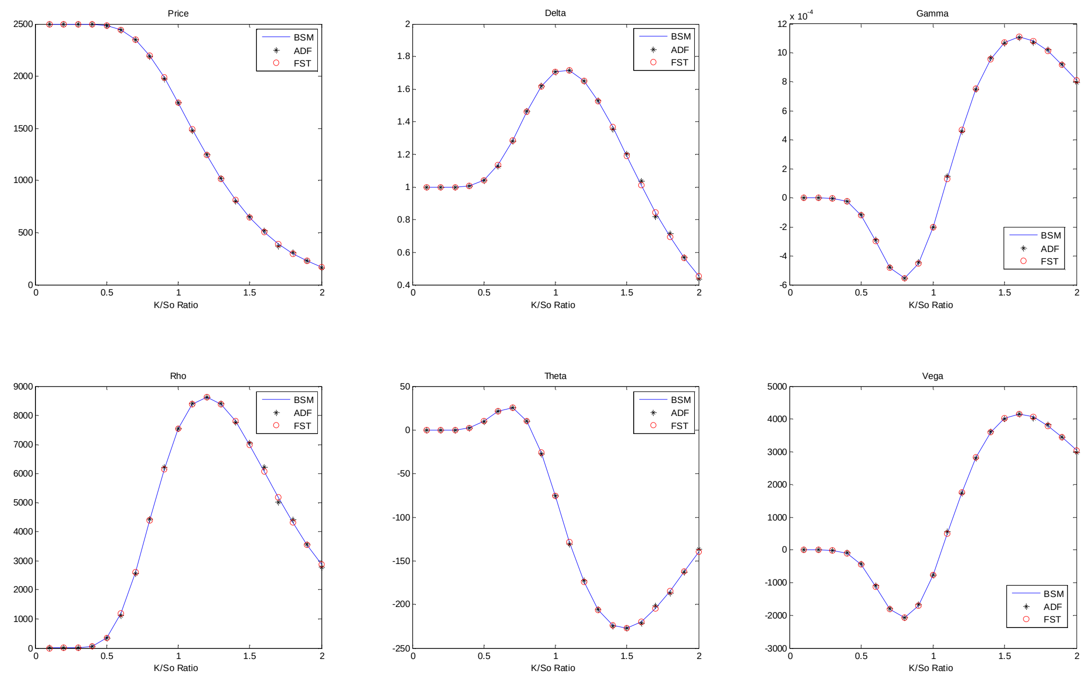


Figure 34: Pricing results of DAON Call according to the Pricing Methodology and the Strike Level.

5.2.1.3 Preliminary analysis on "*path-independent*" option benchmarks

The digital benchmark aims to highlight the pricing capacity of FST method of payoff with discontinuities. The two digital payoffs are widely used in financial literature to show the lack of convergence capacity of Monte Carlo approach. Several works demonstrate that this issue can be solved by using different derivative estimation methods. However this complexity increase propagates in the implementation of more complex structure.

As we will see in §5.2.3, most of Structured Products integrate a large number of digital options and then show a highly discontinuous profile. In the two last sections we show the lack of convergence capacity for "*simple*" digital payoff. Fortunately closed-formulae exist to assess the pricing measures produced by FST and MCM but it won't be the case for more complex structures. Then we can extrapolate that this issue will go worse and worse for payoff structures with compounded digital options.

5.2.2 Path-Dependent Options

In the current section, we will evaluate more complex options socalled "*Path Dependent*". This means that the value of such options is dependent of the followed path between start date and maturity date. The benchmark will encompass two different options: American Option and Barrier Option.

In terms of implementation, both option types will illustrate how to adapt the Iterative Integration process presented in §5.1.2. This is a key stage to understand how to construct FST algorithm adapted to Structured Product Pricing.

5.2.2.1 American Option

We began the "*path-dependent*" benchmark with the american option. This option in its simplest form defines a European payoff which can be exercised at each time bucket between the starting date and the maturity date. The purpose of this benchmark is to illustrate the declination of generic FST algorithm for payoff structures with interstage valuation between the start date and the maturity date.

t	$\mathbb{R}$		$\mathbb{C}$	
n	1) $\mathbf{v}_n = \varphi(\mathbf{S},K) = \max(\mathbf{S}-K,0)$	→	2) $\hat{\mathbf{v}}_n = \mathrm{FT}[\mathbf{v}_n]$	↓
n-1	4) $\mathbf{v}_{n-1} = \mathrm{FT}^{-1}[\hat{\mathbf{v}}_{n-1}]$	←	3) $\hat{\mathbf{v}}_{n-1} = \hat{\mathbf{v}}_n e^{\Psi(n-n+1)}$	↓
	5) $\mathbf{v}_{n-1} = \max(\varphi(\mathbf{S},K),\mathbf{v}_{n-1})$	→	6) $\hat{\mathbf{v}}_{n-1} = \mathrm{FT}[\mathbf{v}_{n-1}]$	↓
n-2	8) $\mathbf{v}_{n-2} = \mathrm{FT}^{-1}[\hat{\mathbf{v}}_{n-2}]$	←	7) $\hat{\mathbf{v}}_{n-2} = \hat{\mathbf{v}}_{n-1} e^{\Psi(n-1-n+2)}$	
	9) $\mathbf{v}_{n-2} = \max(\varphi(\mathbf{S},K),\mathbf{v}_{n-2})$	→	10) $\hat{\mathbf{v}}_{n-2} = \mathrm{FT}[\mathbf{v}_{n-2}]$	⋮
⋮	⋮		⋮	⋮
0	X+1) Price & Greeks See algorithm on §5.2.1.1	←	X) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_n e^{\Psi(n-0)}$	↓

Panel 15: FST Algorithm used to evaluate an American Option.

In Panel 15, we present the adapted algorithm for american option and note the key differences:

1. We divide the time space into several buckets, each representing an arbitrage opportunity,
2. At each time bucket after adding time value, we make a round trip such as $\mathbb{C} \to \mathbb{R} \to \mathbb{C}$,

3. While in $\mathbb{R}$ space, we proceed to the arbitrage estimation with the following payoff $\mathbf{v}_i = \max(\varphi(\mathbf{S}, K), \mathbf{v}_i)$. Please note that this apply on each point of $\mathbf{v}_i$ and not for the whole vector values.
4. We repeat the process until arriving at $t = 0$ then we estimate the priced items with the same estimation process presented in §5.2.1.1.

The american option benchmark is produced with help of four pricing models:

1. CRR: The Cox-Ross-Rubistein method (Cox, et al., 1979) which approximates the dispersion of the underlying Index / Stock by building a tree.
2. FiD: The finite difference model which build a probability grid to approximate the Index / Stock Dispersion,
3. LSM: The Longstaff-Schwartz method (Longstaff, et al., 2001) which develops a backward evaluation method adapted to Monte Carlo Pricing. The arbitrage process is replicated by introducing a regression approach to estimate the expected payoff at $t = i - 1$.
4. FST: And finally the FST method following the algorithm presented in Panel 15.

In Figure 35 we present the results produced by each method regarding an American Call / Put Option. As we know, an American Call Pricing must be identical to a European Call Pricing which can be assessed by proceeding to a comparison with results presented in Figure 23. Regarding the results produced for an American Put Option, we can see that most of results are quite similar apart for the LSM method which has relevant differences in sensitivity factors' estimation.

The analysis of these results shows a high consistancy between CRR, FiD and FST methods with negligible differences for the call valuation. The situation is different with MCM evaluations which show important differences, especially for the Rho, Theta and Vega measures. Regarding the calculation time consumption, the FST method shows the best perfomance with a time consumption from twice lower (FiD) to twenty times lower (CRR).

We also produced the pricing results for the put option to assess the good behavior of pricing models. However we were surprised by some differences on some sensitivity factors pricing. Thus the CRR method shows a null value for most of sensitivity factors while other methods calculated a sensitivity value for Rho and Theta.

The presence of these sensitivity factors seems normal because:

1. A rate increase will impact the drift and so the Index / Stock will tend to higher values than its initial value. Hence the american put must loose value with rate increase.
2. A Time increase will favorize the dispersion and so increase the probability to produce positive payoff.

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	3.00
basis	252.00

Method	Parameters	Values
CRR	Nb X Buckets	756.00
FiD	Nb X Buckets	756.00
	Nb Y Buckets	1 000.00
LSM	Nb Sims	10 000.00
FST	N	8 192.00
	X_min	- 10.00
	X_max	10.00

Option	Payoff	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
American	Call	CRR	370.94	0.60	0.00	3 360.62	- 101.02	1 686.39	49.70
		FiD	369.56	0.58	0.00	3 044.66	- 94.94	1 630.29	5.37
		LSM	373.00	0.56	0.01	26 080.18	- 209.88	7 541.83	14.98
		FST	370.92	0.60	0.00	3 361.51	- 100.71	1 676.61	1.11
	Put	CRR	372.63	- 0.53	0.00	- 2 205.83	- 21.81	1 536.52	49.59
		FiD	372.40	- 0.52	0.00	- 1 941.53	- 25.18	1 532.09	5.37
		LSM	362.63	- 0.44	0.04	2 628.49	- 19.37	945.31	85.66
		FST	372.56	- 0.53	0.00	- 5 099.77	- 21.67	2 690.01	1.10

Figure 35: Example of pricing results of an American Option (Call and Put).

We continue the analysis by producing a benchmark on an American Put Option between CRR and FST according to the Strike level. These results are produced in Figure 36 which represents the calculated values for Price and Sensitivity Factors by the two methods.

As we can see, both methods produce very similar results for each priced items, exception of Gamma where we notice differences for K/S_0 near 1. However the gamma values are very low (less than 0.01) and these differences are representatives of proxy evaluations around 0.

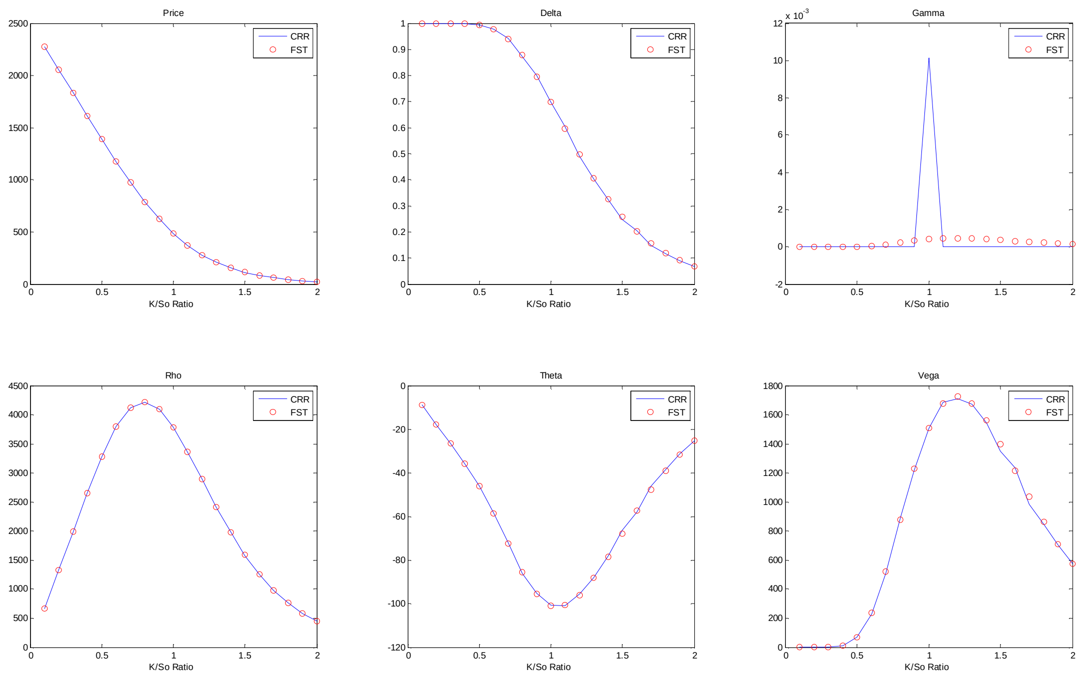


Figure 36: Pricing Results Comparison of an American Put according to Pricing methods and Strike Level

We can conclude the "*American Option*" benchmark that the FST method produces accurate and relevant results but with less time consumption where it is five up to fifty times faster than the methods of reference used in this benchmark.

5.2.2.2 Barrier Option

In this section we will study another type of "*path-dependent*" option with help of Barrier Options. This aims to introduce a new complexity in interstage payoff valuation with the presence of an indicator function. Firstly we will start with the European Barrier Option (exercise date limited to maturity date) and lastly we will finish with the American Barrier Option which adds the interstage arbitrage process.

5.2.2.2.1 European Option with Barrier ("EOB")

In Panel 16, we presented the FST algorithm adapted to a European Call Option with Barrier: the only difference with a simple European Call Option is the introduction of an indicator function into the native payoff structure.

The EOB benchmark is produced with help of the CRR, FST and MCM methods. We identify the several barrier options with the following acronyms:

1. DI means "*Down and In*", i.e. the payoff is activated only when the stock reached down the barrier level.
2. DO means "*Down and Out*" and is the opposite of DI barrier option.
3. UI means "*Up and In*", i.e. the payoff is activated only when the stock reached up the barrier level
4. UO means "*Up and Out*" and is the opposite of UI barrier option

t	$\mathbb{R}$		$\mathbb{C}$	
n	1) $\mathbf{v}_n = \varphi(\mathbf{S},K) = \max(\mathbf{S}-K,0)\times \mathbb{1}_B$ where asset : delivered asset (cash or S_n) B : condition according to the barrier type (DI, DO, UI or UO)	→	2) $\hat{\mathbf{v}}_n = \mathrm{FT}[\mathbf{v}_n]$	↓
0	4) Price & Greeks See algorithm on §5.2.1.1	←	3) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_n e^{\Psi(n-0)}$	

Panel 16: FST Algorithm used to evaluate a European Call Option with Barrier.

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	3.00
basis	252.00

Method	Parameters	Values
CRR	Nb X Buckets	756.00
FST	N	8 192.00
	X_min	-10.00
	X_max	10.00
MCM	Nb Sims	10 000.00

Barreer	Level
DI	2 250.00
DO	2 250.00
UI	3 000.00
UO	3 000.00

Payoff	Option	Barreer	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
European Option with Barreer	Call	DI	CRR	-	-	-	-	-	-	3.67
			FST	-	-	-	-	-	-	0.06
			MCM	-	-	-	-	-	-	5.36
		DO	CRR	370.98	0.60	-0.00	3 360.79	-93.26	672.16	3.67
			FST	370.92	0.60	0.00	3 361.51	-100.71	1 676.61	0.03
			MCM	368.57	0.94	2.20	-115 605.23	1 575.69	-2 545.21	5.32
		UI	CRR	361.18	0.50	0.00	3 368.08	-95.26	673.62	3.67
			FST	360.19	0.59	0.00	3 368.56	-102.62	1 731.18	0.03
			MCM	354.95	0.83	1.19	-146 212.22	3 064.23	-10 009.24	5.33
		UO	CRR	9.80	0.10	-0.00	-7.28	2.00	-1.46	3.67
			FST	10.73	0.00	-0.00	-7.04	1.91	-54.58	0.02
			MCM	10.62	0.04	-0.05	2 191.75	-7.20	2 717.82	5.32
	Put	DI	CRR	264.24	-2.13	-1.53	-3 717.37	-12.56	-743.47	3.67
			FST	259.66	-0.39	0.00	-3 684.60	-13.54	1 880.14	0.03
			MCM	258.20	-1.68	1.37	39 253.46	-100.14	2 156.56	5.34
		DO	CRR	45.77	1.72	1.53	-238.93	9.64	-47.79	3.66
			FST	50.29	-0.02	-0.00	-270.98	10.40	-203.53	0.02
			MCM	50.14	0.18	0.06	4 505.60	6.47	2 347.72	5.33
		UI	CRR	-	-	-	-	-	-	3.67
			FST	-	-	-	-	-	-	0.01
			MCM	-	-	-	-	-	-	5.37
		UO	CRR	310.01	-0.40	0.00	-3 956.30	-2.93	-791.26	3.67
			FST	309.95	-0.40	0.00	-3 955.58	-3.15	1 676.61	0.03
			MCM	308.30	-2.14	1.46	-95 407.99	-222.91	-32 320.58	5.35

Figure 37: Example of pricing results of a European Option with Barrier (Call and Put).

The valuation is related to Call and Put Options with relevant and irrelevant barrier levels to assess that results remain in line with option structure. Thus a Call with a DI Barrier Option can't be activated and generate a payoff instrinsic value at the same time. And the Put with UI Barrier Option represents the same situation adapted to a put payoff.

We presented the EOB benchmark results in Figure 37

1. Most of Price estimations are convergent (the highest difference represents 0.28% of S_0),
2. Partial Derivatives:
    a. CRR and FST are convergent most of the time apart from Vega estimation where FST method calculates higher values and with some sign differences.

   b. The MCM method is convergent on Delta and Gamma estimations and diverges highly on other sensitivity estimations.
3. Time consumption: the FST is 360% to 500% faster than CRR or MCM methods.

Going forward we will review the robustness of FST method by comparing its results with CRR. Thus we produced several calculations based on a moving K/S_0 ratio and Barrier level $B/S_0 = K/S_0 + 0.1$. The results are presented in Figure 38:

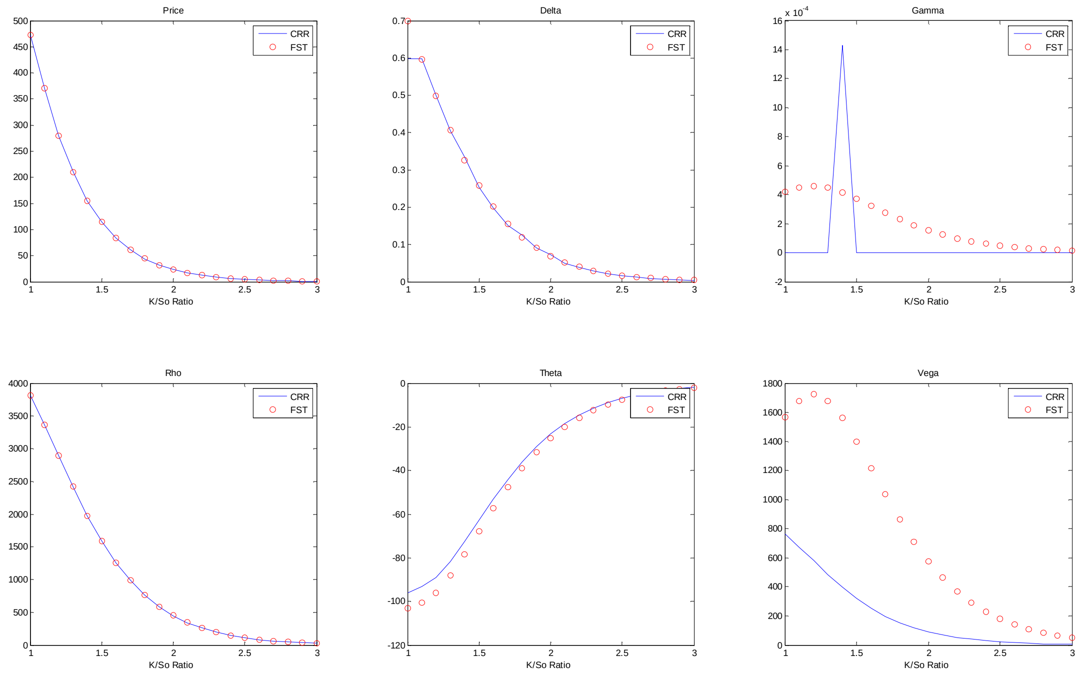


Figure 38: Pricing Results of a European Call with a moving UI Barrier (B/S = K/S + 0.1) depending of Pricing Method and Strike Level.

We can see that Price, Delta and Rho measures show similar results whatever the method and strike levels. Regarding the Theta measure, results are similar with more important values produced by FST method. We can see the same difference type on Gamma as presented for American Option (see §5.2.2.1), i.e. minor differences on low Gamma values. The most striking difference concerns the Vega measure where FST presents Volatility levels twice time greater than those produced by CRR.

At this stage we can say that the FST method seems relevant on most priced items apart from the Vega. However we can't establish the origin of such differences either in terms of level or sign.

5.2.2.2.2 American Option with Barrier (AOB)

Now we will increase the payoff structure complexity by using an American Payoff instead of the European Payoff. The consequences on option structure are the presence of a continuous barrier option combined with an American Evaluation condition. Hence this structure will integrate several interstages all along the evaluation period which will combine an indicator function into the American Payoff evaluation.

The presence of a permanet Barrier imposes to us to to distinguish the "*knock-in*" option types by adding the value of knocked-in options during the insterstage period:

1. The backward valuation will start by adding the terminal payoff value, i.e. the knocked-in option only at $t = n$.

2. we propagate this value to $t = n-1$ by considering this value won't touch a new the barrier level. But we have to integrate the value of knocked-in options at $t = n-1$. We combine the two option values according to the following formula:
   $$\mathbf{v}_{n-1} = \mathrm{FT}^{-1}\left[\hat{\mathbf{v}}_{n-1}\right]\times \mathbb{1}_{\overline{B}} + \mathrm{FT}^{-1}\left[\mathrm{FT}\left[\max\left(\mathbf{S}-K,0\right)\right]e^{\Psi(n-n+1)}\right]\times \mathbb{1}_{\{DI,UI\}}$$
3. Lastly we evaluate the american payoff as follows: $\mathbf{v}_{n-1} = \max\left(\varphi\left(\mathbf{S},K\right),\mathbf{v}_{n-1}\right)$
4. We repeat the process until $t = 0$

Next we described the FST algorithm adapted to an American Option with Barrier where we add the interstage evaluation steps as for a simple American option.

t	$\mathbb{R}$		$\mathbb{C}$
n	1) $\mathbf{v}_n = \varphi(\mathbf{S},K) = \max(\mathbf{S}-K,0)\times\mathbb{1}_B$ where asset : delivered asset (cash or S_n), B: condition according to the barrier type	→	2) $\hat{\mathbf{v}}_n = \mathrm{FT}[\mathbf{v}_n]$ ↓
n-1	4) $\mathbf{v}_{n-1} = \mathrm{FT}^{-1}[\hat{\mathbf{v}}_{n-1}]\times\mathbb{1}_{\overline{B}} + \mathrm{FT}^{-1}\left[\mathrm{FT}[\max(\mathbf{S}-K,0)]e^{\Psi(n-n+1)}\right]\times\mathbb{1}_{\{DI,UI\}}$	←	3) $\hat{\mathbf{v}}_{n-1} = \hat{\mathbf{v}}_n e^{\Psi(n-n+1)}$
	5) $\mathbf{v}_{n-1} = \max(\varphi(\mathbf{S},K),\mathbf{v}_{n-1})$	→	6) $\hat{\mathbf{v}}_{n-1} = \mathrm{FT}[\mathbf{v}_{n-1}]$ ↓
n-2	8) $\mathbf{v}_{n-2} = \mathrm{FT}^{-1}[\hat{\mathbf{v}}_{n-2}]\times\mathbb{1}_{\overline{B}} + \mathrm{FT}^{-1}\left[\mathrm{FT}[\max(\mathbf{S}-K,0)]e^{\Psi(n-n+2)}\right]\times\mathbb{1}_{\{DI,UI\}}$	←	7) $\hat{\mathbf{v}}_{n-2} = \hat{\mathbf{v}}_{n-1} e^{\Psi(n-1-n+2)}$
	9) $\mathbf{v}_{n-2} = \max(\varphi(\mathbf{S},K),\mathbf{v}_{n-2})$	→	10) $\hat{\mathbf{v}}_{n-2} = \mathrm{FT}[\mathbf{v}_{n-2}]$ ↓
⋮	⋮		⋮
0	X+1) Price & Greeks See algorithm on §5.2.1.1	←	X) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_n e^{\Psi(n-0)}$

Panel 17: FST Algorithm used to evaluate an American Call Option with Permanent Barrier.

Now we will evaluate the pricing capacity of FST method by comparing its results with the CRR method adapted to evaluation of AOB. We produced two pricing results according to the barrier proximity to S_0: one with closed barriers (Down = 90% and Up = 110%, see Figure 39) and another with further barriers levels (Down = 50% and Up = 200%, see Figure 40):

1. Price estimations are almost equal whatever the barrier type and barrier level.
2. Sensitivity Factors show more disparate results especially for Rho and Vega pricing results where we can notice sign mismatches and / or great differences (up to ten times the result produced with help of CRR, see vega of a Call DI Option in Figure 39).
3. Regarding the time consumption, FST is 4 to 6 times faster than the CRR method.

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	3.00
basis	252.00

Method	Parameters	Values
CRR	Nb X Buckets	756.00
FST	N	8 192.00
	X_min	-10.00
	X_max	10.00

Barreer	Level
DI	2 250.00
DO	2 250.00
UI	3 000.00
UO	3 000.00

Payoff	Option	Barreer	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
American Option with Barreer	Call	DI	CRR	128.24	-2.63	-2.31	895.27	-55.26	179.05	11.46
			FST	120.17	-0.30	0.00	-2 577.45	-58.06	2 772.88	2.67
		DO	CRR	242.74	3.23	2.31	2 465.53	-36.48	493.11	8.11
			FST	250.75	0.89	-0.00	5 938.96	-42.64	-1 096.28	1.33
		UI	CRR	369.85	0.58	0.00	3 361.89	-92.27	672.38	11.40
			FST	369.76	0.60	0.00	3 379.73	-101.28	1 686.52	2.67
		UO	CRR	153.18	0.77	0.00	461.62	-11.59	92.32	8.04
			FST	135.20	0.16	0.00	789.70	-11.24	21.41	1.41
	Put	DI	CRR	366.79	-0.81	-0.25	-2 190.44	-22.52	-438.09	11.51
			FST	352.49	-0.52	0.00	-4 922.41	-17.21	2 485.13	2.77
		DO	CRR	351.75	0.35	0.83	-1 129.41	-10.39	-225.88	8.12
			FST	344.66	-0.47	0.00	-4 523.42	-10.28	2 117.74	1.36
		UI	CRR	82.31	-0.18	0.00	-488.95	-25.74	-97.79	11.41
			FST	74.62	0.13	0.00	750.43	-22.58	377.29	2.76
		UO	CRR	300.91	-0.46	0.00	-1 279.54	-0.06	-255.91	8.03
			FST	300.61	-0.71	0.00	-6 248.49	-0.39	2 511.03	1.38

Figure 39: Example of pricing results of an American Option with Barrier (Call and Put) with close barriers (Down = 90% of So, Up = 120 % of So).

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	3.00
basis	252.00

Method	Parameters	Values
CRR	Nb X Buckets	756.00
FST	N	8 192.00
	X_min	-10.00
	X_max	10.00

Barreer	Level
DI	1 250.00
DO	1 250.00
UI	5 000.00
UO	5 000.00

Payoff	Option	Barreer	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
American Option with Barreer	Call	DI	CRR	0.00	0.00	0.00	0.00	-0.01	0.00	11.27
			FST	0.00	-0.00	0.00	-0.08	-0.01	0.21	2.66
		DO	CRR	370.98	0.60	0.00	3 360.79	-91.73	672.16	7.89
			FST	370.92	0.60	0.00	3 361.59	-100.70	1 676.39	1.31
		UI	CRR	134.11	0.11	0.00	2 000.65	-97.28	400.13	11.29
			FST	131.51	0.34	0.00	2 177.07	-104.82	2 273.67	2.67
		UO	CRR	365.83	0.60	0.00	3 162.11	-85.69	632.42	7.87
			FST	364.91	0.58	0.00	3 237.75	-93.33	1 504.75	1.38
	Put	DI	CRR	42.86	-0.01	-0.00	-889.17	-30.53	-177.83	11.32
			FST	41.65	-0.12	0.00	-1 057.31	-32.44	1 396.03	2.74
		DO	CRR	372.65	-0.53	0.00	-2 206.87	-19.73	-441.37	7.94
			FST	372.58	-0.53	0.00	-5 099.99	-21.67	2 690.08	1.37
		UI	CRR	0.02	-0.00	0.00	-0.08	-0.04	-0.02	11.24
			FST	0.01	0.00	0.00	0.47	-0.04	1.03	2.76
		UO	CRR	372.64	-0.53	0.00	-2 206.77	-19.69	-441.35	7.87
			FST	372.57	-0.53	0.00	-5 100.48	-21.63	2 689.02	1.37

Figure 40: Example of pricing results of an American Option with Barrier (Call and Put) with Far Barriers (Down = 50% of So, Up = 200 % of So).

The pending question regarding the two benchmarks' results concerns the relevancy of sensitivity factors evaluated by FST method. This phenomenon is also observed with previous benchmarks at lower level. However most of pricing methods reviewed previously use a path diffusion proxy either with help of lattice (CRR) or grid (FiD), while the FST method use a proxy evaluation by interpolating calculated values.

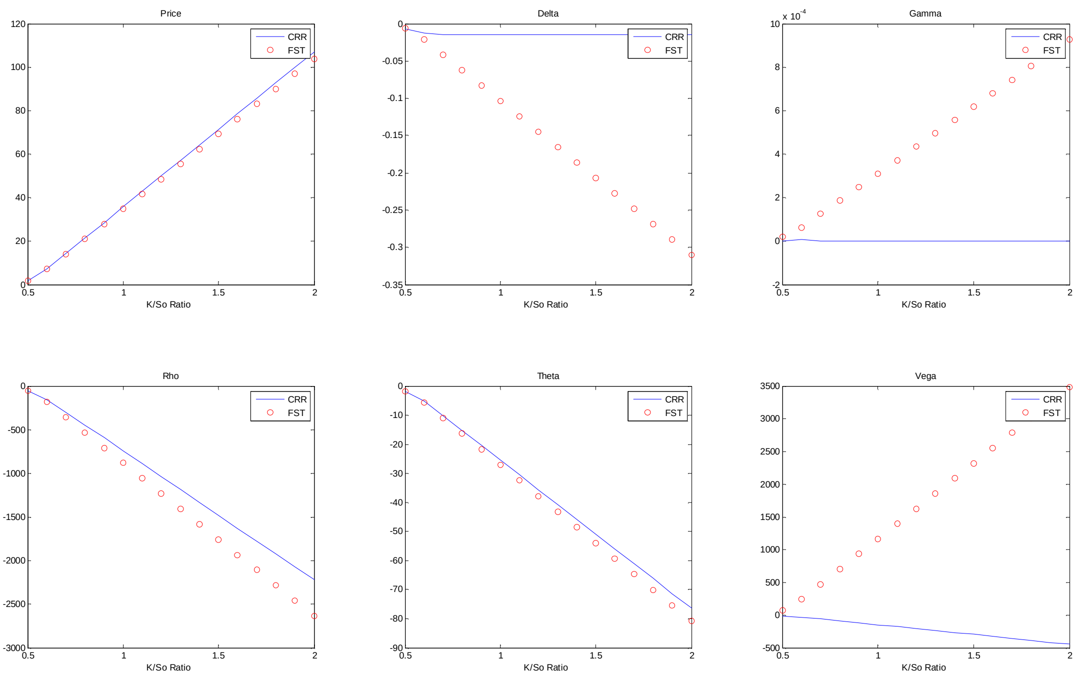


Figure 41: Pricing Results of an American Put with a fixed DI Barrier (B/So = 0.5) depending of Pricing Method and Strike Level

We investigate further the pricing capacity of FST method by producing a benchmark between CRR and FST with several strike levels and a constant Barrier Level ($B = S_0/2$). There we can see that both methods are close enough with less convergent results than those presented in previous sections. Most of priced items are "*near convergent*", i.e. the divergence increases slightly with the Strike Level. The only exception is the Vega measure where there are an opposition of sign, plus an absolute value twice higher than the CRR pricings. This situation is similar to the one presented in section §5.2.2.2.1.

5.2.2.3 Forward Starting Option (FSO)

In this section we will focus on the forward starting option, i.e. options whom strike will be set up in the future. This is a common feature in Structured Products' Primary Market because it cancels the basis effect and so the implication of Delta and Gamma.

Please remind that FST Method is a backward evaluation method and so adapting generic algorithm implies to define a new dimension according the future value of the strike k. Thereby we move from a vectorial valuation with size $(N_s, 1)$ to a matrix valuation with size (N_s, N_k) during the evaluation period $[t_s, t_n]$ where N_s and N_k represents respectively the number of points in discrete vectors $\mathbf{S}$ and $\mathbf{K}$, t_n represents the maturity date and t_s the strike date. The next step is to estimate the option price for each $k \in \mathbf{K}(1, N_k)$ to constitute the value vector $\mathbf{v}_{t_s}$ where the option price "*At The Money*" (ATM) is the fixed strike k. Lastly we get the value vector $\mathbf{v}_{t_0}$ by actualizing the vector $\mathbf{v}_{t_s}$. To give a better intuition of the adapted algorithm, we represented in Figure 42 the pricing process according to the employed dimensions and the detailed version is presented in Panel 18 for a Forward Starting European Option.

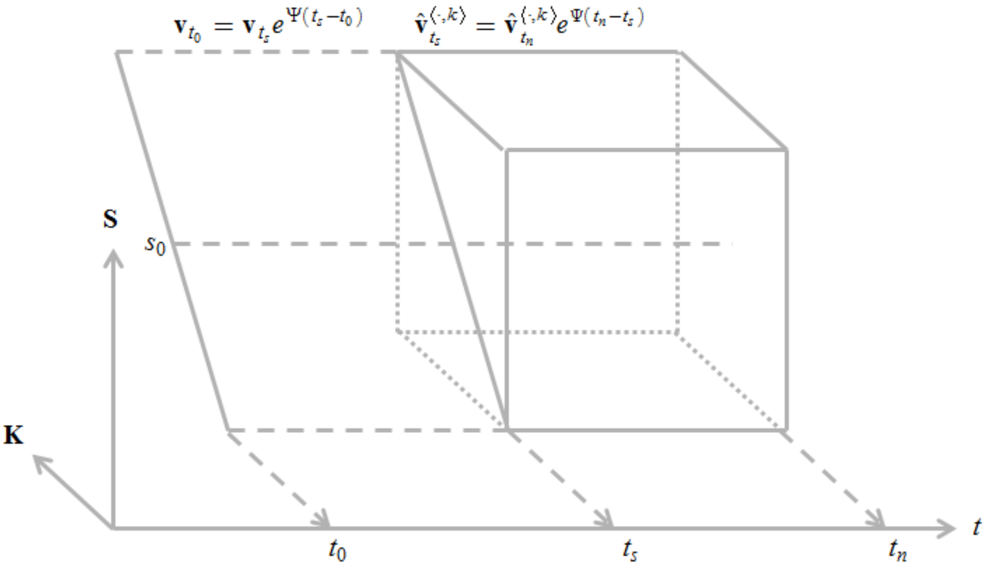


Figure 42: Pricing Representation of Forward Starting Options

t	$\mathbb{R}$		$\mathbb{C}$
n	1) $\mathbf{v}_{t_n}^{\langle\cdot,k\rangle} = \varphi(\mathbf{S}, k) = \max(\mathbf{S} - k)$ where size of $\mathbf{v}_{t_n}^{\langle\cdot,\cdot\rangle} = (N_s, N_k)$	→	2) $\hat{\mathbf{v}}_{t_n}^{\langle\cdot,k\rangle} = \mathrm{FT}\left[\mathbf{v}_{t_n}^{\langle\cdot,k\rangle}\right]$ ↓
s	4) $\mathbf{v}_{s}^{\langle\cdot,k\rangle} = \mathrm{FT}^{-1}\left[\hat{\mathbf{v}}_{s}^{\langle\cdot,k\rangle}\right]$ $\mathbf{v}_{s}^{k} = \mathrm{interpolate}\left(\mathbf{v}_{s}^{\langle\cdot,k\rangle}, \mathbf{K}, k\right)$	←	3) $\hat{\mathbf{v}}_{t_s}^{\langle\cdot,k\rangle} = \hat{\mathbf{v}}_{t_n}^{\langle\cdot,k\rangle} e^{\Psi(t_n - t_s)}$
	5) repeat the steps 1 to 4 for each $k \in \mathbf{K}(1, N_k)$		
	6) $\mathbf{v}_{t_s} = \left[\mathbf{v}_{t_s}^{1}, \mathbf{v}_{t_s}^{2}, \ldots, \mathbf{v}_{t_s}^{k}, \ldots \mathbf{v}_{t_s}^{N_k}\right]$	→	7) $\hat{\mathbf{v}}_{t_s} = \mathrm{FT}\left[\mathbf{v}_{t_s}\right]$ ↓
0	X+1) Price & Greeks See algorithm on §5.2.1.1	←	X) $\mathbf{v}_{t_0} = \mathbf{v}_{t_s} e^{\Psi(t_s - t_0)}$

Panel 18: FST Algorithm used to evaluate a European Forward Starting Option.

Next we will produce the European FSO Benchmark with help of four methods: BSM (see forward formulae in §7.3), ADF (based on fwd BSM closed formula), FST and MCM. We produced two benchmark results according to the number of simulations employed in MCM evaluations which are either 1 million paths (Figure 43) or 5 millions paths (Figure 44). Regarding the FST Method, we define the assumption $N_s = N_k$ and use only $N = 1000$ (preliminary tests showed the best ratio "*Accurate Results / Time consumption*").

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T1	1.00
T2	3.00
basis	1.00

Method	Parameters	Values
MCM	Nb Sims	1 000 000
FST	N	1 000.00
	X_min	-10.00
	X_max	10.00

T1	Forward Period
T2	Exercise Period

Option	Payoff	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
European	Call	BSM	370.92	0.15	-	3 361.51	-100.71	1 676.61	0.00
		FST	370.93	0.57	0.00	3 361.49	-100.71	1 676.57	6.57
		ADF	370.92	0.15	-	3 361.51	-100.71	1 676.61	0.01
		MCM	369.68	0.16	0.00	5 156.44	-67.07	1 741.91	2.05
	Put	BSM	309.95	0.12	-	-3 955.58	-3.15	1 676.61	0.00
		FST	309.96	-0.39	0.00	-3 955.61	-3.14	1 676.57	6.57
		ADF	309.95	0.12	-	-3 955.58	-3.15	1 676.61	0.01
		MCM	310.44	0.14	-0.00	-4 385.57	13.26	1 683.43	2.07

Figure 43 Example of pricing results of a Forward Starting European Option (Call and Put) (low simulation number).

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T1	1.00
T2	3.00
basis	1.00

Method	Parameters	Values
MCM	Nb Sims	5 000 000.00
FST	N	1 000.00
	X_min	-10.00
	X_max	10.00

T1	Forward Period
T2	Exercise Period

Option	Payoff	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
European	Call	BSM	370.92	0.15	-	3 361.51	-100.71	1 676.61	0.00
		FST	370.93	0.57	0.00	3 361.49	-100.71	1 676.57	6.51
		ADF	370.92	0.15	-	3 361.51	-100.71	1 676.61	0.01
		MCM	370.43	0.14	0.00	2 763.81	-75.18	1 702.54	10.44
	Put	BSM	309.95	0.12	-	-3 955.58	-3.15	1 676.61	0.00
		FST	309.96	-0.39	0.00	-3 955.61	-3.14	1 676.57	6.52
		ADF	309.95	0.12	-	-3 955.58	-3.15	1 676.61	0.01
		MCM	310.23	0.13	-0.00	-3 723.27	-3.75	1 725.84	10.46

Figure 44: Example of pricing results of a Forward Starting European Option (Call and Put) (higher simulation number).

A common market assumption says that FSOs have a null Delta and Gamma due to the lack of basis effect (i.e. the strike is not fixed). This assumption is partially right regarding the closed formula of the European FSO: the Delta corresponds to a formula independent of S_t, and consequently the Gamma has a null value. This point is assessed by both benchmarks which produced a null Gamma and a low Delta, and hence certified the adapted FST algorithm presented in Panel 18.

Concerning the produced results, Both Benchmarks show a strong convergence either in term of Price or Sensitivity Factor evaluations. However we can notice that MCM requires an important number of simulations to get stable estimations. Meanwhile the FST method shows a strong accuracy even with a low number of discretization points. But we can see that the matrix approach is highly time consuming with a calculation time half time to MCM with 5 million paths. A code analysis of the Matlab code shows that 80% of time is consumed by the Mathworks' "*interp1(…)*" function and the remaining by the adapted code (i.e. 1.3 second). Hence there are possibilities to improve the FST performance by developing a more accurate interpolation method.

We extended this analysis by producing a more detailed benchmark between BSM, ADF and FST methods depending on the Strike Level. These results are presented in Figure 45 and we can notice a general convergence between the three methods apart from the Delta and Gamma measures. However the differences are negligible reported to the FSO Price (less than a basis point).

So we can estimate that the FST method is relevant on pricing of Forward Starting European Option and we assume we can extend this point on other option types. Nevertheless an assessment must be produced to validate completely this assumption.

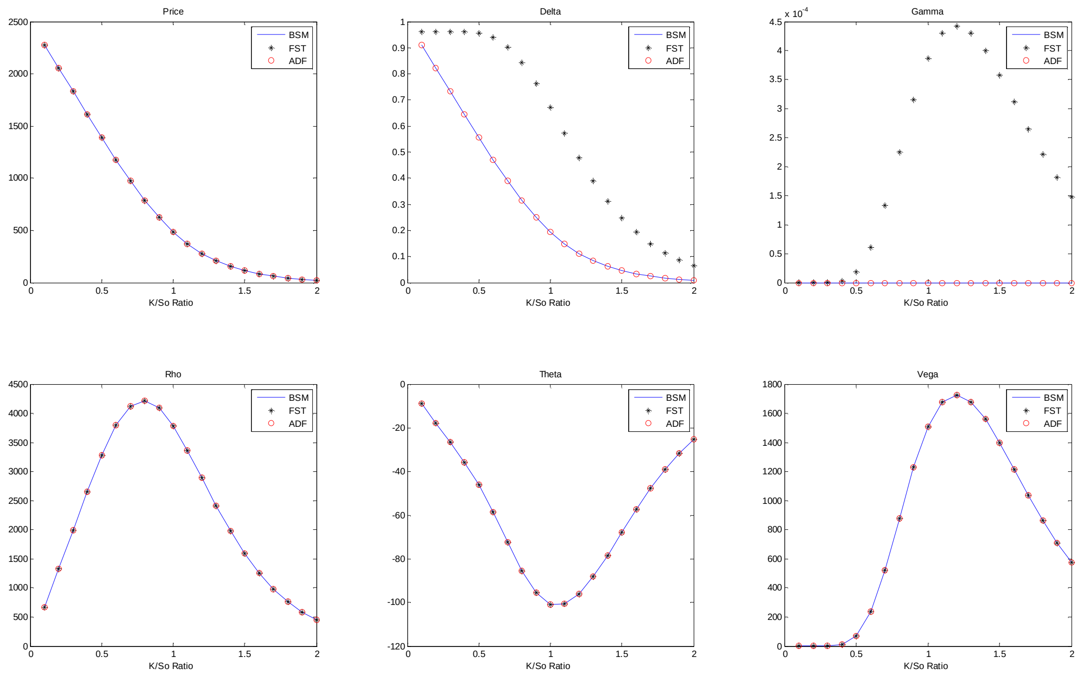


Figure 45: Pricing Results of European Forward Starting Call according to Pricing Methods and Strike Level.

5.2.2.4 Automatic Early Redemption Option (AERO)

We continue the "*path-dependent*" option review with a bond option structure called "*Auto Callable*" option which define a bond with automatic redemption when $S \geq K$ for each $t \in \{1, 2, \ldots, n-1\}$ where n is the bond maturity.

In next panel, we described the adapted FST algorithm for AERO Structure where we define $n-1$ evaluation steps to agregate the value of the bond according to the option trigger.

t	$\mathbb{R}$		$\mathbb{C}$
n	1) $\mathbf{v}_n = \varphi(\mathbf{S}, K) = \text{nominal + coupon}$	→	2) $\hat{\mathbf{v}}_n = \text{FT}[\mathbf{v}_n]$ ↓
n-1	4) $\mathbf{v}_{n-1} = \text{FT}^{-1}[\hat{\mathbf{v}}_{n-1}]$	←	3) $\hat{\mathbf{v}}_{n-1} = \hat{\mathbf{v}}_n e^{\Psi(n-n+1)}$
	5) $\mathbf{v}_{n-1} = \text{coupon} + \mathbb{1}_{\mathbf{S}>K} \times \text{nominal} + \mathbb{1}_{\mathbf{S}\leq K} \times \mathbf{v}_{n-1}$	→	6) $\hat{\mathbf{v}}_{n-1} = \text{FT}[\mathbf{v}_{n-1}]$ ↓
n-2	8) $\mathbf{v}_{n-2} = \text{FT}^{-1}[\hat{\mathbf{v}}_{n-2}]$	←	7) $\hat{\mathbf{v}}_{n-2} = \hat{\mathbf{v}}_{n-1} e^{\Psi(n-1-n+2)}$
	9) $\mathbf{v}_{n-1} = \text{coupon} + \mathbb{1}_{\mathbf{S}>K} \times \text{nominal} + \mathbb{1}_{\mathbf{S}\leq K} \times \mathbf{v}_{n-1}$	→	10) $\hat{\mathbf{v}}_{n-2} = \text{FT}[\mathbf{v}_{n-2}]$ ⋮
⋮	⋮		⋮
0	X+1) Price & Greeks See algorithm on §5.2.1.1	←	X) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_n e^{\Psi(n-0)}$

Panel 19: FST Algorithm used to evaluate an Automatic Early Redemption Option.

We produced the AERO Benchmark with help of two methods only (FST and MCM) because there are no closed-form solution and no well-established pricing model. This situation gives a taste of the pricing issue regarding the structured products. Please note that the presented AERO is an Equity-linked Structure regarding the exercise of the redeemed option.

The benchmark results are produced in Figure 46 and correspond to the pricing results according to three time positions: Spot (i.e. $t = t_0$), Living (i.e. $t = t_1 = t_0 + 1\ \text{year}$) and Forward (i.e. $t = t_{-1} = t_0 - 1\ \text{year}$). To give a better idea of AERO impact, we add the calculation of the Net Actualized Value (NAV) of bond which redeemed at a given coupon date. Hence the Bond's NAV corresponds to the sum of actualized perceived coupon(s) and actualized nominal according to the date value. For instance, the NAV redemption date $t_r = 8$ corresponds to the value of 8Y vanilla bond priced at t_0, while the NAV at redemption date $t_r = 1$ corresponds to the value of redeemed bond 1 year later than the issue date t_0.

Parameters	Values
So	2 500.00
K	2 750.00
r	4.00%
Vol	20.00%
T	8.00
T1	1.00
T2	1.00
T3	7.00
basis	1.00
Nominal	100.00
Coupon	6.00

Method	Parameters	Values	
FST	N	8 192	(1000 for Fwd)
	X_min	- 10	
	X_max	10	
MCM	Nb Sims	100 000	

Spot	= T
Living	= T3 = T - T2
Forward	= T1 + T = T1 + T2 + T3

Redemption Date	1	2	3	4	5	6	7	8
NAV Nominal	96.08	92.31	88.69	85.21	81.87	78.66	75.58	72.61
NAV Coupons	5.76	11.30	16.62	21.74	26.65	31.37	35.90	40.26
NAV Bond	101.84	103.62	105.32	106.95	108.52	110.03	111.48	112.88

Product	Pricing Type	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
AERO	Spot	MCM	106.20	0.00	0.00	- 371.81	0.31	1.07	32.37
		FST	106.20	- 0.01	0.00	- 975.90	4.52	28.52	0.03
	Living	MCM	105.88	- 0.00	- 0.00	- 383.93	0.12	2.25	31.52
		FST	105.86	- 0.01	0.00	- 841.51	4.52	20.26	0.03
	Forward	MCM	102.05	0.01	- 0.00	- 458.20	0.60	2.19	34.24
		FST	101.96	- 0.01	0.00	- 936.95	4.32	29.43	8.22

Figure 46: Example of pricing results of a Bond with AERO according the Pricing Method and time position (Spot, Living or Forward Starting Option).

The next demonstration step is to compare the FST capacity versus other methods. Unfortunately we are limited to Monte Carlo Method and we know it may not be the most adequate. Although we produced the benchmark and summarized the results in Figure 47 where we can see the following points:

1. A strong convergence on Price estimation between the both methods,
2. Similar results on derivates versus S or T (see the scale),
3. Rho measures show strong differences between MCM and FST: the first one seems oriented with a step level around -100 until $K/S_0 \approx 0.75$ and after it decreases significantly until -700, while the FST method estimates a more important Rho with the same effect as seen previously.
4. Concerning Vega, MCM shows estimation near null whatever the situation while the FST method defines a "*wave*" effect depending on the ratio value. With a ratio below one, the option is in the money so a higher volatility increases the probability not to trigger the AERO effect and hence it increases the bond value (see static bond values presented in Figure 46). And we do the opposite rationale with a ratio above one: the AERO option is out the money so high volatility increases the probability to trigger the option and so the bond loses value.

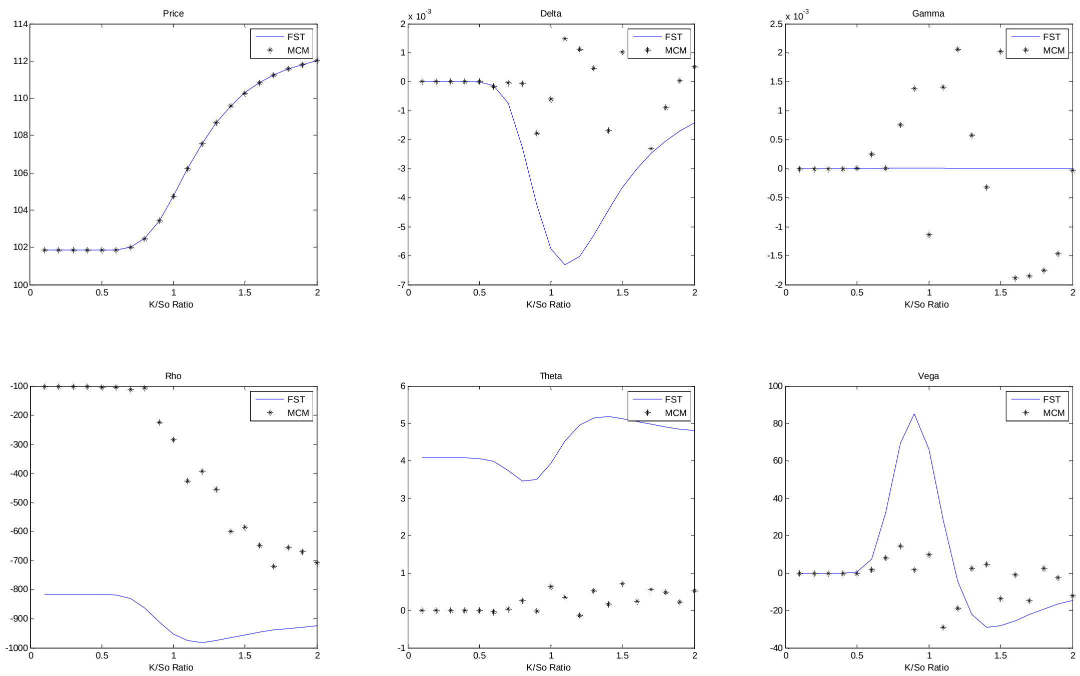


Figure 47: Results of a Bond with AERO according to Pricing Methods and Strike Level (simulation number = 1 million paths).

Next we finished the benchmark by producing a comparison between FST pricings according to the pricing date (i.e. spot, living or forward). The rationale is to highlight the potential impacts depending on time position and the presence (or not) of a basis effect.

We can notice that most measured items are mostly similar (Delta, Gamma, Theta and Vega). However the two others (Price and Rho) have to be analyzed carefully. Indeed a Bond with AERO is a common liquidity support for Structured Products issued in primary market.

And we can highlight two important points which have strong consequences on deal negotiation:

1. A lower bond price: this allows getting better price and so increases the intermediate fee (there we have upfront discount of near 4%).
2. The Forward Rho equals nearly its Spot counterpart: this Sensitivity Factor will concern interest rate movements but also credit spread approximated as risk premium. So selecting the counterpart will have also strong consequences on the final price delivered and so the intermediate fee.

Hence an alerted investor will keep in mind these two parameters while proceeding to a new investment due to their influence on negotiated price.

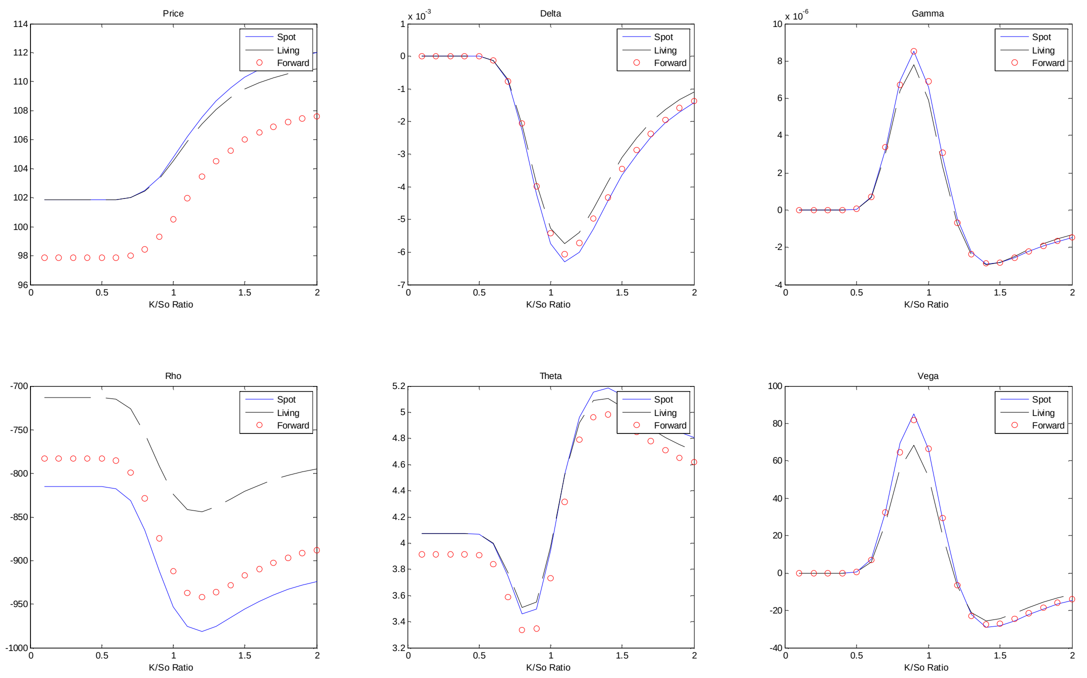


Figure 48: FST Pricing Results of a Bond with AERO, according to the ratio level "Strike/Current Underlying Level" (K/So) and the Scenario Type.

5.2.2.5 Preliminary Conclusions on "*Path-Dependent*" Options

We presented in this section a set of "*Path-Dependent*" Options to assess the Pricing Capacity of the FST Method regarding the increase of Payoff Structure Complexity. To reach this goal, we compared its results with those produced by methods of reference for a given Option Structure.

Regarding the presented results, we can say that FST method is convergent on Price estimation and most of the underlying Sensitivity Factors. However we saw in particular cases some divergences especially while estimating the Vega of American Option with Continuous Barrier. We will conclude the FST efficiency but remain aware that this point will require a further investigation to validate this assessment.

In the last two sub sections, we presented uncommon option structures with the FSO and AERO Options and the linked issue which is the limitation of referred method to the Monte Carlo Method. Thus we adapted and extended the inital work presented by Vladimir Surkov to evaluate such structures and assessed the FST efficiency with the benchmark results. Regarding the AERO assessment, we presented also the economic issues linked to this structure, especially during the negociation phase with Forward Starting Products.

The next stage is to evaluate real Structured Products which integrate some of the options presented in §5.2.1 and §5.2.2.

5.2.3 Examples of Structured Product Pricing

Now we arrive at the final stage of the FST Tests by producing benchmarks based on real Strutured Products. Due to confidentiality measures, the presented structures have been changed but they remain similar enough to be the assessment of real versions.

The Structured Products presented in this part are:

1. BMB: A Note with 8Y maturity with a laddered coupon delivered at maturity, defined by a set of digitals and framed by floor and cap option (i.e. min = 0%, max = 75%),
2. BSB: A Note with 8Y maturity and a structured coupon with AERO for each date $t \in \{1,2,\ldots,7\}$, a memorized coupon option and a final Put DI Barrier at maturity,
3. BFT: A Note with 3Y maturity, an AERO for each working day until maturity and an final Put DI Barrier at maturity.

5.2.3.1 8 years Ladder Equity-Linked Structure with Guaranteed Capital at Maturity (BMB)

Let's start the evaluation of Structured Products with the BMB Structure which is represented in Figure 49 according to its financial characteristics:

1. Maturity : Eight years,
2. Underlying Index: Euro Stoxx 50 (STX5),
3. Capital: Protected, , i.e. 100% delivered at maturity whatever the variation of STX5
4. Performance:
   a. A coupon delivered at maturity,
   b. The delivered coupon is laddered between 0% and 75%, with 15% intermediate cumulative thresholds according the final performance of STX5. For instance, if STX5 performance is 135%, the delivered coupon will be 45%.

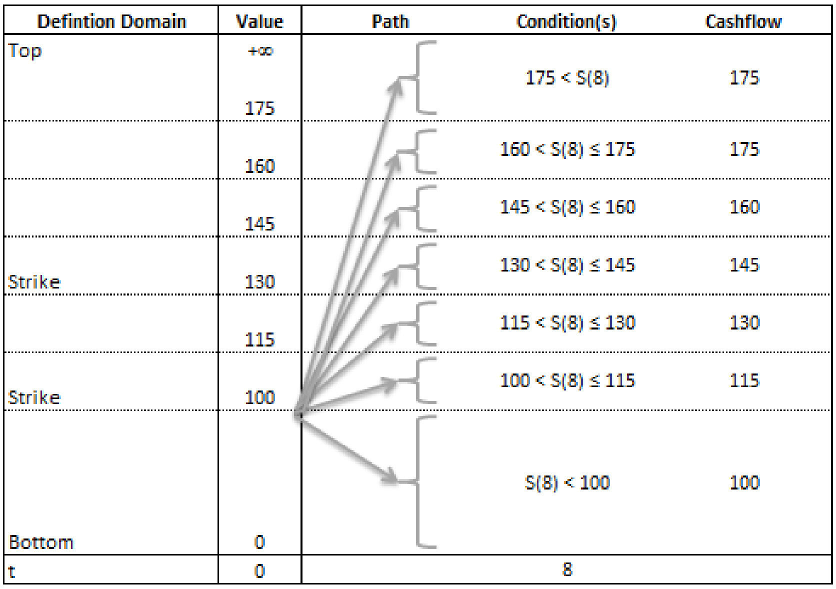


Figure 49: Option Structure of BMB Product.

Similarly to previous benchmarks, we presented the FST algorithm adapted to this Option Structure in Panel 20.

t	$\mathbb{R}$	$\mathbb{C}$
8	$\mathbf{v}_8 = \text{nominal} + \text{coupon} \times \sum_{k=0}^{4} \mathbb{1}_{C_k}$ → 1) with $C_k = (1 + k \times 0.15) \leq \frac{S_8}{S_0}$	2) $\hat{\mathbf{v}}_8 = \text{FT}[\mathbf{v}_8]$ ↓
0	4) Price & Greeks See algorithm on §5.2.1.1 ←	3) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_8 e^{\Psi \times 8}$

Panel 20: FST Algorithm used to evaluate a BMB Product.

Now we present the BMB results produced by the FST and MCM methods according to the Pricing Type (Figure 50):

1. Spot: Pricing at issued date,
2. Living: Pricing one year after the issued date (i.e. T = 7)
3. Forward: Pricing one year before the issue date (i.e. T = 9).

Parameters	Values
So	2 500
r	4.00%
Vol	20.00%
T	8
T1	1
T2	1
T3 = T - T2	7
basis	1

Method	Parameters	Values
MCM	Nb Sims	1 000 000
FST	N	1 000
	X_min	-10
	X_max	10

Spot	= T
Living	= T3 = T - T2
Forward	= T1 + T = T1 + T2 + T3

Product	Pricing Type	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
BMB	Spot	MCM	96.57	0.01	0.01	-583.55	2.15	-128.17	6.13
		FST	96.59	0.01	-0.00	-481.42	2.95	-43.27	0.01
	Living	MCM	99.54	0.02	-0.01	-82.22	4.04	-75.64	5.51
		FST	99.52	0.02	-0.00	-416.92	2.90	-36.03	0.01
	Forward	MCM	92.76	-0.00	0.01	-392.18	2.78	33.99	8.96
		FST	92.74	0.01	-0.00	-462.10	2.83	-41.31	5.68

Figure 50: Example of pricing results of a BMB Product according the time position (Spot, Living or Forward).

We can see that both pricing methods converge for each priced items, even if we can see relevant differences on Rho and Vega. Please note that the MCM's results are volatile enough due to the presence of digital options in fine. This is not surprising as we saw this phenomenon previously (see §5.2.1.2). On performance side, we can see that FST is 600% more efficient on Spot and Living Pricing scenarios, while it is 50% less efficient than the MCM method on forward scenario due to the time consumption by "*interp1()*" function.

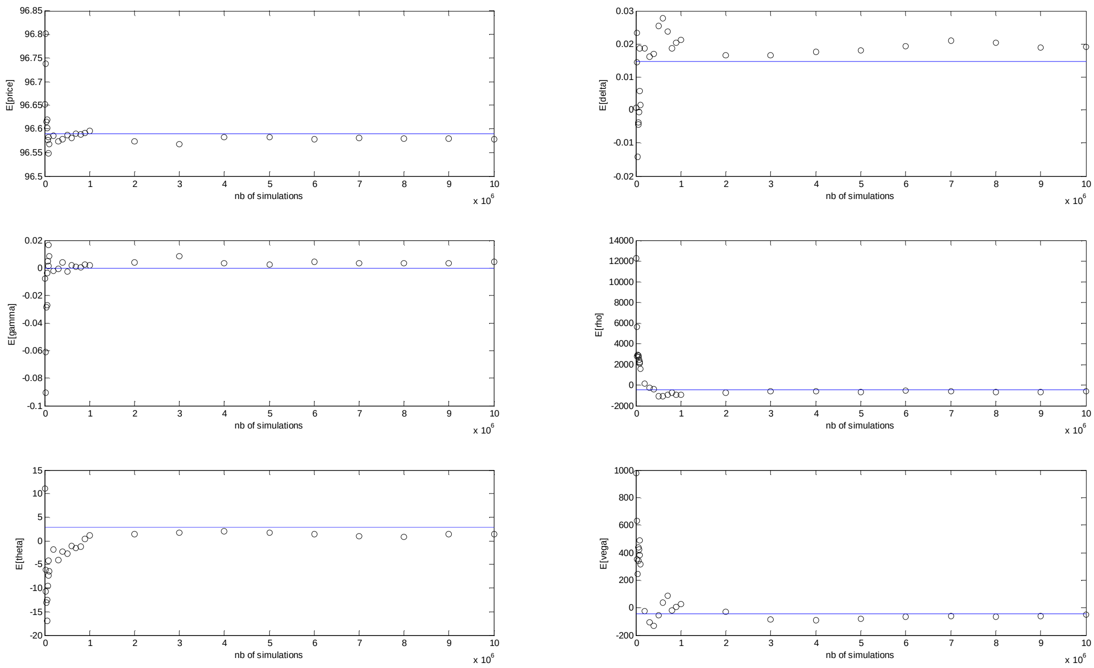


Figure 51: MCM convergence speed on mean pricing values (Price and Greeks) for a BMB Product

Now we investigate the convergence capacity by producing a comparison between FST and MCM regarding the number of simulations employed by MCM (from 10k to 10M, see Figure 51). There we can see most of priced items converge fast enough but Rho and Theta require important simulations number to produce results similar to FST method.

By combining these results with those produced in Figure 50, the MCM requires at least 5 million of simulations to produce results as efficient as FST Method. Hence the time consumption is multiply by 5 and so FST becomes a better pricing method either in terms of estimation or time performance.

5.2.3.2 8 Years Equity- Linked Structure with Memorized Coupons, Automatic Early Redemption and Final Put DI Options (BSB)

Now we will study a more complex option structure with the following financial characteristics:

1. Maturity : Eight years,
2. Underlying Index: Euro Stoxx 50 (STX5),
3. Capital: Partially protected, i.e. $100 \times \left(1 + \frac{S_8 - S_0}{S_0} \mathbb{1}_{S_8 < \text{DI Barrier}}\right)$ where DI Barrier = 60%.
4. Coupon: a 6% coupon is memorized for each $t \in \{1, 2, \ldots, 8\}$ if $S_t \geq S_0$.
5. Redemption:
   a. *Automatic Early Redemption*: BSB may be redeemed earlier for each $t \in \{1, 2, \ldots, 7\}$ if $S_t \geq S_0$. Then the delivered amount is $100 + 6 \times t$
   b. *Final Redemption* ($t = 8$): $100 \times \left(1 + \frac{S_8 - S_0}{S_0} \mathbb{1}_{S_8 < \text{DI Barrier}}\right)$ + $48 \times \mathbb{1}_{S_8 \geq S_0}$ where DI Barrier = 60%.

The option structure is summarized in Figure 52 where we represented the generated cash flows depending on the evolution of STX5 index.

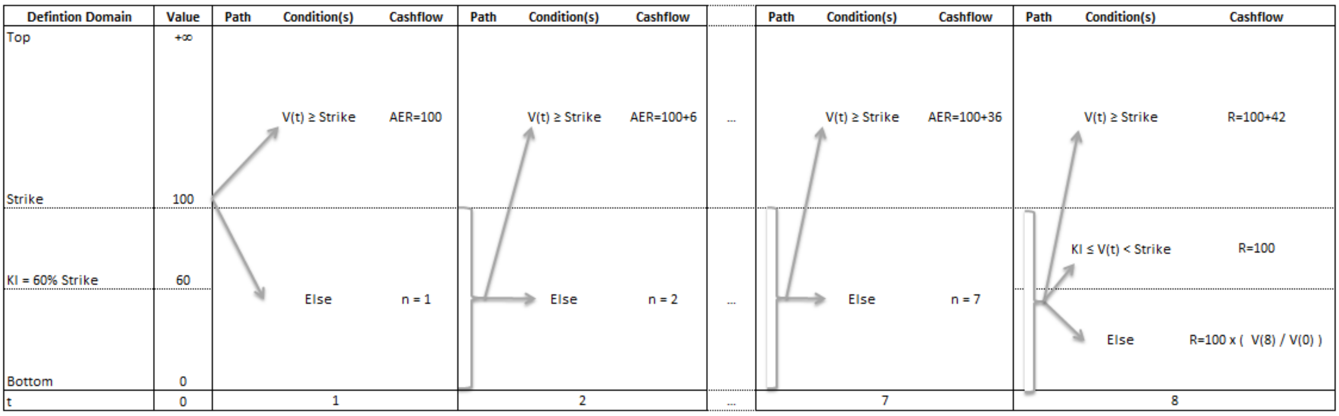


Figure 52: Option Structure of BSB Product

As usual, we present now the FST algorithm adapted to this Option Structure in Panel 21, where we adapt the backward approach by integrating step by step the different options into the value vector.

t	$\mathbb{R}$		$\mathbb{C}$
8	1) $\mathbf{v}_8 = \text{nominal} \times \left(1 + \frac{\mathbf{s} - S_0}{S_0} \mathbb{1}_{\mathbf{s}<\text{KI}}\right) + \text{n}_8 \times \text{coupon} \times \mathbb{1}_{\text{Strike} \leq \mathbf{s}}$ with $\text{n}_k = k(\text{memorized coupons})$	→	2) $\hat{\mathbf{v}}_8 = \text{FT}[\mathbf{v}_8]$ ↓
7	4) $\mathbf{v}_7 = \text{FT}^{-1}[\hat{\mathbf{v}}_7]$	←	3) $\hat{\mathbf{v}}_7 = \hat{\mathbf{v}}_8 e^{\Psi}$
	5) $\mathbf{v}_7 = \mathbb{1}_{\mathbf{S}>\text{Strike}} \times (\text{nominal} + n_7 \times \text{coupon}) + \mathbb{1}_{\mathbf{S} \leq K} \times \mathbf{v}_7$	→	6) $\hat{\mathbf{v}}_7 = \text{FT}[\mathbf{v}_7]$ ↓
⋮	⋮		⋮
0	X+1) Price & Greeks See algorithm on §5.2.1.1	←	X) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_1 e^{\Psi}$

Panel 21: FST Algorithm used to evaluate a BSB Product.

In Figure 53 we present the pricing results for FST and MCM methods, depending on the Pricing Type (Spot, Living or Forward). As we can see, the results are similar whatever the priced item, apart from Theta and Vega estimations which present relevant differences (from one to five).

Parameters	Values
So	2 500.00
r	4.00%
Vol	20.00%
T	8.00
T1	1.00
T2	1.00
T3 = T - T2	7.00
basis	1.00

Method	Parameters	Values
MCM	Nb Sims	1 000 000
FST	N	1 000.00
	X_min	- 10.00
	X_max	10.00

Spot	= T
Living	= T3 = T - T2
Forward	= T1 + T = T1 + T2 + T3

Product	Pricing Type	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
BSB	Spot	MCM	96.69	- 0.00	- 0.01	- 477.90	- 1.45	- 131.21	20.95
		FST	95.98	0.01	- 0.00	- 495.32	5.50	- 241.86	0.02
	Living	MCM	96.55	- 0.00	0.01	- 283.26	3.31	- 156.06	20.12
		FST	95.93	0.01	- 0.00	- 431.07	5.51	- 213.23	0.01
	Forward	MCM	92.97	- 0.01	0.00	- 109.82	3.58	- 62.92	24.34
		FST	92.44	0.01	- 0.00	- 482.06	5.36	- 235.85	7.05

Figure 53: Example of pricing results of a BSB Product according the time position (Spot, Living or Forward).

Analysing the results of MCM method according to the simulation numbers input (see Figure 53) shows its difficulties to converge efficiently on particular sensitivity factors even for high simulations numbers. For instances, Rho and Vega estimated by MCM converges toward zero while FST estimates negative values.

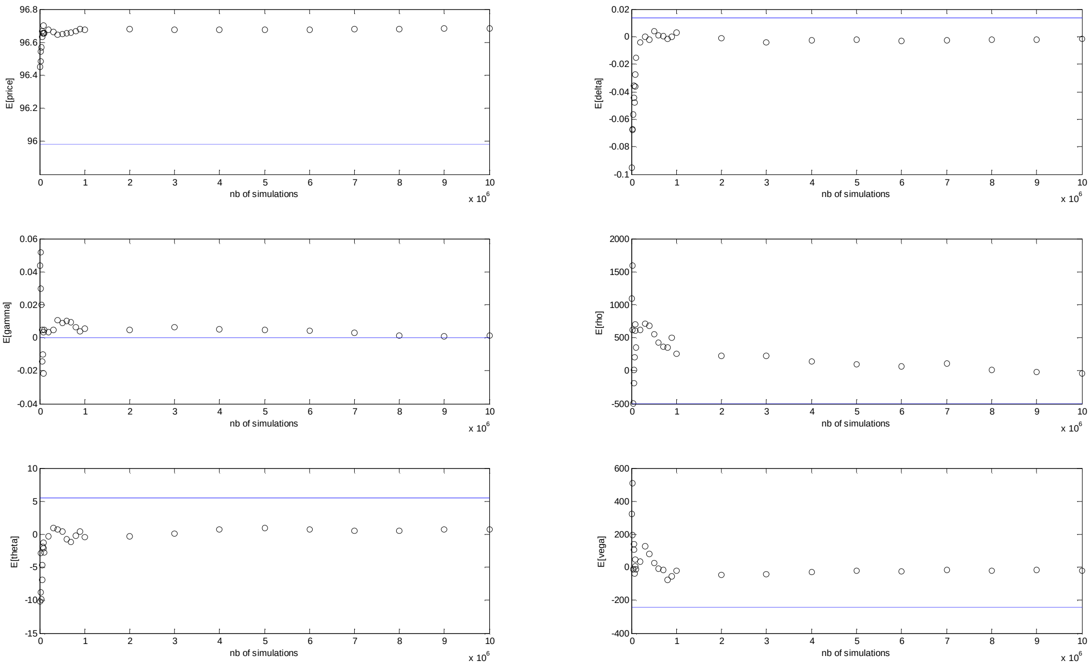


Figure 54 MCM convergence speed on mean pricing values (Price and Greeks) for a BSB Product

On operational side, MCM will produce random values which will gravitate around zero with potential change of sign. In term of Risk Management, this lack of stability is an important issue: we can't define a stable risk measure with such volatile values.

5.2.3.3 3 Years Equity-Linked Trigger Return Structure (BFT)

We finish the benchmark dedicated to Structured Products with a SP similar to previous except for the time basis which is based on daily evaluation.

Here we summarize its financial characteristics:

1. Maturity : Eight years,
2. Underlying Index: Euro Stoxx 50 (STX5),
3. Time basis: daily (1 year = 252 working days).
4. Capital: Partially protected, i.e. $100 \times \left(1+\frac{S_{756}-S_0}{S_0}\mathbb{1}_{S_{756}<\text{DI Barrier}}\right)$ where DI Barrier = 70%.
5. Coupon: An 8% coupon delivered in case of exercise of AER Option, i.e. for each $t \in \{1,2,\ldots,755\}$ when $S_t \geq S_0$.
6. Redemption:
   a. *Automatic Early Redemption*: BSB may be redeemed earlier for each $t \in \{1,2,\ldots,755\}$ if $S_t \geq S_0$. Then the delivered amount is $100+8\times\frac{t}{252}$
   b. *Final Redemption*: ($t=756$): $100 \times \left(1+\frac{S_8-S_0}{S_0}\mathbb{1}_{S_8<\text{DI Barrier}}\right)$ where DI Barrier = 70%.

Figure 55 sketches the Structured Product evolution according to t and variations of STX5:

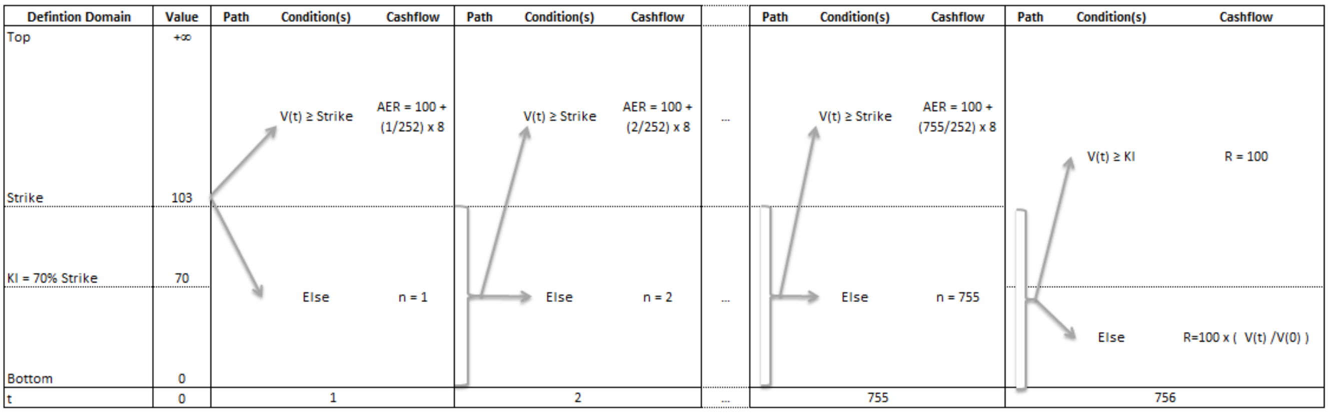


Figure 55: Option Structure of BFT Product.

Now we present the FST Algorithm adapted to the BFT Structured Product in Panel 22. As we can see, the backward algorithm is straight forward where the most important trick remains in the composition of the vector of integrated values (see step n°4).

t	$\mathbb{R}$		$\mathbb{C}$
756	1) $\mathbf{v}_{756} = \text{nominal} \times \left(1 + \frac{\mathbf{s} - S_0}{S_0} \mathbb{1}_{\mathbf{s}<\text{KI}}\right)$	$\longrightarrow$	2) $\hat{\mathbf{v}}_{756} = \text{FT}\left[\mathbf{v}_{756}\right]$ ↓
755	4) $\mathbf{v}_{755} = \text{FT}^{-1}\left[\hat{\mathbf{v}}_{755}\right]$	$\longleftarrow$	3) $\hat{\mathbf{v}}_{755} = \hat{\mathbf{v}}_{756} e^{\Psi^{\text{basis}=252}}$
	5) $\mathbf{v}_{755} = \mathbb{1}_{\mathbf{S} \leq \text{Strike}} \times \mathbf{v}_{755} + \mathbb{1}_{\mathbf{S} > \text{Strike}} \times \left(\text{nominal} + \frac{755}{252} \times \text{coupon}\right)$	$\longrightarrow$	6) $\hat{\mathbf{v}}_{755} = \text{FT}\left[\mathbf{v}_{755}\right]$ ⋮
⋮	⋮		⋮
0	X+1) Price & Greeks See algorithm on §5.2.1.1	$\longleftarrow$	X) $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_1 e^{\Psi^{\text{basis}=252}}$ ↓

Panel 22: FST Algorithm used to evaluate a BFT product.

Below we present the benchmark results produced by the FST and MCM methods according to the Pricing Type. We can see that most of priced items show convergent results whatever the pricing method used. The only exceptions are Rho and Vega which present significant differences in amount and/or sign. Please note that the MCM was limited to a thousand paths due to the daily discretization. Indeed a superior number of simulations will induce a Matlab error and produce no results. Hence the MCM results show a high volatile profile on particular Sensitivity Factor estimations.

Regarding the performance of each method, we can see that the daily estimation creates an overhead for both methods, with a neat advantage for the FST method for the first two scenarios (Spot and Living). However the FST performance decreases a lot while estimating the Forward BFT. Indeed the Forward FST algorithm requires to perform "*N+1*" pricings (see §5.2.2.3) and so the time consumption is multiplied by one thousand (Forward FST belongs to the $o(n)$ algorithm category). However the MCM used only Ten Thousand paths which are few to get accurate results. With a minimum of one million paths, the MCM consumption will increase at least by 100 and push the forward BFT estimation near to 1100 seconds, i.e. 20 minutes.

Parameters	Values
So	2 500.00
r	4.00%
Vol	20.00%
T	8.00
T1	1.00
T2	1.00
T3 = T - T2	7.00
basis	1.00

Method	Parameters	Values
MCM	Nb Sims	10 000.00
FST	N	1 000.00
	X_min	- 10.00
	X_max	10.00

Spot	= T
Living	= T3 = T - T2
Forward	= T1 + T = T1 + T2 + T3

Product	Pricing Type	Method	Price	Delta	Gamma	Rho	Theta	Vega	Time
BFT	Spot	MCM	99.08	- 0.00	- 0.05	- 1 483.21	4.23	270.81	7.45
		FST	99.08	0.01	- 0.00	- 192.07	7.92	- 161.42	0.19
	Living	MCM	99.02	- 0.00	0.01	280.34	41.86	- 111.77	4.96
		FST	99.13	0.01	- 0.00	- 132.03	7.95	- 106.63	0.14
	Forward	MCM	94.45	- 0.00	- 0.01	1 077.38	31.95	261.78	11.09
		FST	94.86	0.01	- 0.00	- 181.51	6.42	- 120.07	181.87

Figure 56: Example of pricing results of a BFT Product according the time position (Spot, Living or Forward).

We lead further our investigations by providing a convergence study on MCM depending on the number of employed simulations (see Figure 57). We can notice that BFT Product presents the same lack of convergence capacity on Rho and Vega as for BFT Product. Thereby these two sensitivity factors converge to zero value while FST converges to negative values. This fact raises a Risk Management issue due for BSB Product to the volatile value of these Sensitivity Factors.

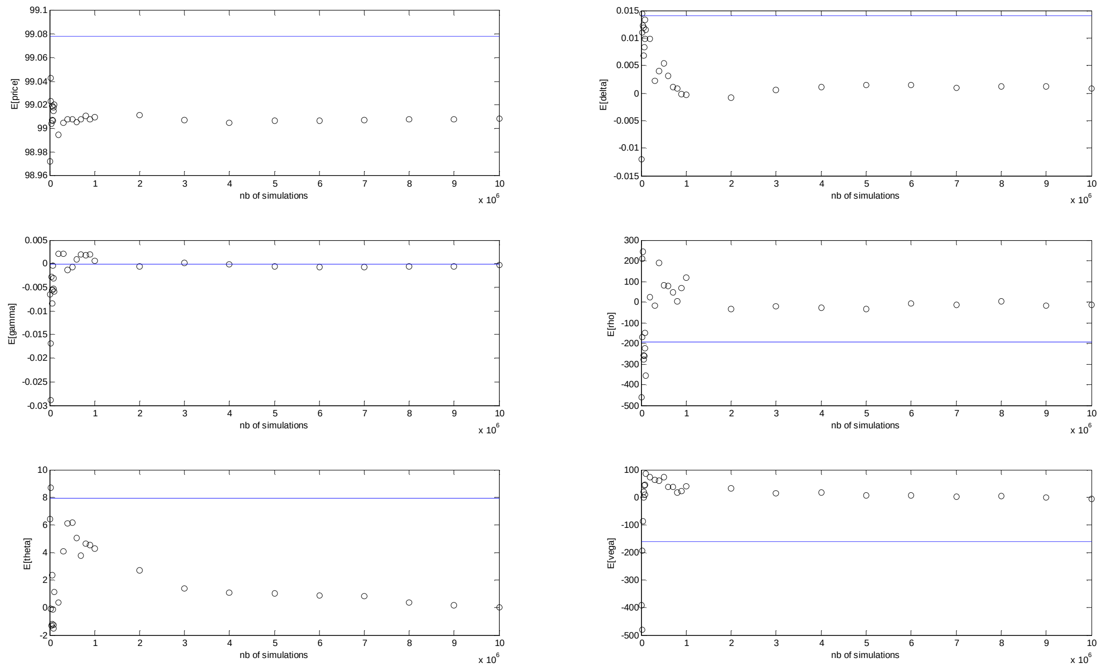


Figure 57: MCM convergence speed on mean pricing values (Price and Greeks) for a BFT Product.

5.2.4 Conclusions on Structured Product Benchmark

We presented in this part a set of structured products provided by official market counterparties, and we provided a comparison of pricing results between FST and MCM methods. Due to their inner complexity, most of SP can be priced with only a reference method which presents some issues, especially on sensitivity factor estimations.

These issues have lots of consequence such as the product negotiation ("*is the price fair?*"), the risk management ("*how to manage a neutral position with volatile estimations?*") or the accountancy of such products ("*How to provide a fair value accounting with a complex OTC Product*?"). Moreover reaching these goals requires a dedicated IT System with the consequences on budget dedicated to its uses and maintenance.

The study of FST method highlights:

1. It is accurate and fast enough whatever the option structure and can be used in trading context,
2. Its results are stable for a given calibration. Hence repeating the valuation process will produce the same results whatever the time spent since the last produced valuation. This fact solves the issues in risk management and accountancy.
3. The FST algorithms are easy to implement and to maintain once the algorithm is defined. Moreover the FFT is deployable on Distributed Grids and so FST can manage a huge volume of calculations.

With such results, FST method is a good candidate on complex option structure pricing in financial environment. However this must be moderated by the following points:

1. Scope Extension of Current Study: This study is limited to mono equity asset options. Regarding the papers produced by Vladimir Surkov and Al., the FST method was applied on several different cases: Options with multi asset underlyings (Jackson, et al., 2008), different asset categories (IR, Commodities, see (Jaimungal, et al., 2010) and (Jaimungal, et al., 2011)) or different random generator processes (GBM with Jump for instance, see (Surkov, 2009)). We limited the benchmark for the sake of simplicity and to focus on the FST capacity to handle Complex Structure Pricing. However that method can be extended and the interested readers may investigate these points further with the referred papers in §8.
2. FST Requirements: The use of FST method requires a good mathematical knowledge to define accurate algorithms and avoid misconceptions or misunderstandings. Having a Look on the FST application on IR Options highlights these difficulties (Jaimungal, et al., 2010). This is an important constraint that will limit a broad use of this method comparing to the simplicity of MCM.
3. Limited Application with Particular Option: The FST method is a backward approach and its application on payoffs with forward valuation can be complex enough. As example, we presented the case of Forward Starting European Option in §5.2.2.3 and so highlighted this situation. This difficulty will increase a lot with Asian Options and use of "spaces within spaces" approach will be reflected on high degradation of performances.

From our point of view, FST method is a good candidate to complete and challenge MCM method but not to replace it completely. We estimate it is better to have two pricing methods than one when you have to produce quantitative studies of complex option structures. FST presented a set of qualities which outreach some MCM limitations. We will use it in the next part to help measuring the SP sensitivities and so managing the underlying risk.

5.3 PIPELINE RISK FRAMEWORK (PRF) IMPLEMENTATION

Now let's come back to the application context presented during the introduction where we decided to use a VaR approach as backbone of the Pipeline Risk Framework.

To get an accurate risk measure, the following items are necessary:

1. Define the positions at Risk,
2. Estimate its sensitivities regarding the underlying risk factor(s) variations,
3. Estimate the dispersion at risk of underlying factors for a given confidence levels.

➔Thus thee combined elements will supply a measure of pipeline risk.

As explained during the introduction, we won't investigate the forecast of risk exposure because it is highly linked to the customers' behavior, a sensitive and highly confidential business topic. So we will mark the positions at risk statically by assuming that the whole or a part of these positions is submitted to price variations. This assumption helps to solve #1.

In sections §4.4 and §5.2, we presented and studied the FST method to evaluate its pricing capacities of Sensitivity Factors. We concluded that this study must be completed but already proves that FST method is relevant to estimate sensitivity factors. Thus we decided to use it to solve partially point #2 because we defined the methodology but not how to calibrate FST method with market data to get market estimations.

And a last point to solve is the estimation of the dispersion at risk of underlying factors, presented in point #3. And there remains an important issue because the model to use depends highly of the underlying factor's nature.

So we will present in the following sections the solutions we agreed but not as detailled as for the FST Method. Indeed technical details are part of Barclays Internal Risk Management and so submitted to confidentiality. We will present only the main facts without detailing the technical implementations.

5.3.1 Mathematical Expression of Pipeline Risk Measure

Let's start from the VaR formula adapted to the Pipeline Risk, defined as follows:

$$PR_{\alpha,t} \approx \sum_{k=1}^{s} RE_t^k \times \sum_{i=1}^{n} \frac{\partial V_t^k}{\partial x_i} \times \Delta x_{i,T-t,\alpha} \tag{16}$$

With

k : Index dedicated to Structured Products with $k \in \{1,...,s\}$ and s, the total number of Structured Products,

i : Index dedicated to Underlying Factors with $i \in \{1,...,n\}$ and n the total number of Underlying Factors,

$PR_{\alpha,t}$: The Pipeline Risk generated by all Structured Products estimation at time t for a given confidence level α,

RE_t^k : The Risk Exposure at time t (i.e. the remaining inventory) for a given Structured Product # k

x_i : Underlying Factor # i

V_t^k : Estimated price of a given Structured Product # k at time t

$\frac{\partial V_t^k}{\partial x_i}$: Estimated Sensitivity Factor versus underlying factor x_i for a given Structured Product k at time t

$\Delta x_{i,T-t,\alpha}$: Estimated Variation at Risk for a given Underlying Factor x_i at maturity date $T-t$, based on a confidence level α.

We will use this simplified formula with the following assumptions:

1. The Risk Exposure RE_t^k is static,
2. The potential loss will be estimated only with first order derivatives,
3. The potential correlations between Underlying Factors are neglected.

We are aware that these assumptions may be challenged and current approach can be improved on most of these points.

However we agreed on this simple approach for the following reasons:

1. Risk Communication: the Pipeline Risk Measure will be employed also as communication tool. Hence it must be understandable by Executive Staff independently of its professional background. The more complicated a model is, the more difficult it is to explain results and variations in front of an uninformed auditory. Risk may be complex and our duty is to present the situation in clear terms to give the full insight to executive officers and help them to be fully informed.
2. Model Resilience: Another important reason is the model resilience in distressed market conditions. Indeed most of high level models are fragile and have a bad resistance in worst market conditions for several reasons such as spoiled data, outbound model conditions and so on. Regarding the Risk Management function, our duty is to be able to estimate our current positions to establish the necessary actions plan to mitigate the risks, in all market conditions.
3. Backup and Maintenance: this is a pragmatic reason based on the fact that this model will be distributed and employed among Barclays Business Units. So the model must be understandable by different people among several countries. And the more people use a model, the more bugs are detected. Solving these bugs will consume time according to its complexity.

For all these reasons, we decided to use this simplified version of Pipeline Risk Measure. Later we will investigate the tracking record of produced results to improve (potentially) this measure.

5.3.2 Risk factors forecastings

Now we have to select the most relevant underlying factors to get the most relevant measure. To do this, we have to analyse the contributions of each Underlying Factors in Structured Product Pricings (see §5.2.3).

Thus we can observe from the produced benchmark that:

- On Primary Market (Forward), the most important risk contributions are generated by Rho, Theta and Vega. As expected, the sensitivity factors based on state variables have few effects due to the lack of basis risk.
- On Secondary Market (Spot and Living), the most relevant sensitivity factors remains the same with the introduction of the basis risk with the fixing of strike levels of reference (i.e. introduction of Delta and Gamma). However we can notice that this basis risk is less significant than the other risks.

So we decided to integrate at this stage only “*Interest Rates*” and “*Equity Volatility*” as Underlying Risk Factors. However we have to segregate the IR contribution in two categories: zero risk interest rate and funding conditions. Indeed the first one depends essentially on the continuous negotiations on IR market while the second one is specific to an issuer’s signature.

5.3.2.1 Interest & Funding Risk Factors

The Pipeline Risk Framework (PRF) requires to forecast the future forward interest rate for a given risk exposure. From a practical approach, the risk exposure is carried for a limited time period (a few days to a few months) where interest and funding rates are fixed at purchase. Hence we have to forecast their forward counterparts to estimate the future rate deviations and the potential loss at risk.

Several techniques exist to estimate the “*Forward Interest Rate at Risk*” (FIRAR) from econometric methods to IR market model. In the case of Pipeline Risk estimation, we need a resilient and accurate method with the simplest implementation possible. For these reasons, we selected the Libor Market Model (LMM) developed by (Brace, et al., 1997) due to its wide use by market places and its capacity to produce results comparable to those produced by the Black Closed Formula on Cap Pricing (see (Black, 1976)).

Basically the LMM demonstrates that Forward Rates follow a log-normal process under a forward measure such as:

$$\frac{dF(t,T_1,T_2)}{F(t,T_1,T_2)} = \sigma_{t,T_1,T_2} \times dW^F_{t,T_1,T_2} - \frac{\sigma^2_{t,T_1,T_2}}{2} dt \qquad (17)$$

Where

- t is the time point of evaluation, T_1 the beginning of the forward period, T_2 is the end of forward period, and $t < T_1 < T_2$
- $F(t,T_1,T_2)$ represents the Forward Interest Rate at evaluated at time t for the period between T_1 and T_2,
- σ_{t,T_1,T_2} is the instantaneous volatility at time t for the period between T_1 and T_2,
- dW^F_{t,T_1,T_2} is a standard Brownian motion under the forward measure F

Regarding the literature, two LMM exist according to the employed measure (forward or spot). The first one is the basic model defined by Brace and is adapted for forward period near to t, while the second one is adapted to evaluate forward periods far from t (e.g. distant caplets). Regarding the requirements of PRF, we only need to estimate near forward periods and so we decided to keep the formula defined with help of the forward measure. A rigorous reader can find more details regarding the LMM theory in (Brigo, et al., 2006) and its technical counterpart in (Gatarek, et al., 2007).

Next steps deal with the adaptation of LMM to produce the necessary forecasts required by the PRF.

The first step is to find a proxy valuation to estimate the FIRAR for a given confidence level. Thus we used the Wiener Process definition which states that $dW \sim> N(0,dt) = \sqrt{dt} \times N(0,1)$. So dW can be approximated such as $dW \approx \sqrt{dt} \times t_\alpha$ where t_α is the cumulative distribution value for a given confidence level α (e.g. if $\alpha = 99.5\%$ then $t_\alpha \approx 3.35$). Hence we can forecast the FIRAR with the following adapted formula:

$$FIRAR(T_1,T_1,T_2) \approx F(t,T_1,T_2) \times e^{\sigma_{t,T_1,T_2} \times \sqrt{T_1 - t} \times t_\alpha - \frac{\sigma^2_{t,T_1,T_2}}{2} \times (T_1 - t)} \qquad (18)$$

The second step concerns the way to calibrate such a model and we use the historical data of Euro Zero Coupon Bonds because these financial instruments are the reference while dealing with Structured Products. These data are extracted with help of Reuter Xtra 3000 and used to estimate appropriate forward interest rates and to estimate its historical volatility.

For the sake of simplicity, we adapted this approach to estimate the "*Forward Funding Rate at Risk*" (FFRAR) by using Credit Default Swap Spread instead of Euro Zero Coupon Bonds. We are aware that these choices can be challenged because:

1. It is well known that CDS price are sensitive to switching regimes which are not integrated in a such model,
2. And CDS are very questionable as representative of market risk estimation. Indeed these instruments are subject to high speculation positions which violate the "Lack of Arbitrage Opportunity", the core principle of pricing theory.

This is a common use done by market practitioners to get a simple proxy of this parameter but CDS raised several criticisms on the validity as "*Credit Risk Measure*" (CRM). For instance, an important criticism is the presence of high speculative positions generated by few counterparts and hence produce biased CRM. This phenomenon is increased with the narrowness of CDS market on particular issuers by comparison with the respective bond market.

Other approaches exist however they are far more complex because they required refined issuers data and sophisticated mathematical models to estimate the underlying CRM:

1. The first approach is to produce its own CRM however the biggest issue of these approaches is to get relevant, refined and up-to-date data. Or these data are essentially confidential and require lots of resources to be collected in terms of IT, staff and Time consumption. For instance, Barclays' Group Credit Risk has its own credit grading system dedicated to pocket rooms' allocation and monitoring. But it can't be used to estimate a funding risk measure due to a lack of reactivity.

2. A second approach is to use a Credit Analyst Provider (Rating Agencies, credit quantitative analyst such as Reuters' Starmine) either to feed an internal model or to get direct applicable results. The biggest issues with such approach is 1) the cost of such services, 2) a strong dependency with provider's results / data and 3) a lack of understanding / control of underlying methodologies.

   Please note that both approaches demand more attention and require a last step to get a relevant FFRAR by mapping the underlying CRM to an IR Benchmark according to the issuer's credit grading and the investment's maturity.

3. A third approach is to produce an IR benchmark based on a set of relevant financial debt instruments (bonds essentially) to map the funding risk measure like previously. A first issue is that the measure estimation is based on a statistical approach and it doesn't take account of issuer's specificities / current situation. A second issue is that IR benchmark doesn't integrate OTC debt instruments such as Notes and so may produce a bias on the funding risk level by itself.

We are aware that using CDS Spreads as proxy is not the best solution but our main concern is to produce a first implementation of this framework with the funding risk integration. Hence we decide to keep this approach and investigate further this subject to improve the PRF following an iterative process.

5.3.2.2 Equity Volatility Risk Factor

Now we will focus on the forecast of volatility of equity indices / stocks to estimate a "*Forward Equity Volatility at Risk*" (FEVAR). A quick review of the literature shows that the best way to estimate FEVAR is the use of GARCH Model, the de facto volatility model developped by (Bollerslev, 1986).

A preliminary study focused on the definition of an econometric model adapted to the forecast of such "asset" class and we proceeded following the next stages.

1. The first stage aims to analyze statisticaly historical volatility data to demonstrate if they are stationary (KPSS Test), and if they integrates ARCH effects (either with Philipps-Perron unit root Test or Engle's Test.
2. The second stage is to determine 1) what kind of econometrical component we have to integrate and 2) what will be their parameter's values. To do this, we estimate the best fitted model by recombining graphic results (see Figure 58) and statistical results (see Figure 59). Hence we decided to select an ARMA(2,2) + GARCH(1,1) Model to estimate FEVAR (see Panel 23 for a mathematical presentation).

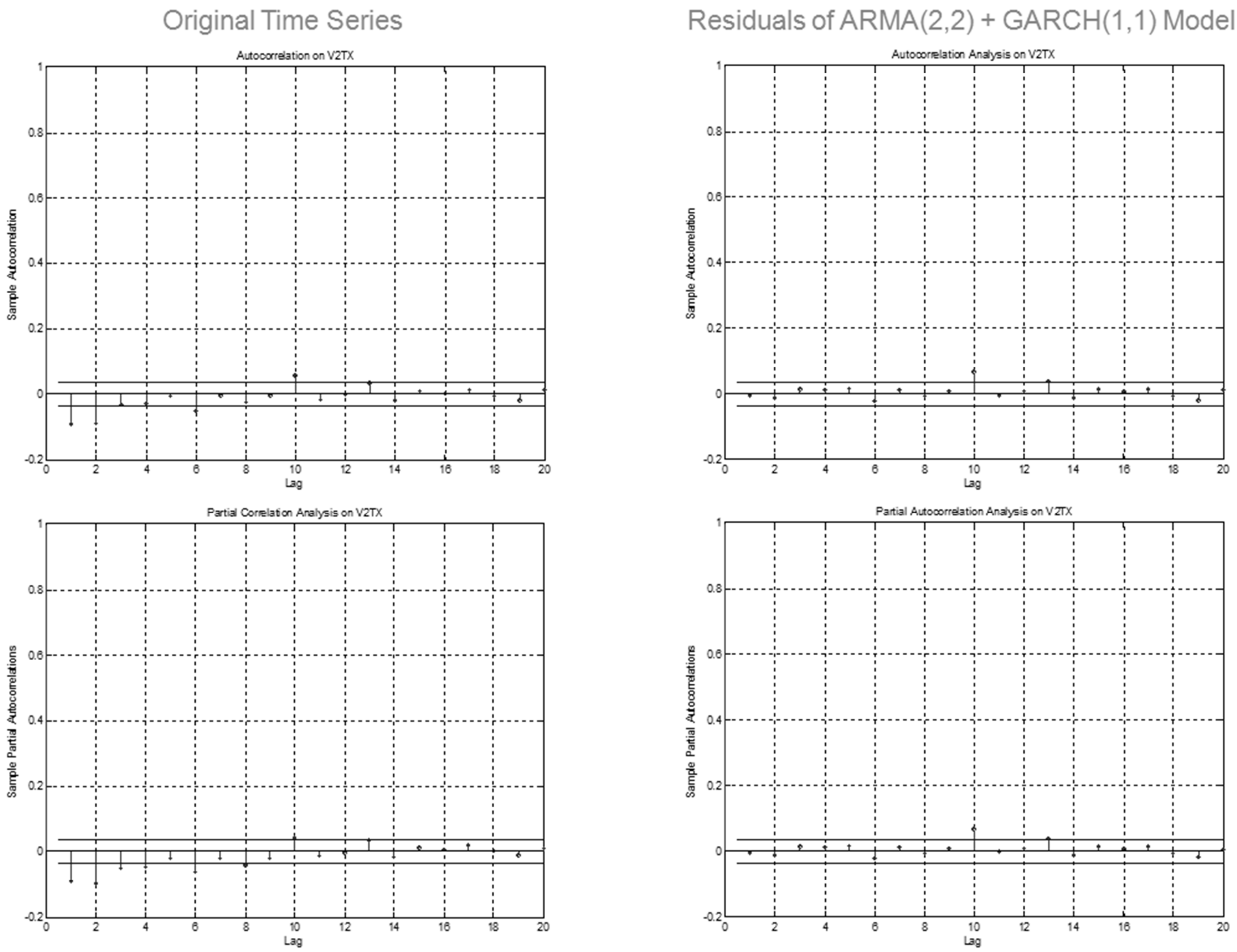


Figure 58: Auto- Correlogram (up) and Partial Auto-Correlogram (down) results of EuroStoxx 50 Volatility data. We compare the analysis of original time series (left) with the selected model to estimate FEVAR (right)

1) AR(2) Model

Mean: ARMAX(2,0,0); Variance: GARCH(0,0)

Conditional Probability Distribution: Gaussian
Number of Model Parameters Estimated: 4

Parameter	Value	Standard Error	T Statistic
C	-9.2585e-005	0.001129	-0.0820
AR(1)	-0.1004	0.014742	-6.8107
AR(2)	-0.097944	0.013405	-7.3067
K	0.0035102	5.317e-005	66.0185

Akaike Criteria for V2TX : -8586.574954

2) MA(2) Model

Mean: ARMAX(0,2,0); Variance: GARCH(0,0)

Conditional Probability Distribution: Gaussian
Number of Model Parameters Estimated: 4

Parameter	Value	Standard Error	T Statistic
C	-7.2798e-005	0.00088935	-0.0819
MA(1)	-0.11333	0.014638	-7.7421
MA(2)	-0.10968	0.013885	-7.8994
K	0.003499	5.3242e-005	65.7200

Akaike Criteria for V2TX : -8596.303703

2) ARMA(2,2) Model

Mean: ARMAX(2,2,0); Variance: GARCH(0,0)

Conditional Probability Distribution: Gaussian
Number of Model Parameters Estimated: 6

Parameter	Value	Standard Error	T Statistic
C	-4.9769e-005	0.00050445	-0.0987
AR(1)	-0.14397	0.17427	-0.8261
AR(2)	0.49387	0.12619	3.9136
MA(1)	0.031708	0.16869	0.1880
MA(2)	-0.6146	0.13765	-4.4651
K	0.0034767	5.3141e-005	65.4244

Akaike Criteria for V2TX : -8611.851604

3) ARMA(2,2) + GARCH(1,1)

Mean: ARMAX(2,2,0); Variance: GARCH(1,1)

Conditional Probability Distribution: Gaussian
Number of Model Parameters Estimated: 8

Parameter	Value	Standard Error	T Statistic
C	-0.00061964	0.00033355	-1.8577
AR(1)	-0.096497	0.48758	-0.1979
AR(2)	0.59477	0.3719	1.5993
MA(1)	-0.0098272	0.48128	-0.0204
MA(2)	-0.69437	0.41432	-1.6759
K	0.00016605	2.3751e-005	6.9914
GARCH(1)	0.87534	0.012956	67.5643
ARCH(1)	0.076622	0.0082074	9.3357

Akaike Criteria for V2TX : -8934.452405

Akaike Criteria	VIX	VSTOXX	VCAC
AR(2)	- 14,316.12	- 8,586.57	- 7,833.20
MA(2)	- 14,331.71	- 8,596.30	- 7,840.84
ARMA(2,2)	- 14,370.36	- 8,611.85	- 7,852.50
ARMA(2,2) + GARCH(1,1)	- 14,750.67	- 8,934.45	- 8,176.98

Figure 59: Comparison of Econometric Models with help of Matlab Econometric Toolbox.

Model	Definition
Autoregressive Moving Average $\mathrm{ARMA}(R,M)$	$$X_t = c + \varepsilon_t + \sum_{i=1}^{R}\varphi_i X_{t-i} + \sum_{j=1}^{M}\theta_j \varepsilon_{t-j}$$ With $\varphi_1,\ldots,\varphi_R$: model parameters c : constant $\theta_1,\ldots,\theta_M$: model parameters μ : expectation of X_t (i.e. $\mathbb{E}(X_t)$) $\varepsilon_t,\varepsilon_{t-1},\ldots,\varepsilon_{t-M}$: error terms (or innovations)
Generalized Autoregressive Conditional Heteroskedacity $\mathrm{GARCH}(P,Q)$	$$\varepsilon_t = \sigma_t z_t$$ with $z_t \xrightarrow{iid} N(0,1)$ $$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{Q}\alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{P}\beta_j \sigma_{t-j}^2$$ With following constraints: $\sum_{i=1}^{Q}\alpha_i + \sum_{j=1}^{P}\beta_j < 1$ $\alpha_0 > 0$ $\alpha_i \geq 0$ $\beta_j \geq 0$

Panel 23: Econometric Components used to forecast Equity Volatility Indices.

Next we implemented a FEVAR Estimation Solution based on Excel, Reuters Xtra 3000 and Matlab which proceeds as follows:

1. Historical data are extracted from Reuters Xtra 3000 and stored into an Excel spreadsheet,
2. These data are sent into Matlab to calibrate the Volatility Model,
3. After calibration, Matlab proceeds to 50k simulations to estimate the FEVAR Returns with a confidence level of $\alpha = 99.95\%$ (Barclays' internal rule) on a time horizon of 252 working days,
4. These returns are sent back to Excel which will estimate the FEVAR levels according to the defined time horizon.

Figure 60 presents the FEVAR estimations according to the econometric model type and we compared these results with historical data (see Figure 61) to estimate the reliability of FEVAR estimations. As we can expect, FEVAR estimations recreate the Lehman Brother Bankruptcy as worst case situation. With such results, we adopted the FEVAR Estimation Model and integrated it as PRF component.

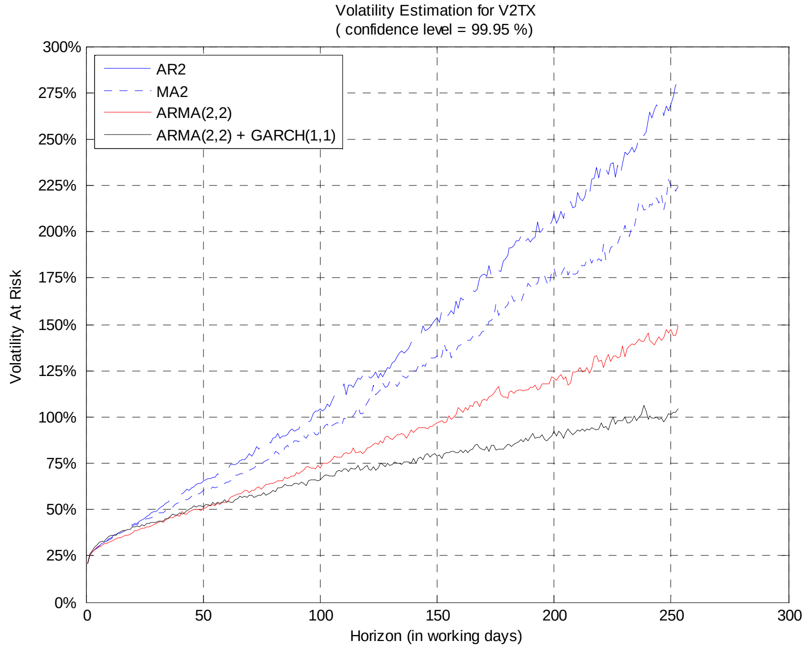


Figure 60: FEVAR estimation for EuroStoxx 50 Volatility according the model type.

Incr. / Decr.	VSTOXX - 21 WD	VSTOXX - 42 WD	VSTOXX - 63 WD	VSTOXX - 84 WD	VSTOXX - 105 WD	VSTOXX - 126 WD
Min	-33.86%	-36.26%	-39.74%	-45.67%	-43.93%	-47.79%
Begin Date	21/11/2008	27/10/2008	09/10/2008	15/10/2008	15/10/2008	14/10/2008
End Date	22/12/2008	24/12/2008	09/01/2009	13/02/2009	16/03/2009	15/04/2009
Max	52.99%	63.49%	62.63%	63.48%	67.95%	65.29%
Begin Date	11/09/2008	19/08/2008	18/07/2008	19/06/2008	21/05/2008	17/04/2008
End Date	10/10/2008	16/10/2008	16/10/2008	16/10/2008	16/10/2008	16/10/2008

Incr. / Decr.	VSTOXX - 147 WD	VSTOXX - 168 WD	VSTOXX - 189 WD	VSTOXX - 210 WD	VSTOXX - 231 WD	VSTOXX - 252 WD
Min	-51.40%	-59.39%	-56.59%	-59.66%	-61.20%	-61.46%
Begin Date	10/10/2008	08/10/2008	08/10/2008	07/10/2008	07/10/2008	07/10/2008
End Date	14/05/2009	12/06/2009	13/07/2009	11/08/2009	09/09/2009	08/10/2009
Max	58.80%	58.75%	55.59%	70.27%	63.30%	67.12%
Begin Date	17/03/2008	15/02/2008	11/01/2008	14/12/2007	15/11/2007	16/10/2007
End Date	16/10/2008	16/10/2008	10/10/2008	16/10/2008	16/10/2008	16/10/2008

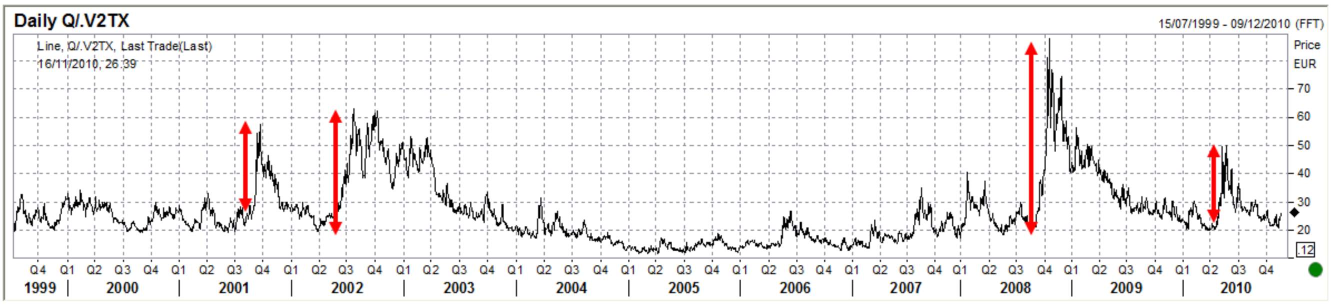


Figure 61: Historical analysis of EuroStoxx 50 Volatility according the lag size.

5.3.3 FST Model Calibration

In section §5.1, we presented the overall discretization process and the way we put it in place to get relevant pricings in the section §5.2. All pricings produced in this section are based on theoretical data and now we will present how to calibrate the FST Pricing Model to get Market Evaluations.

The first part of this section will be dedicated to FST parameters by themselves. Indeed we highlighted that FST discretization is driven by three parameters: $x_{\min}$, $x_{\max}$ and N . So we will estimate the potential pricing errors on a European Call valuation between the FST and the BSM methods, either with one or two variable parameters. Thus we will investigate how to define appropriate levels for FST parameters regarding the pricing context.

The second part of this section will focus on how to calibrate the FST model with market data. We will continue on the simple example of GBM process with drift due to its simplicity and its understanding by most of market practitioners.

5.3.3.1 Fourier Discretization Parameters

We begin this section with the study of the log error between FST and BSM while varying one pricing parameter (i.e. strike, rate, time, or volatility) whatever the pricing measure. We fixed the FST parameters with $x_{\max} = -x_{\min} = 10$ and $N = 8192$ (basic conditions used by Vladimir Surkov) and results are presented in Figure 62 according to the measure number (see figure legend). As we can see most of pricing errors are negligible except for high volatility values. Thus we can see the pricing errors exceed the unit error (i.e. $\log(\Delta) > 0$) when volatility rates are superior to 80%. This represents extreme market conditions where most of pricing models reach the limit conditions.

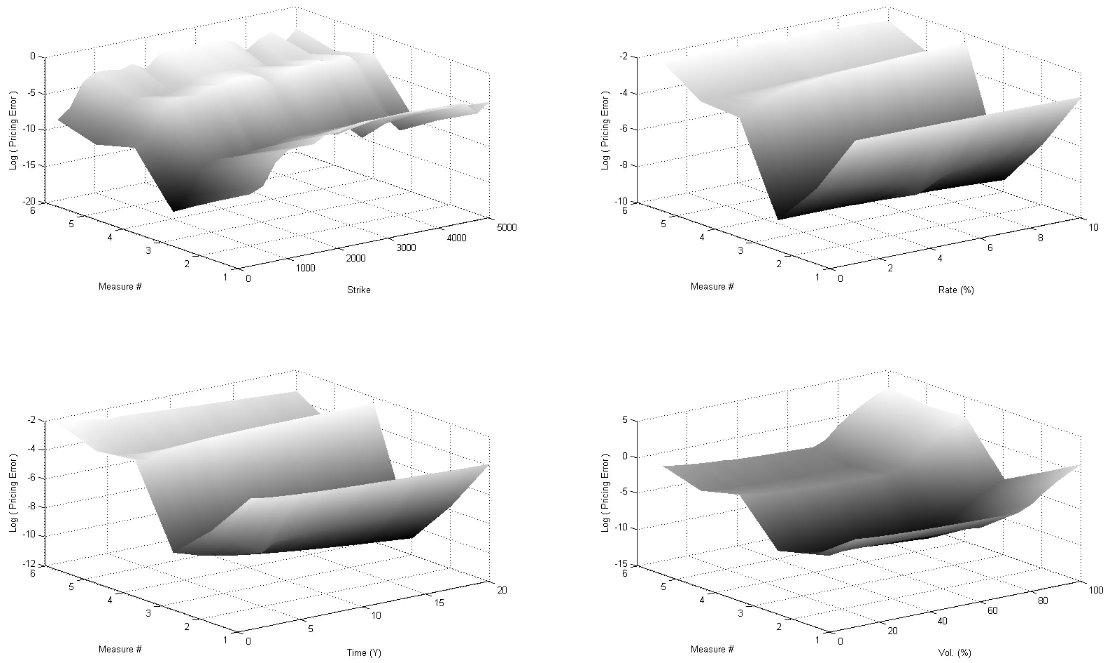


Figure 62: Log Error on pricing results for one variable parameter (Strike, Rate, Time or Volatility). Measures are respectively Price (1), Delta (2), Gamma (3), Rho (4), Theta (5) and Vega (6).

Going forward, we will investigate the accuracy of FST model versus the BSM model while varying two market parameters. We produced this analysis for each possible pair. However we presented only results produced with the "*Time and Volatility*" couple (see Figure 63). Indeed other couples present accurate results while this couple presents strong pricing errors on each measure when $\sigma \geq 40\%$. Vladimir Surkov noted this point in his thesis and proposed the use of higher values for FST parameters to get more accurate results.

We investigated this point by varying $x_{\max} \in \{10, 20, 30\}$ (with the inner relationship $x_{\min} = -x_{\max}$) and $N \in \{8000, 16000, 32000, 64000\}$. Results are presented in Figure 64 and we can highlight the following facts:

1. Log Error decreases significantly when $x_{\max}$ increases. Thus we can see that the pricing divergence is confined below the unit error while $T \leq 10$ years and $\sigma \leq 65\%$ when $x_{\max} = 30$
2. N seems to have little influence on FST pricing quality or acts as 2nd order pricing factor.

These results are very acceptable because they are coherent with Market Realities: indeed σ tends to fall when T increases, and market conditions as those tested were never observed even in case of distressed situations. Hence we will use higher values for $x_{\max}$ when pricing products with long maturity and / or with high volatility conditions.

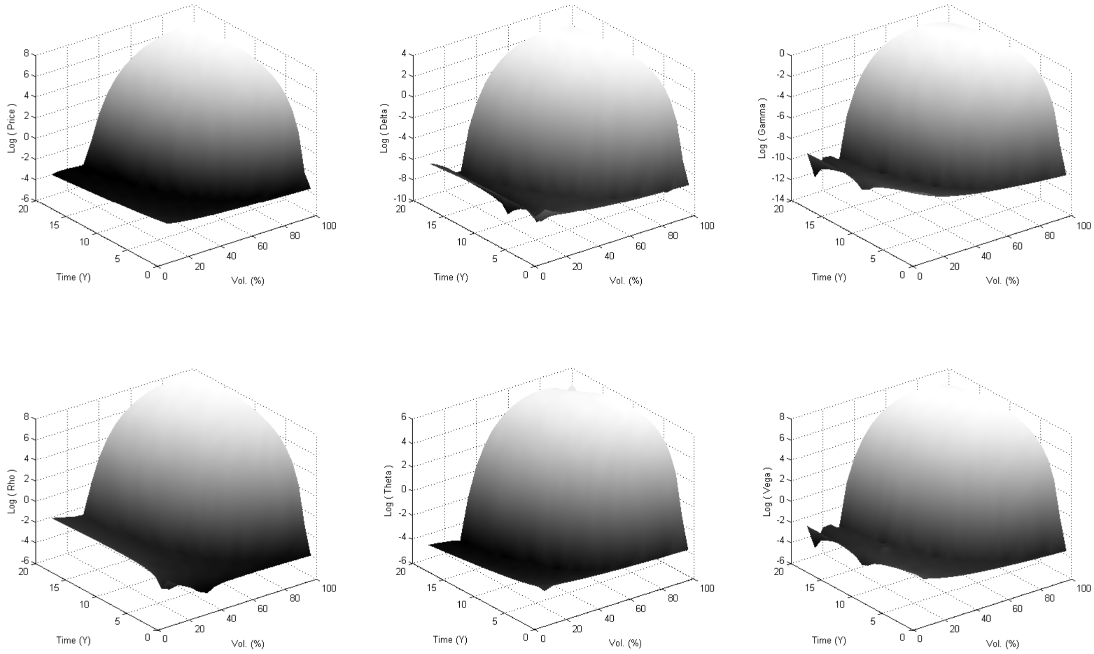


Figure 63: Log Error of pricing results according to variations of "Time" and "Volatility" parameters

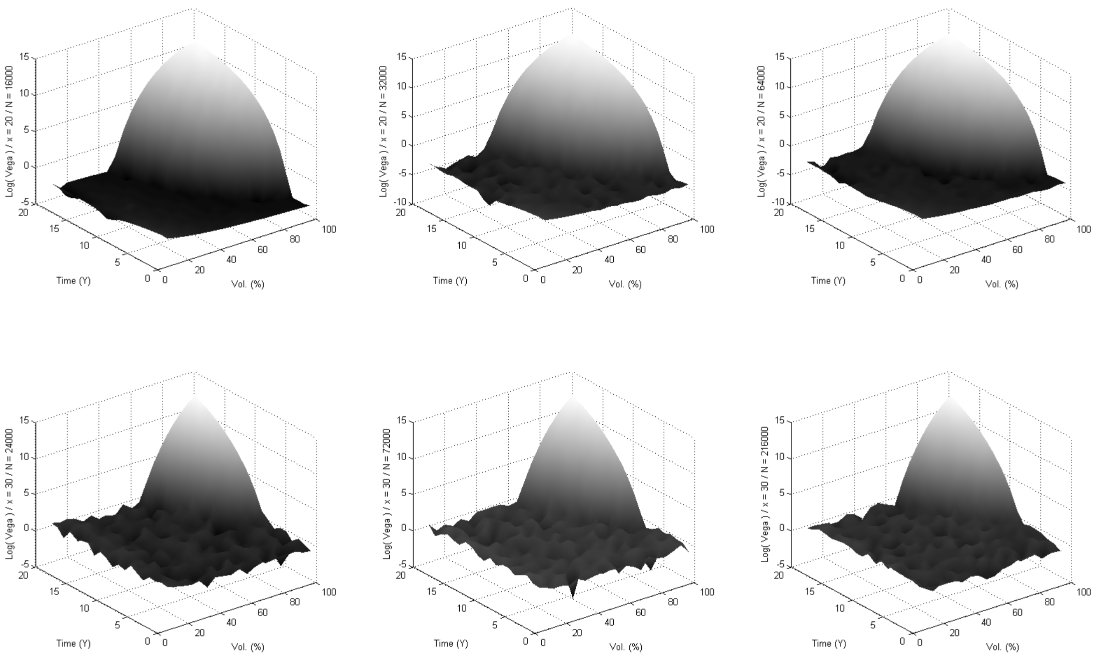


Figure 64: Log Error of pricing results while varying the couple "Time and Volatility" and the FST parameters x_max and N.

5.3.3.2 Market Data Integration

Fitting a pricing model to the Market data is a very important issue for professional investors due to its economical consequences. All researches on this area deal with :

- Either the development of underlying market data structure (e.g. Implied Volatility (IV), Volatility Skewness...),
- Or the development of models based on alternative random processes (e.g. Heston model, Variance Gamme, Ku's model, ...),
- Or the use of new mathematical techniques (e.g. optimization processes when calibrating on option prices).

We will briefly present in this section how to proceed to the market calibration with the FST model following the next stages:

1. Which market data are required during the integration process,
2. What are their structures regarding their nature and the data sources,
3. And how to integrate these data into the FST Pricing Model.

Please note that a "*model to market*" fitting process is strictly linked to the steps corresponding to the "*time value integration*" (i.e. the use of underlying random process(es)). For instance, calibrating a GBM is strictly different from a Variance-Gamma process because the two processes don't share the same parameters (see Panel 5 for mathematical details). Thus we will present the Market Calibration Process of FST Model based on GBM, the reference random process used all along this dissertation. So keep in mind that it must be adapted when using another random process, to integrate its specificities.

The first stage requires to identify the "*time value integration*" steps into FST Model which is limited to the mono/multi application(s) of equation (8). Below we recall the unidimensional definition of the adapted GBM's Characteristic Exponent Factor (CEF):

$$\Psi(\omega) = \Psi_{LK}(\omega) - r = i\left(r - q - \frac{\sigma^2}{2}\right)\omega - \frac{\sigma^2\omega^2}{2} - r$$

This represents the only entry point in FST Model which involves market parameters and we can easily identify that calibrating the CEF will require Interest Rates, Dividend Rates, and Implied Volatility rates.

Looking further at these data structures give the following comments:

- IR is compounded of Risk Free Rates and Risk Premium Rates. The first ones are generaly approximated by Zero Coupon Rates and the second ones by CDS Spreads of the underlying counterpart. Both data are easily available through Reuters, Bloomberg or Markit under vector format.
- Dividend Rates are more difficult to approximate because they are calculated values limited to most important indices / stocks. Morevoer data providers are limited and supply a vector with a limited time horizon. Nevertheless this issue is easily solved when "No Return" Equity Indices are employed as underlyings which is the general case of most Equity Linked Structured Products.
- Implied Volatility Rates are the least available data because these data are calculated by reverting the BSM closed formula. So this calculation implies having updated bid-offer option values which are specific and volatile. These data can be retrieved either from an internal or consolidate derivatives book and so may project differences regarding the price dispersion. In current dissertation, we use the implied volatility surface provided by Bloomberg for a given index (paying service) under a matrix format. We assumed these data are consistent with those used by our main counterparts (mostly based on their volatility surface based on its internal derivate books).

Please note these are raw data and required intermediate transformations and calculations to get their forward counteparts. This process is easy on IR Data and we detail only the extraction of forward implied volatility surface.

The classical approach to calibrate a model with Implied Volatility Surface (IVS) is:

1. <u>Get option values for given maturities and strikes</u>: these data are provided by BloomBerg for a given equity indice,

2. Estimate local IV values: we get this estimation for each node with help of the option value and the BSM reversed formula.
3. Extrapolate the IVS: to do this, we used the "*gridfit.m*" matlab function provided by (D'Errico, 2005) and the result is represented by Figure 65.

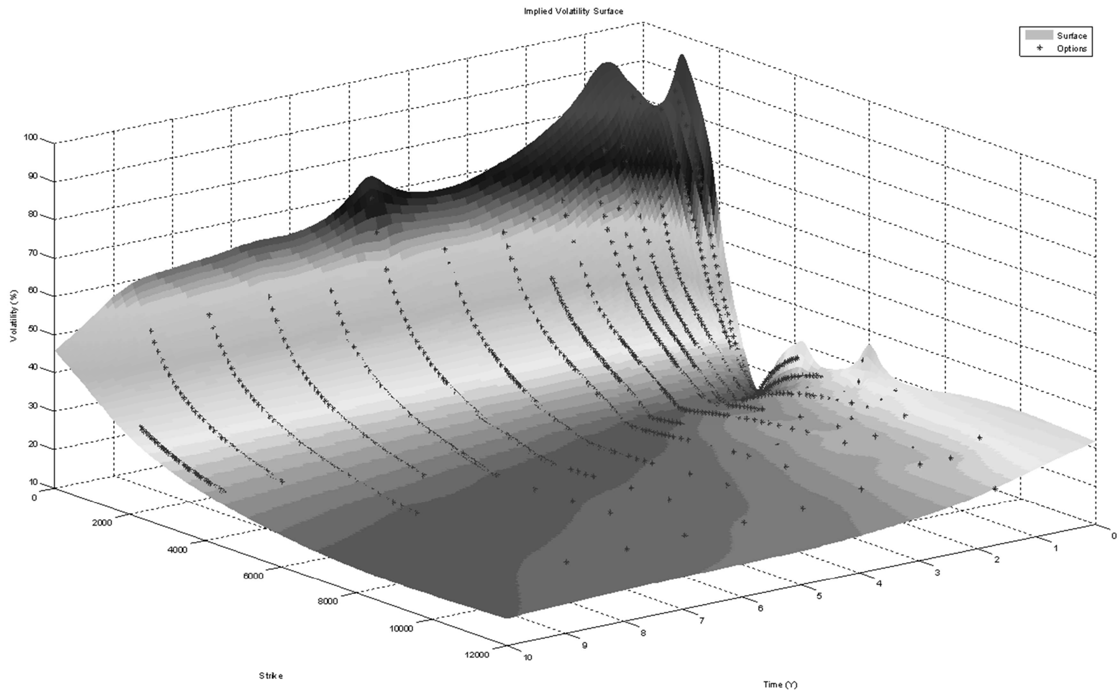


Figure 65: Plot of Implied Volatilities. Market Data are extracted From Bloomberg and interpolated with help of "gridfit.m" function; see (D'Errico, 2005).

4. Estimate the Forward IVS for a given time step. This stepd is solved with help of the Bootstrap formula, adapted to Volatility specificities and presented in equation (19)

$$\sigma^2_{n-1,n} = \frac{\tau_{0,n} \times \sigma^2_{0,n} - \sum_{k=1}^{n-1} \tau_{k-1,k} \times \sigma^2_{k-1,k}}{\tau_{n-1,n}} \text{ for } n \geq 2 \tag{19}$$

where

$\sigma^2_{n-1,n}$ is the forward implied volatitily rate between time buckets t_{n-1} and t_n

$\tau_{0,n}$ is the time period between t_0 and t_n,

$\sigma^2_{0,n}$ is the implied volatility rate between t_0 and t_n,

$\tau_{k-1,k}$ is the time period between t_{k-1} and t_k,

$\sigma^2_{k-1,k}$ is the implied volatility rate between t_{k-1} and t_k,

$\tau_{n-1,n}$ is the time period between t_{n-1} and t_n,

However this methodology raises an important flaw while pricing in $\mathbb{R}$ with the potential presence of negative forward variance points, i.e. volatility points with complex values. This kind of issue is common on the shortest maturities where arbitrages are the most intense. For instance, we extrapolated the Forward Implied surface with a 1 Month time step and represented the results in Figure 66. As you can see, we can detect the presence of complex forward volatilities for low strike levels which will disrupt any calculation process. To solve this issue, praticionners use smoothing calculations to estimate proxy points compatible with $\mathbb{R}$ - based Pricing Methods.

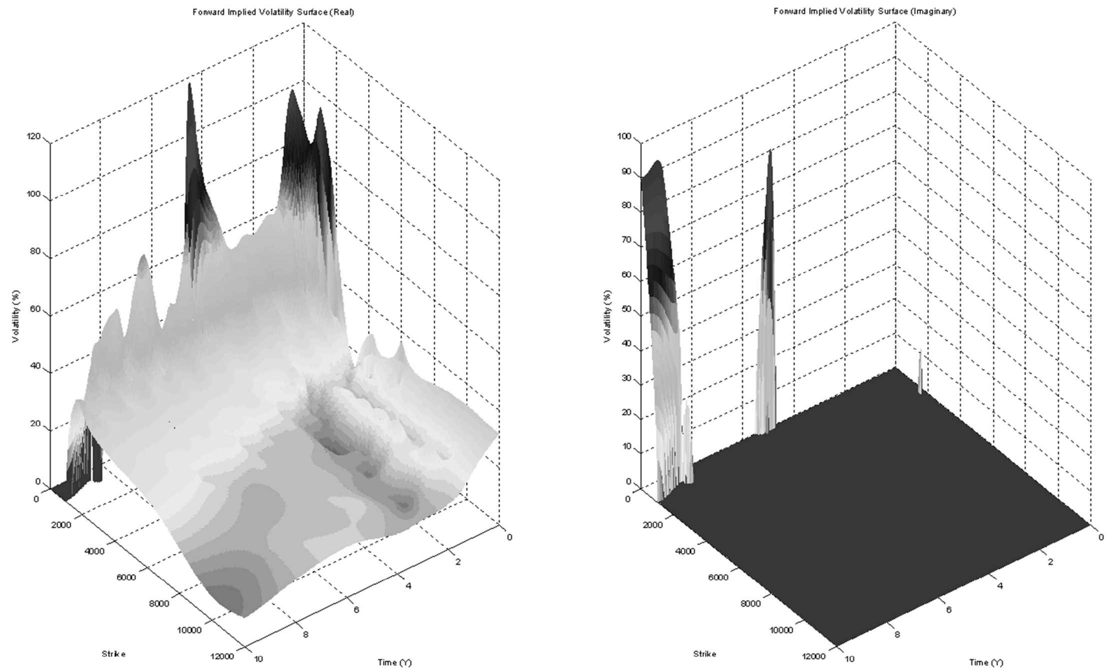


Figure 66: Forward Implied Volatility Surface, calculated from extrapolated values, with real values on left and imaginary values on right.

Regarding the FST Model, this issue seems to have no real impact because Time Value Integration is proceeded in $\mathbb{C}$ with help of Fourier Transform where the presence of complex values are not annoying but part of the valuation process.

Thus we adapted the previous process from step #4 because we proceed the bootstrap process as follows:

1. Calibrate Characteristic Exponent Factors (CEF) with spot values for a given maturity (IR, Volatility).
2. Determinate the Forward Characteristc Exponent Factors (fCEF) with help of a bootstrap process similar to those employed in forward IR estimation.

Mathematically we can estimate at time $t = n$ the complex value of $\tilde{\mathbf{v}}_{n-\tau}$ in Fourier Space such as:

$$\tilde{\mathbf{v}}_{n-\tau} = \tilde{\mathbf{v}}_n \times e^{\tau \Psi^{\langle n-\tau,n \rangle}}$$

Based on the vector value $\tilde{\mathbf{v}}_n$ and for given delays τ and κ with $\kappa < \tau$, this equation can be also expressed as

$$\begin{aligned} \tilde{\mathbf{v}}_{n-\tau} &= \tilde{\mathbf{v}}_n \times e^{(\tau-\kappa)\Psi^{\langle n-\tau+\kappa,n-\kappa \rangle}} \times e^{\kappa \Psi^{\langle n-\kappa,n \rangle}} \\ &= \tilde{\mathbf{v}}_n \times e^{(\tau-\kappa)\Psi^{\langle n-\tau+\kappa,n-\kappa \rangle} + \kappa \Psi^{\langle n-\kappa,n \rangle}} \end{aligned}$$

Next we can estimate the fCEF with help of equation (20):

$$\Psi^{\langle n-\kappa,n \rangle} = \frac{\tau \Psi^{\langle n-\tau,n \rangle} - (\tau-\kappa)\Psi^{\langle n-\tau+\kappa,n-\kappa \rangle}}{\kappa} \tag{20}$$

Hence estimating fCEF is an easy task which will require only accurate data to estimate CEF.

To assess this calibration process, we proceeded to a pricing benchmark based on European Call Option with financial characteristics summarized in Panel 24.

<table>
<tr><th>Parameter</th><th>Value(s)</th></tr>
<tr><td>Initial Value S_0</td><td>2500</td></tr>
<tr><td>Maturity</td><td>$t \in \{1,2,3,4,5\}$</td></tr>
<tr><td>Strike</td><td>$K \in \left\{ \frac{S_0}{2}, 1.2 \times S_0, 2 \times S_0 \right\}$</td></tr>
<tr><td>Interest Rates</td><td>
<table>
<tr><th>Interest Rates</th><th>0</th><th>0.2</th><th>0</th><th>0</th><th>0</th></tr>
<tr><td>Spot IR</td><td>1.99%</td><td>2.47%</td><td>2.81%</td><td>3.05%</td><td>3.25%</td></tr>
<tr><td>Forward IR</td><td>1.99%</td><td>2.96%</td><td>3.49%</td><td>3.77%</td><td>4.05%</td></tr>
</table>
</td></tr>
<tr><td>Volatility Scenarios</td><td>
<table>
<tr><th>Market Data Type</th><th>Scenario ID</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td rowspan="3">Spot Vol.</td><td>1</td><td>30.00%</td><td>31.62%</td><td>33.17%</td><td>33.54%</td><td>33.91%</td></tr>
<tr><td>2</td><td>30.00%</td><td>22.36%</td><td>24.49%</td><td>25.00%</td><td>25.50%</td></tr>
<tr><td>3</td><td>26.17%</td><td>25.52%</td><td>25.29%</td><td>25.14%</td><td>25.32%</td></tr>
<tr><td rowspan="3">Forward Vol.</td><td>1</td><td>30.00% + 0.00% i</td><td>10.00% + 0.00% i</td><td>10.00% + 0.00% i</td><td>5.00% + 0.00% i</td><td>5.00% + 0.00% i</td></tr>
<tr><td>2</td><td>30.00% + 0.00% i</td><td>0.00% + 20.00% i</td><td>10.00% + 0.00% i</td><td>5.00% + 0.00% i</td><td>5.00% + 0.00% i</td></tr>
<tr><td>3</td><td>26.17% + 0.00% i</td><td>0.00% + 5.79% i</td><td>0.00% + 3.43% i</td><td>0.00% + 2.74% i</td><td>3.08% + 0.00% i</td></tr>
</table>
</td></tr>
<tr><td>Pricing Method</td><td>1. BSM Closed Formula and calibrated on Spot Values
2. FST Method with direct integration (one step) and calibrated Spot Values (CEF),
3. FST Method with stepping integration and calibrated according the bootstrap process presented in equation (20)</td></tr>
</table>

Panel 24: Main Financial Characteristics used to assess the FST Calibration Process

The Idea of this benchmark is to validate the calibration process described previously but also to highlight the lack of impact from complex forward implied volatilities. Thereby we defined three scenarios for forward implied volatility curves:

1. A forward volatilitiy curve with real values only,
2. Another with a node with complex value,
3. And a last one based on market data which presents three successive nodes with complex values.

To give a complete benchmark, we selected several maturities from 1Y to 5Y and different strike values with an Out-of-The-Money (OTM), a Deep In-The-Money (DITM) and a Deep Out-of-The-Money (DOTM). These last two strike levels are well known for their pricing issues for complex models. Lastly we will compare the BSM results with either FST with direct integration (calibrated on spot values) or with stepping integration (calibrated on forward values, through the use of fCEF).

We presented the pricing results according to the selected volatitility scenario (#1 = Panel 25 , #2 = Panel 26 and #3 = Panel 27) and we can conclude that:

1. There is no relevant differences whatever the selected volatility scenario,
2. The presence of forward volatility points with complex value has no impact on FST pricing with stepping integration.

Thus we have a simple and elegant calibration process which supplies reliable pricing results and solves an important issue for $\mathbb{R}$-based pricings. We only presented a calibration example based on curve, however this process can be easily adapted to IVS to capture the skewness.

Hence we will integrate this calibration process in the final Pipeline Risk Framework.

Maturity	Strike	Method	Price	Delta	Gamma	Rho	Theta	Vega
1	1250	BSM	1 276.09	0.99	0.00	1 209.52	- 30.16	40.98
		FST (direct)	1 276.09	0.99	0.00	1 209.52	- 30.16	40.98
		FST (stepped)	1 276.09	0.99	0.00	1 209.52	- 30.16	40.98
	2500	BSM	149.83	0.35	0.00	719.39	- 152.85	923.76
		FST (direct)	149.83	0.35	0.00	719.39	- 152.85	923.76
		FST (stepped)	149.83	0.35	0.00	719.39	- 152.85	923.76
	5000	BSM	4.48	0.02	0.00	40.81	- 17.50	111.28
		FST (direct)	4.48	0.02	0.00	40.81	- 17.50	111.28
		FST (stepped)	4.48	0.02	0.00	40.81	- 17.50	111.28
2	1250	BSM	1 325.59	0.97	0.00	2 199.97	- 46.12	239.02
		FST (direct)	1 325.59	0.97	0.00	2 199.97	- 46.12	239.02
		FST (stepped)	1 325.59	0.97	0.00	2 199.97	- 46.12	239.02
	2500	BSM	316.56	0.47	0.00	1 720.60	- 132.50	1 406.68
		FST (direct)	316.56	0.47	0.00	1 720.60	- 132.50	1 406.68
		FST (stepped)	316.56	0.47	0.00	1 720.60	- 132.50	1 406.68
	5000	BSM	50.93	0.11	0.00	458.45	- 58.93	673.70
		FST (direct)	50.93	0.11	0.00	458.45	- 58.93	673.70
		FST (stepped)	50.93	0.11	0.00	458.45	- 58.93	673.70
3	1250	BSM	1 389.58	0.95	0.00	2 953.12	- 52.55	449.58
		FST (direct)	1 389.58	0.95	0.00	2 953.12	- 52.55	449.58
		FST (stepped)	1 389.58	0.95	0.00	2 953.12	- 52.55	449.58
	2500	BSM	474.02	0.55	0.00	2 676.63	- 119.95	1 715.73
		FST (direct)	474.02	0.55	0.00	2 676.63	- 119.95	1 715.73
		FST (stepped)	474.02	0.55	0.00	2 676.63	- 119.95	1 715.73
	5000	BSM	140.81	0.22	0.00	1 227.04	- 82.37	1 281.89
		FST (direct)	140.81	0.22	0.00	1 227.04	- 82.37	1 281.89
		FST (stepped)	140.81	0.22	0.00	1 227.04	- 82.37	1 281.89
4	1250	BSM	1 452.07	0.94	0.00	3 586.87	- 52.50	599.37
		FST (direct)	1 452.07	0.94	0.00	3 586.87	- 52.50	599.37
		FST (stepped)	1 452.07	0.94	0.00	3 586.87	- 52.50	599.37
	2500	BSM	602.13	0.60	0.00	3 561.69	- 108.33	1 935.43
		FST (direct)	602.13	0.60	0.00	3 561.69	- 108.33	1 935.43
		FST (stepped)	602.13	0.60	0.00	3 561.69	- 108.33	1 935.43
	5000	BSM	236.69	0.30	0.00	2 083.06	- 89.11	1 746.22
		FST (direct)	236.69	0.30	0.00	2 083.06	- 89.11	1 746.22
		FST (stepped)	236.69	0.30	0.00	2 083.06	- 89.11	1 746.22
5	1250	BSM	1 514.00	0.93	0.00	4 107.23	- 50.98	715.73
		FST (direct)	1 514.00	0.93	0.00	4 107.23	- 50.98	715.73
		FST (stepped)	1 514.00	0.93	0.00	4 107.23	- 50.98	715.73
	2500	BSM	721.24	0.64	0.00	4 368.84	- 99.47	2 095.35
		FST (direct)	721.24	0.64	0.00	4 368.84	- 99.47	2 095.35
		FST (stepped)	721.24	0.64	0.00	4 368.84	- 99.47	2 095.35
	5000	BSM	339.33	0.37	0.00	2 981.89	- 91.24	2 118.49
		FST (direct)	339.33	0.37	0.00	2 981.89	- 91.24	2 118.49
		FST (stepped)	339.33	0.37	0.00	2 981.89	- 91.24	2 118.49

Panel 25: Pricing Results of a European Call Option, based on volatility scenario #1 (Theoretical Forward Volatility Curve with only real values)

Maturity	Strike	Method	Price	Delta	Gamma	Rho	Theta	Vega
1	1250	BSM	1 276.09	0.99	0.00	1 209.52	- 30.16	40.98
		FST (direct)	1 276.09	0.99	0.00	1 209.52	- 30.16	40.98
		FST (stepped)	1 276.09	0.99	0.00	1 209.52	- 30.16	40.98
	2500	BSM	149.83	0.35	0.00	719.39	- 152.85	923.76
		FST (direct)	149.83	0.35	0.00	719.39	- 152.85	923.76
		FST (stepped)	149.83	0.35	0.00	719.39	- 152.85	923.76
	5000	BSM	4.48	0.02	0.00	40.81	- 17.50	111.28
		FST (direct)	4.48	0.02	0.00	40.81	- 17.50	111.28
		FST (stepped)	4.48	0.02	0.00	40.81	- 17.50	111.28
2	1250	BSM	1 312.08	0.99	0.00	2 345.37	- 32.43	60.96
		FST (direct)	1 312.08	0.99	0.00	2 345.37	- 32.43	60.96
		FST (stepped)	1 312.08	0.99	0.00	2 345.37	- 32.43	60.96
	2500	BSM	187.78	0.40	0.00	1 607.91	- 96.09	1 362.91
		FST (direct)	187.78	0.40	0.00	1 607.91	- 96.09	1 362.92
		FST (stepped)	187.78	0.40	0.00	1 607.91	- 96.09	1 362.92
	5000	BSM	8.34	0.03	0.00	134.52	- 15.20	242.15
		FST (direct)	8.34	0.03	0.00	134.53	- 15.20	242.16
		FST (stepped)	8.34	0.03	0.00	134.53	- 15.20	242.16
3	1250	BSM	1 360.48	0.98	0.00	3 265.31	- 39.34	213.50
		FST (direct)	1 360.48	0.98	0.00	3 265.31	- 39.34	213.50
		FST (stepped)	1 360.48	0.98	0.00	3 265.31	- 39.34	213.50
	2500	BSM	324.54	0.49	0.00	2 720.66	- 96.03	1 727.17
		FST (direct)	324.54	0.49	0.00	2 720.66	- 96.03	1 727.17
		FST (stepped)	324.54	0.49	0.00	2 720.66	- 96.03	1 727.17
	5000	BSM	48.04	0.11	0.00	686.36	- 39.84	818.09
		FST (direct)	48.04	0.11	0.00	686.36	- 39.84	818.09
		FST (stepped)	48.04	0.11	0.00	686.36	- 39.84	818.09
4	1250	BSM	1 411.25	0.97	0.00	4 054.80	- 41.58	340.39
		FST (direct)	1 411.25	0.97	0.00	4 054.80	- 41.58	340.39
		FST (stepped)	1 411.25	0.97	0.00	4 054.80	- 41.58	340.39
	2500	BSM	434.88	0.55	0.00	3 775.98	- 90.63	1 978.04
		FST (direct)	434.88	0.55	0.00	3 775.98	- 90.63	1 978.04
		FST (stepped)	434.88	0.55	0.00	3 775.98	- 90.63	1 978.04
	5000	BSM	102.79	0.19	0.00	1 450.59	- 52.94	1 339.90
		FST (direct)	102.79	0.19	0.00	1 450.59	- 52.94	1 339.90
		FST (stepped)	102.79	0.19	0.00	1 450.59	- 52.94	1 339.90
5	1250	BSM	1 463.96	0.96	0.00	4 717.21	- 42.22	452.47
		FST (direct)	1 463.96	0.96	0.00	4 717.21	- 42.22	452.47
		FST (stepped)	1 463.96	0.96	0.00	4 717.21	- 42.22	452.47
	2500	BSM	541.99	0.60	0.00	4 776.02	- 86.16	2 161.30
		FST (direct)	541.99	0.60	0.00	4 776.02	- 86.16	2 161.30
		FST (stepped)	541.99	0.60	0.00	4 776.02	- 86.16	2 161.30
	5000	BSM	171.99	0.26	0.00	2 380.90	- 61.65	1 810.62
		FST (direct)	171.99	0.26	0.00	2 380.90	- 61.65	1 810.62
		FST (stepped)	171.99	0.26	0.00	2 380.90	- 61.65	1 810.62

Panel 26: Pricing Results of a European Call Option, based on volatility scenario #21 (Theoretical Forward Volatility Curve with only real values except on maturity point #3)

Maturity	Strike	Method	Price	Delta	Gamma	Rho	Theta	Vega
1	1250	BSM	1 275.02	1.00	0.00	1 219.62	- 26.43	16.91
		FST (direct)	1 275.02	1.00	0.00	1 219.62	- 26.43	16.91
		FST (stepped)	1 275.02	1.00	0.00	1 219.62	- 26.43	16.91
	2500	BSM	115.13	0.31	0.00	664.99	- 128.93	884.50
		FST (direct)	115.13	0.31	0.00	664.99	- 128.93	884.51
		FST (stepped)	115.13	0.31	0.00	664.99	- 128.93	884.51
	5000	BSM	1.45	0.01	0.00	16.80	- 6.95	50.54
		FST (direct)	1.45	0.01	0.00	16.80	- 6.95	50.54
		FST (stepped)	1.45	0.01	0.00	16.80	- 6.95	50.54
2	1250	BSM	1 314.82	0.99	0.00	2 307.35	- 35.90	115.20
		FST (direct)	1 314.82	0.99	0.00	2 307.35	- 35.90	115.20
		FST (stepped)	1 314.82	0.99	0.00	2 307.35	- 35.90	115.20
	2500	BSM	231.20	0.43	0.00	1 665.53	- 109.02	1 385.86
		FST (direct)	231.20	0.43	0.00	1 665.53	- 109.02	1 385.87
		FST (stepped)	231.20	0.43	0.00	1 665.53	- 109.02	1 385.87
	5000	BSM	18.29	0.05	0.00	235.68	- 27.81	390.20
		FST (direct)	18.29	0.05	0.00	235.69	- 27.81	390.20
		FST (stepped)	18.29	0.05	0.00	235.69	- 27.81	390.20
3	1250	BSM	1 362.26	0.98	0.00	3 240.31	- 40.36	236.43
		FST (direct)	1 362.26	0.98	0.00	3 240.31	- 40.36	236.43
		FST (stepped)	1 362.26	0.98	0.00	3 240.31	- 40.36	236.43
	2500	BSM	338.21	0.50	0.00	2 721.76	- 98.33	1 727.45
		FST (direct)	338.21	0.50	0.00	2 721.76	- 98.33	1 727.45
		FST (stepped)	338.21	0.50	0.00	2 721.76	- 98.33	1 727.45
	5000	BSM	54.72	0.12	0.00	741.97	- 43.64	870.36
		FST (direct)	54.72	0.12	0.00	741.97	- 43.64	870.36
		FST (stepped)	54.72	0.12	0.00	741.97	- 43.64	870.36
4	1250	BSM	1 411.72	0.97	0.00	4 047.77	- 41.74	345.23
		FST (direct)	1 411.72	0.97	0.00	4 047.77	- 41.74	345.23
		FST (stepped)	1 411.72	0.97	0.00	4 047.77	- 41.74	345.23
	2500	BSM	437.59	0.55	0.00	3 773.15	- 90.93	1 977.51
		FST (direct)	437.59	0.55	0.00	3 773.15	- 90.93	1 977.51
		FST (stepped)	437.59	0.55	0.00	3 773.15	- 90.93	1 977.51
	5000	BSM	104.63	0.19	0.00	1 463.65	- 53.56	1 348.98
		FST (direct)	104.63	0.19	0.00	1 463.65	- 53.56	1 348.98
		FST (stepped)	104.63	0.19	0.00	1 463.65	- 53.56	1 348.98
5	1250	BSM	1 463.19	0.96	0.00	4 729.28	- 42.05	445.87
		FST (direct)	1 463.19	0.96	0.00	4 729.28	- 42.05	445.87
		FST (stepped)	1 463.19	0.96	0.00	4 729.28	- 42.05	445.87
	2500	BSM	538.30	0.60	0.00	4 784.11	- 85.88	2 162.46
		FST (direct)	538.30	0.60	0.00	4 784.11	- 85.88	2 162.46
		FST (stepped)	538.30	0.60	0.00	4 784.11	- 85.88	2 162.46
	5000	BSM	168.91	0.26	0.00	2 363.29	- 60.98	1 801.02
		FST (direct)	168.91	0.26	0.00	2 363.29	- 60.98	1 801.02
		FST (stepped)	168.91	0.26	0.00	2 363.29	- 60.98	1 801.02

Panel 27: Pricing Results of a European Call Option, based on volatility scenario #3 1 (Forward Volatility Curve extracted from market data and with several complex value nodes)

5.3.4 Pipeline Risk Framework

Now we present in this section the production process defined to the Pipeline Risk linked to a given structured products portfolio. Thus the Panel 28 summarizes its main tasks with a quick description of most important sub-stages and the softwares employed to implement the solutions.

This process defines three important tasks:

1. Pricing ans Sensitivity Factors estimation: this process focuses essentially on the pricing matters and used the FST Method to estimate the price and sensitivity factors of defined structured products. This implementation is centered on an Excel spreadsheet solution which collect data from Bloomberg (Equity options, Interest Rates and CDS data), drive the calculations on Matlab and publish the pricing parameters to produce accurate audit trails.
2. "*Forward Equity Volatility at Risk*" estimation (FEVAR): The next step focuses on the estimation of Volatility Log Return at Risk estimation. It involves Reuters to retrieve volatility data and Matlab to produce the GARCH calibration and forecastings. The final sub-steps consist to retrieve Matlab's results to produce the audit trail.
3. Pipeline Risk Calculation and Report Production: this is the last stage of PRF to produce a real-time estimation of Pipeline Risk. It requires data produced by the two previous tasks but also market data such as interest rates, CDS and Volatility indices. Thus it involves only Excel and Reuters and all intermediate calculations are produced by Excel. The final goal is to produce a daily report linked to the underlying risk exposure.

#	Task Description	Software
1	1) Pricing Model Calibration	
	2) Retrieve Market Volatility Data	Bloomberg
	3) Estimate Spot Implied Volatility Surface and Calibrate the FST Model	MATLAB
	4) Fill Pipeline Risk Report (see 3.1)	
2	1) FEVAR Estimation	
	2) Retrieves Historical Volatility Data	
	3) Estimates GARCH Parameters and Produce Simulations to Estimate FEVAR Returns	MATLAB
	4) Fill Pipeline Risk Report (see 3.1)	
3	1) Pipeline Risk Report	
	2) Retrieves Historical Market Data • Zero Coupon Rates, • CDS Spread, • Equity Implied Volatility	
	3) Estimates GBM Parameters	
	4) Fill Structured Products Parameters	See Figure 67, #1
	5) Automatic Calculations	See Figure 67, #2 and #4
	6) Manage the Risk Exposure regarding the Measure Level	See Figure 67, #3

Panel 28: Overview of the Pipeline Risk Production Process

We present in Figure 67 an anonymized example of Pipeline Risk Report used in Barclays Bank Plc to manage risk exposures during the primary market phase. This report defines several areas:

1. The first area aims to introduce the essential parameters of a given Structured Product in terms of Business Unit, Risk Exposure Size (Traded and Sold), Periods, Sensitivity Factors and Fixed Values.
2. The second area give a detailled view of underlying risk with several metrics such as remaining sales period or risk amount for each risk factors,
3. The third area supplies an agregated view of Pipeline measure according the business unit and position in pipeline workflow,
4. And the fourth area supplies the averaged values of risk factor estimations at the end of the sales period.

We presented in §3.2.4 the principles followed in Barclays Bank Plc (BBPLC) to asses, authorize, scale and manage the risk exposures generated by bussinesses. These principles are defined to get compliant with Basel 2 rules and their implementations are monitored by the "*Financial Services Authority*" (FSA), i.e. the BBPLC's official regulator. These rules aims to estimate the solvency level of a given bank regarding its risk exposures with help of the "*Economic Capital*" (EC). Its estimation varies according to the business and/or the product type, and involves intermediate values to get its final estimation. For instance in retail banking, a mortgage activity requires to calculate the "*Risk-Weighted Asset*" (RWA) intermediate value to get the EC estimation. In case of Market Activities, the allocated EC is based essentially on agregated "*Daily Value-at-Risk*" (DVaR) estimation of underlying risk exposures.

BBPLC's Risk Framework involves that EC allocated to market-linked risk exposures have to be valuated with help of DVaR based on a 99.95% confidence treshold. The Pipeline Risk Framework supplies all of this and so is elligible to EC calculation with the adapted confidence treshold.

The current implementation, designed by the author, supplies an EC estimation following the production process described in Panel 28, but also a set of advanced real-time risk indicators. These indicators detect early signs of market crisis and allow an efficent risk exposure management to avoid the materialization of substantial losses. It is in use only in BBPLC Paris Branch for two years and underwent two important period of distressed market, known as "*The European Crisis*" in mass media. With help of these advanced indicators, we detected the early signs of market crisis and managed efficiently the risk exposure to avoid the materialization of substantial losses

The future steps for current framework are:

1. to standardize the framework following the BBPLC Model Design Policy,
2. to get the necessary assement from Group Market Risk to be elligible to EC calculcation,
3. To homogenize the pipeline risk governance and management in each european country with help of this framework.

<u>Update from the Author</u>: At this moment, this framework is reviewed by Group Market Risk to get the final assessment to be Group EC model.

BBPLC Paris Branch **11/03/2011** | α: 99.95% | Selected Model: 3 - GBM + GARCH

Inputs	1	2	3	4	5	6	7	8	9	TOTAL
Issuer :	BU #1	BU #1	BU #1	BU #1	BU #1	BU #1	BU #1	BU #1		
Nom	BOG / T1	BTB / T1	BO / T1	BFT / T1	BO / T2	BFT / T2	BSB / T1	BSB / T2	To complete	
Status	04 - Selling	04 - Selling	04 - Selling	04 - Selling	04 - Selling	03 - Dealing	04 - Selling	04 - Selling	01 - None	
Counterpart	Barclays	SocGen	RBS plc	Credit Suisse	RBS plc	Credit Suisse	BNP-Paribas	BNP-Paribas	None	
Underlying Index	BGF - WGF	STOXX 50	None	STOXX 50	None	STOXX 50	STOXX 50	STOXX 50	None	
Traded notionnal	10 000 000	15 000 000	20 000 000	15 000 000	10 000 000	15 000 000	10 000 000	5 000 000		100 000 000
Saled Notionnal	149 000	2 096 138	13 135 575	8 229 900	-	-	-	-		23 610 613
Unsold risk parameter	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%		
Beginnig sales period	t	t	t	t	t	t	t + 34	t + 34		
Sales End Date	t + 94	t + 119	t + 94	t + 94	t + 94	t + 94	t + 91	t + 91		
Maturity (in months)	96	96	96	36	96	36	96	96		
Rhô (IR)	- 3.58	- 0.35	- 7.50	- 0.70	- 7.50	- 0.70	- 0.72	- 0.72		
Rhô (Funding)	- 3.58	- 3.50	- 8.15	- 0.70	- 8.80	- 0.70	- 0.72	- 0.72		
Vega	- 0.02	- 0.34		- 0.25		- 0.25	- 0.57	- 0.57		
Rate Fixing	3.020%	3.090%	3.090%	2.140%	3.461%		3.451%	3.451%		
Funding Rate Fixing	1.150%	1.610%	2.180%	0.680%	2.129%		1.218%	1.218%		
Underlying Vol. fixing	17.69%	19.80%		21.53%			27.51%	27.51%		

1

Product	BOG / T1	BTB / T1	BO / T1	BFT / T1	BO / T2	BFT / T2	BSB / T1	BSB / T2	To complete	Total
Sales State	Open position	Open position	Open position	Open position	Open position	Waiting acquisition	Open position	Open position	None	
Traded Notional	10 000 000	15 000 000	20 000 000	15 000 000	10 000 000	15 000 000	10 000 000	5 000 000	-	100 000 000
Sold Notional	149 000	2 096 138	13 135 575	8 229 900	-	-	-	-	-	23 610 613
Remaining Notional (€)	9 851 000	12 903 862	6 864 425	6 770 100	10 000 000	15 000 000	10 000 000	5 000 000	-	76 389 387
Remaining Notional (%)	98.51%	86.03%	34.32%	45.13%	100.00%	100.00%	100.00%	100.00%	-	76.39%
Target Sales by day	252 590	230 426	176 011	173 592	256 410	384 615	166 667	83 333	-	1 723 645
Risky amount (€)	5 000 000	7 500 000	6 864 425	6 770 100	5 000 000	7 500 000	5 000 000	2 500 000	-	46 134 525
Risky amount (%)	50.00%	50.00%	34.32%	45.13%	50.00%	50.00%	50.00%	50.00%	-	46.13%
Pipeline Interest rate	270 197	45 248	741 090	97 988	400 517	84 802	51 309	25 655	-	1 716 806
Funding	555 442	936 413	1 071 221	64 233	864 993	74 149	137 429	68 715	-	3 772 594
Volatility	36 749	969 973	-	582 249	-	585 815	915 043	457 522	-	3 547 351
Total	**862 387**	**1 951 634**	**1 812 310**	**744 469**	**1 265 510**	**744 766**	**1 103 782**	**551 891**	**-**	**9 036 750**
Pipeline / Notional Ratio	**8.62%**	**13.01%**	**9.06%**	**4.96%**	**12.66%**	**4.97%**	**11.04%**	**11.04%**	**0.00%**	**9.04%**

2

Pipeline: Workflow Position by BUs	BU #1	BU #2	BBPLC Paris	Pipeline Limit
Stopped / Finished	-	-	-	
Open position	8 291 984	-	**8 291 984**	**10 000 000**
Waiting Acquisition	744 766	-	744 766	
Simulation	-	-	-	
Total	9 036 750	-	**9 036 750**	**10 000 000**

3

Portfolio Components-at-Risk		
name	%	σ
Interest Rate	4.584%	1.639%
Funding Spread	4.107%	4.635%
Volatility Rate	57.339%	5.049%

4

Figure 67: Example of Pipeline Risk Report used in Barclays Bank Plc Paris Branch (Area #1 = Structured Products Parameters, Area #2 = Intermediate Measures, Area #3 = Consolidated Pipeline Risk Measure segregated by Workflow State and Business Unit (BU), Area #4 = Weighted Average Risk Factor Values).

6. CONCLUSIONS

In this dissertation, we presented that pipeline risk issue is generated by the temporary inventory management of structured products, which are complex market products with important valuation issues. We produced a risk analysis of this situation and showed that pipeline risk is driven by two macro random processes: at first the customer behaviour and at second the influence market movements on prices. After a review of most important constraints, we decided to produce a risk framework which must be able to estimate market price and sensitivity factors of each structured product, and forward risk factors to produce a "*Value-at-Risk*" (VaR) measure. Due to its complexities and all necessary requirements, we couldn't produce a relevant measure of forward impacts of customers' behavior and hence we considered it as static. However we keep in mind to produce "*What-If*" based scenarios to integrate it in future developments.

Producing the Pipeline Risk Framework required to integrate several model approaches to estimate each component. The core technical component dealt essentially with by the presentation of the "*Fourier Space Time-stepping*" (FST) method and how to use it to price derivatives and option structures.

Thereby we proceeded to the FST assessment in three steps:

- The first step aimed to present the essential theoretical points of FST method, such as how to solve the PIDE in Fourier Space and how to produce accurate estimations of the price and sensitivity factors for a given derivative / option structure.
- The second step consisted in the production of a pricing benchmark where we increased step by step the underlying complexity of the priced derivative to finish with real structured products estimations. During this benc
- The third step presented the calibration process of FST method regarding the parameters used in Discrete Fourier and the market parameters. In this part, we highlighted one important issue according to the market data integration: the presence of complex valued nodes while estimating forward implied volatilities. This is an important issue for $\mathbb{R}$ based pricing methods because it introduces a bias during the Time value integration.

All these assessments reavealed several important strenghts of FST method in option pricing:

1. Its capacity to integrate discontinuities introduced either by the option structure (presence of one or more indicator functions), the stochastic process(es) replicating the underlying(s) (i.e. Levy processes) or the calibration process (e.g. complex valued forward volatility).
2. It produces accurate estimations with low time consumption, as showed all along the pricing benchmark. Moreover we presented each FST algorithms used and most of them are easily implementable and maintainable. The only exception is the implementation of American option with continuous barrier which requires more effort during the conception of its algorithm.
3. The last one is the pricing stability: a pricing produced with fixed parameters will remain constant whenever it will be produced.

Hence we demonstrated that FST method is a good alternative to Monte Carlo approach for well known derivatives and a complementary method while pricing complex option structures. However all results produced in this dissertation were based on a restricted assumption space, i.e. mono equity underlying with a GBM stochastic process. So this limits the scope of this conclusion, and its extension requires further investigations on the FST pricing capacity with multiple underlyings (e.g. correlation issues), different assets (i.e. interest rates, commodities, ...) or different stochastic processes (e.g. Variance Gamma, GBM with Jumps, ...).

However Vladimir Surkov investigated most of these points in his papers and demonstrated the FST capacity to encompass these issues. For examples, he showed how to integrate an IR specific stochastic process, how to price a spread option or how to integrate different stochastic processes. These results combined with mathematical and implementation principles presented all along this dissertation assess the extension capacity of FST method to solve the issues presented in previous paragraph.

Now let's have a look on the potential consequences on business practices if FST method is broadly adopted. Thus these impacts can be divided into three categories:

1. Cost of IT infrastucture,
2. Pooling needs of several internal businesses,
3. And Business evolution in a post "*Solvency 2*" (S2) environment.

1) Cost of IT infrastructure:

The first impact is the consequence of flaws of Monte Carlo Method while pricing (and hedging) structured products: producing unstable estimations and requiring high time consumption. These two flaws can be "*avoided*" with use of technical tricks but the drawbacks is an expensive IT structure. Indeed most of financial institutions developed IT solutions based on distributed calculations and / or grid computing. These requires powerful computers / servers, dedicated softwares and skilled IT teams to manage this highly competitive environment. As we saw during the pricing benchmark, FST method is ten to one hundred faster than Monte Carlo while producing accurate and stable estimations. Please note that FST method is only a logic solution and its performance can be improved further with accurate hardware / distributed software environment. Hence use of FST method allows to outreach these limits while decreasing the cost of the IT infrastructure.

2) Pooling needs of several internal businesses

The second impact is the capacity to provide a repository to fulfil several business needs at once. Indeed we can give several examples with a more or less common need:

1. A front office team needs fast and accurate results, fitted to "*spot*" market conditions,
2. Prices used to feed general ledgers and to produce official results must be stable and have a reliable audit trail to be provided to the firm's writers.
3. Estimating potential VaRs for given risk exposures requires stable estimations of price and sensitivity factors,
4. Or forecasting future incomes / losses / impairments used in strategic decisions requires reliable assumptions and methodologies inline with business reality.

All these needs can be fulfilled with help of FST method if implemented inside a firm's business processes. This will centralize all calculations into one single repository, managed by fewer people to maintain / evolve them. And the final consequences will be coherent figures whatever the business perspective and a relevant decrease of HR and IT costs.

3) Business evolution in a post-S2 environment

The last impact concerns the future evolution of bank practices induced by the implementation of S2 regulatory rules by all main european insurance firms. These firms collect an important part of individual investments and are critical customers for bank institutions. In few words, these new solvency rules require that insurers must have a good knowledge and control of risks under their responsabilities. Thereby risks are divided into three pillars (similarly to Basel 2 rules) where all the most important quantification aspects are collected into the first pillar. And this pillar defines with two major risk measures:

1. The "*Minimal Capital Requirements*" (MCR) which represents the minimal required capital to exercise legally the insurance activity,
2. The "*Solvency Capital Requirement*" (SCR) which represents the necessary capital to hedge a global risk exposure, estimated on a 99.50% confidence level.

These new risk measures will change deeply the investment behaviour of insurance firms because:

1. Investment teams will have to estimate finely underlying risks according to the investment risk profile and risks embedded into current assets portfolios,
2. New "*Investment Key Indicators*" (IKIs) will be produced to integrate these new internal measures. For instance, we can imagine an "*Asset yield / SCR*" ratio used during the asset selection phase.
3. And the investment follow-up will require to update the risk key indicators on a regular basis (at least quarterly).

Structured Products are eligible investment products for insurers for their BAU asset management. However they present three important issues regarding these new solvency rules:

1. Transaction nature: It is an “*Over The Counter*” (OTC) operation with only one counterpart, thereby the notion of “*Mark To Market Price*” is sullied because a given structured product can be negotiated only in a limited market constituted by a single contributor. Hence this implies that both counterparties must agree on each detail, including the pricing methodology, to estimate that a fair price has been negotiated.
2. Preliminary risk assessment by insurers during investment phase: the asset management team must evaluate the new investments and assess that they are inline with firm’s policies, guidances and risk measures. This last item is an internal risk measure which can’t be apprehended by external counterparties.
3. Post-Report: It is related to the risk indicators provided by the product’s issuer and S2 rules require these figures to be reliable and stable. So dealing with counterparties requires also to put in place post-trade agreements and services to get these figures and the required audit trail.

We saw previously that the random nature of Monte Carlo creates unstable results which are not adapted to a post S2 environment. Moreover this numerical approach produces an opacity, generated mostly from market data used during the calibration phase. This could artificially hide additional margins that are not compliant with the insurers’ commitment to act on the behalf of policyholders’ best interest.

Hence all these points will downgrade current investment bank business model, based essentially on products uniformization. Indeed submitting an investment product in such an environment will be more difficult because it may be adapted to one insurer and not to another, regarding their respective risk measures and IKIs. So the investment bank model has to evolve to be inline with insurers’ needs. This evolution will be essentially related to trade and post-trade agreements, but also on the underlying calculation processes used to evaluate the fairness of an investment price. Thereby we can imagine the evolution of ISDA documentation with the integration of a standard “*calculation clause*” to establish the basis of price evaluation. And knowing now the strenghts of FST method, this method can be part of this evolution as method of reference.

We will conclude this dissertation with the fact that the FST method developed by V. Surkov is an interesting alternative to Monte Carlo pricing method, because it presents lots of strenghts we highlighted all along this dissertation. And we think we demonstrated that it is farly elligible to be part of business evolution because it clarifies the pricing process of derivative-based products (whatever their complexities) and it can be easily adapted in any business environment with simple tools.

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7. ANNEXE

7.1 NOTATIONS

Stochastic Processes	
$X(t), \mathbf{X}(t)$	Log Spot Price
$S(t), \mathbf{S}(t)$	Spot Price
$W(t), \mathbf{W}(t)$	Brownian motion
$N(t), \mathbf{N}(t)$	Poisson process governing the arrival of jumps or losses
$J(t), \mathbf{J}(t)$	Jump processes

Model Parameters	
r	Risk free interest rate
$\gamma, \boldsymbol{\gamma}$	Brownian motion with drift
σ, Σ	Brownian motion volatility and variance-covariance matrix
ρ	Correlation of Brownian motions
$\tilde{v}$	Poisson random measure
$\nu, \boldsymbol{\nu}$	Lévy density
ϑ	Stochastic volatility level
$\kappa, \boldsymbol{\kappa}$	Mean-reversion speed
$\theta, \boldsymbol{\theta}$	Mean-reversion level
λ	Jump arrival rate
$\tilde{\mu}, \tilde{\sigma}$	Merton jump-diffusion model
η_p, η_+, η_-	Kou jump-diffusion model
VG	Variance Gamma model
$CGMY$	Carr-Geman-Madan-Yor model

Option Parameters	
$\varphi(\mathbf{S})$	Payoff function
K	Strike Price
T	Time to maturity
B	Barrier

Option Value	
$V(t,\mathbf{x})$	Option Value function
$v(t,\mathbf{x})$	Discount-adjusted, log-transformed option value function
$\mathbf{V}_m$	Option value on a discrete grid at time $t = m$
$\mathbf{v}_m$	Discount-adjusted, log-transformed option value on a discrete grid at time $t = m$
$\mathbf{v}^{\langle N,M \rangle}$	Discount-adjusted, log-transformed option value on a discrete grid as a function of N space point and M time points

Others	
$\mathcal{F}[\star](\omega),\hat{\star}$	Continuous Fourier transform of $\star$
$\mathcal{F}^{-1}[\star](\mathbf{x})$	Continuous inverse Fourier transform of $\star$
$\mathcal{L}$	Infinitesimal generator
$\mathcal{D},\mathcal{J}$	Diffusion and integral (jump) components of the infinitesimal generator
$\mathbb{I}$	Identity vector / matrix
$\Psi(\omega)$	Characteristic exponent
$\mathbb{P}$	Risk-neutral pricing measure
$\mathbb{Q}$	Real-world pricing measure
$\mathbb{E}_t^{\star}$	Expectation under measure given information at t
$f_{\star}$	Probability density function of $\star$

7.2 SENSITIVITY FACTOR TYPES

Panel 29: First order derivatives:

Name	Definition
Delta	It measures the sensitivity to changes in the underlying asset's price. Delta is the first derivative of the value V of the option with respect to the underlying instrument's price S such as: $\Delta = \frac{\partial V}{\partial S}$
Vega	It measures sensitivity to underlying asset's volatility. Vega is the derivative of the option value with respect to the volatility of the underlying such as $\nu = \frac{\partial V}{\partial \nu}$
Theta	It measures the sensitivity of the value of the derivative to the passage of time $\Theta = -\frac{\partial V}{\partial \tau}$
Rho	It measures sensitivity to the applicable interest rate. Rho is the derivative of the option value with respect to the risk free rate. Except under extreme circumstances, the value of an option is least sensitive to changes in the risk-free-interest rates. For this reason, Rho is the least used of the first-order Greeks with $\rho = \frac{\partial V}{\partial r}$

Panel 30: 2nd order derivatives

Name	Definition
Charm	It measures the instantaneous rate of change of delta over the passage of time. Charm can be an important Greek to measure/monitor when delta-hedging a position over a weekend. Charm is a second-order derivative of the option value, once to price and once to time. It is also then the (negative) derivative of theta with respect to the underling's price. Hence $\text{Charm} = \frac{\partial \Delta}{\partial \tau} = -\frac{\partial \Theta}{\partial S} = \frac{\partial^2 V}{\partial S \partial \tau}$ is also known as "*delta decay*"
dVega/dTime	It measures the rate of change in the Vega with respect to the passage of time. dVega/dTime is the second derivative of the value function: once to volatility and once to time such as $\frac{\partial \nu}{\partial \tau} = \frac{\partial^2 V}{\partial \sigma \partial \tau}$
Gamma	It measures the rate of change in the delta with respect to changes in the underlying price. Gamma is the second derivative of the value function with respect to the underlying price. Gamma is important because it corrects for the convexity of value and it is measured as $\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$
Vanna	It is a second order derivative of the option value, once to the underlying spot price and once to volatility. It is mathematically equivalent to the sensitivity of Delta with respect to change in volatility; or alternately, the partial derivative of Vega with respect to the underlying instrument's price. Vanna can be a useful sensitivity to monitor when maintaining a delta- or Vega-hedged portfolio as Vanna will help the trader to anticipate changes to the effectiveness of a delta-hedge as volatility changes or the effectiveness of a Vega-hedge against change in the underlying spot price. Its measure is $\text{Vanna} = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \nu}{\partial S} = \frac{\partial^2 V}{\partial S \partial \sigma}$

Volga	It measures second order sensitivity to volatility. Volga is the second derivative of the option value with respect to the volatility, or stated another way. Volga measures the rate of change to Vega as volatility changes. With positive Volga, a position will become long Vega as implied volatility increases and short Vega as it decreases, which can be scalped in a way analogous to long gamma. And an initially Vega-neutral, long-Volga position can be constructed from ratios of options at different strikes. Volga is positive for options away from the money, and initially increases with distance from the money (but drops off as Vega drops off). (Specifically, Volga is positive where the usual d1 and d2 terms are of the same sign, which is true when d2 > 0 or d1 < 0.). Hence we define $\text{Volga} = \frac{\partial \nu}{\partial \sigma} = \frac{\partial^2 V}{\partial \sigma^2}$ and it is also known as "*Vomma*" or "*Vega Convexity*"

Panel 31: 3rd order derivatives

Name	Definition
Color	It measures the rate of change of gamma over the passage of time. Color is a third-order derivative of the option value, twice to underlying asset price and once to time. Color can be an important sensitivity to monitor when maintaining a gamma-hedged portfolio as it can help the trader to anticipate the effectiveness of the hedge as time passes. We define $\text{Color} = \frac{\partial \Gamma}{\partial \tau} = \frac{\partial^3 V}{\partial S^2 \partial \tau}$
Speed	It measures the rate of change in Gamma with respect to changes in the underlying price. Speed is the third derivative of the value function with respect to the underlying spot price. Speed can be important to monitor when delta-hedging or gamma-hedging a portfolio. $\text{Speed} = \frac{\partial \Gamma}{\partial S} = \frac{\partial^3 V}{\partial S^3}$ is also sometimes referred to as the "*Delta of the Gamma*".
Ultima	It measures the sensitivity of the option Volga with respect to change in volatility. Ultima is a third-order derivative of the option value to volatility. We define $\text{Ultima} = \frac{\partial \text{Vomma}}{\partial \sigma} = \frac{\partial^3 V}{\partial \sigma^3}$
Zomma	It measures the rate of change of gamma with respect to changes in volatility. Zomma is the third derivative of the option value, twice to underlying asset price and once to volatility. Zomma can be a useful sensitivity to monitor when maintaining a gamma-hedged portfolio as Zomma will help the trader to anticipate changes to the effectiveness of the hedge as volatility changes. $\text{Zomma} = \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial \text{Vanna}}{\partial S} = \frac{\partial^3 V}{\partial S^2 \partial \sigma}$

Panel 32: Special Sensitivity Factors

Name	Definition
Alma	Measures the sensitivity of option price to change in the jump arrival rate.
Fugit	The fugit is the optimal date to exercise an American or Bermudan option. It is useful to compute it for hedging purpose, for example you can represent flows of an American swaption like the flows of a swap starting at the fugit multiplied by delta then use these to compute sensitivities.
Lambda	It is the percentage change in option value per percentage change in the underlying price, a measure of leverage, sometimes called gearing. $\lambda = \frac{\partial V}{\partial S} \times \frac{S}{V}$

7.3 CLOSED FORMULAE OF PRICE AND SENSITIVITY FACTORS

Common factors to all closed formulae:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\tau = T - t$$

$$b = r - q$$

Panel 33: Closed Formula for a European Option

European Option	Call	Put
Price	$C = S \cdot N(d_1) - Ke^{-r\tau} N(d_2)$	$C = Ke^{-r\tau} N(-d_2) - S \cdot N(d_1)$
Delta	$\frac{\partial C}{\partial S} = N(d_1)$	$\frac{\partial P}{\partial S} = -N(-d_1)$
Gamma	$\frac{\partial^2 C}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}} = \frac{\partial^2 P}{\partial S^2}$	
Rho	$\frac{\partial C}{\partial r} = \tau Ke^{-r\tau}\phi(d_2)$	$\frac{\partial C}{\partial r} = -\tau Ke^{-r\tau}\phi(-d_2)$
Theta	$\frac{\partial C}{\partial t} = -\frac{S \cdot \sigma \cdot \phi(d_1)}{2\sqrt{\tau}} - rKe^{-r\tau}N(d_2)$	$\frac{\partial P}{\partial t} = -\frac{S \cdot \sigma \cdot \phi(d_1)}{2\sqrt{\tau}} + rKe^{-r\tau}N(-d_2)$
Vega	$\frac{\partial C}{\partial \sigma} = S \cdot \phi(d_1) \cdot \sqrt{\tau}$	

Panel 34: Closed Formula for an "Asset or Nothing" Digital Option

Payoff	Call	Put
Price	$C = S \cdot N(d_1)$	$P = S \cdot N(-d_1)$
Delta	$\frac{\partial C}{\partial S} = N(d_1) + \frac{\phi(d_1)}{\sigma\sqrt{\tau}}$	$\frac{\partial P}{\partial S} = N(-d_1) - \frac{\phi(-d_1)}{\sigma\sqrt{\tau}}$
Gamma	$\frac{\partial^2 C}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}}\left[1 - \frac{d_1}{\sigma\sqrt{\tau}}\right]$	$\frac{\partial^2 P}{\partial S^2} = -\frac{\phi(-d_1)}{S\sigma\sqrt{\tau}}\left[1 - \frac{d_1}{\sigma\sqrt{\tau}}\right]$
Rho	$\frac{\partial C}{\partial r} = \sqrt{\tau}\phi(d_1)$	$\frac{\partial P}{\partial r} = -\phi(-d_1)\sqrt{\tau}$
Theta	$\frac{\partial C}{\partial t} = \frac{S \cdot \phi(d_1)}{2\tau} \cdot \left[d_1 - \frac{2\left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right]$	$\frac{\partial P}{\partial t} = -S\frac{\phi(-d_1)}{2\tau}\left[d_1 - \frac{2\left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right]$
Vega	$\frac{\partial C}{\partial \sigma} = -\frac{d_2 S}{\sigma} \cdot \phi(d_1)$	$\frac{\partial P}{\partial \sigma} = \frac{d_2 S}{\sigma}\phi(-d_1)$

Panel 35: Closed Formula for a "Cash or Nothing" Digital Option

Payoff	Call	Put
Price	$C = cash \cdot e^{-rT} N(d_2)$	$P = cash \cdot e^{-rT} N(-d_2)$
Delta	$\frac{\partial C}{\partial S} = cash \cdot e^{-r\tau} \cdot \frac{\phi(d_2)}{S\sigma\sqrt{\tau}}$	$\frac{\partial P}{\partial S} = -cash \cdot e^{-r\tau} \cdot \frac{\phi(-d_2)}{S\sigma\sqrt{\tau}}$
Gamma	$\frac{\partial^2 C}{\partial S^2} = -cash \cdot e^{-r\tau} \cdot \frac{\phi(d_2)}{S\sigma\sqrt{\tau}} \cdot \left[\frac{1}{S} + d_2\right]$	$\frac{\partial^2 P}{\partial S^2} = cash \cdot e^{-r\tau} \cdot \frac{\phi(-d_2)}{S\sigma\sqrt{\tau}} \cdot \left[\frac{1}{S} + d_2\right]$
Rho	$\frac{\partial C}{\partial r} = cash \cdot e^{-r\tau}\left[\frac{\sqrt{\tau} \cdot \phi(d_2)}{S} - \tau N(d_2)\right]$	$\frac{\partial P}{\partial r} = -cash \cdot e^{-r\tau}\left[\frac{\sqrt{\tau} \cdot \phi(-d_2)}{S} + \tau N(-d_2)\right]$
Theta	$\frac{\partial C}{\partial t} = \frac{cash \cdot \phi(d_2) \cdot e^{-r\tau}}{2\tau}\left[d_1 - \frac{(r + \sigma^2)\tau}{\sigma\sqrt{\tau}}\right] - rC$	$\frac{\partial P}{\partial t} = -\frac{cash \cdot \phi(-d_2) \cdot e^{-r\tau}}{2\tau}\left[d_1 - \frac{(r + \sigma^2)\tau}{\sigma\sqrt{\tau}}\right] - rP$
Vega	$\frac{\partial C}{\partial \sigma} = -\frac{cash \cdot d_1 \cdot e^{-r\tau}\phi(d_2)}{\sigma}$	$\frac{\partial P}{\partial \sigma} = \frac{cash \cdot d_1 \cdot e^{-r\tau}\phi(-d_2)}{\sigma}$

Panel 36: Closed Formula for a Forward Starting European Option

Payoff	Call	Put
Price	$C = S_0 e^{(b-r)t} \times \left[e^{(b-r)\tau} N(d_1) - \alpha e^{-r\tau} N(d_2) \right]$	$P = S_0 e^{(b-r)t} \times \left[\alpha e^{-r\tau} N(-d_2) - e^{(b-r)\tau} N(-d_1) \right]$
Delta	$\frac{\partial C}{\partial S_0} = e^{(b-r)t} \times \left[e^{(b-r)\tau} N(d_1) - \alpha e^{-r\tau} N(d_2) \right]$	$\frac{\partial P}{\partial S_0} = e^{(b-r)t} \times \left[\alpha e^{-r\tau} N(-d_2) - e^{(b-r)\tau} N(-d_1) \right]$
Gamma	$\frac{\partial^2 C}{\partial S^2} = 0$	$\frac{\partial^2 P}{\partial S^2} = 0$
Rho	$\frac{\partial C}{\partial r} = \alpha \cdot \tau \cdot S_0 \cdot e^{(b-r)t-r\tau} \cdot N(d_2)$	$\frac{\partial P}{\partial r} = -\alpha \cdot \tau \cdot S_0 \cdot e^{(b-r)t-r\tau} \cdot N(-d_2)$
Theta	$\frac{\partial C}{\partial t} = -\frac{S_0 \cdot \sigma \cdot e^{(b-r)T}}{2\sqrt{\tau}} \cdot \phi(d_1) - \alpha \cdot r \cdot S_0 \cdot e^{(b-r)t-r\tau} \cdot N(d_2)$	$\frac{\partial P}{\partial t} = -\frac{S_0 \cdot \sigma \cdot e^{(b-r)T}}{2\sqrt{\tau}} \cdot \phi(d_1) + \alpha \cdot r \cdot S_0 \cdot e^{(b-r)t-r\tau} \cdot N(-d_2)$
Vega	$\frac{\partial C}{\partial \sigma} = \sqrt{\tau} \cdot S_0 \cdot e^{(b-r)} \cdot \phi(d_1)$	$\frac{\partial P}{\partial \sigma} = \sqrt{\tau} \cdot S_0 \cdot e^{(b-r)} \cdot \phi(-d_1)$

With $\alpha = \frac{K}{S_0}$

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