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Stability criteria for Timoshenko columns with intermediate and end concentrated axial loads

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Abstract

This paper presents exact stability criteria and buckling loads of Timoshenko columns under intermediate and end concentrated loads. The new buckling solutions show clearly the erosive effect of transverse shear deformation on the buckling capacity of the column under compressive loads, especially in stocky and built-up columns with high values of shear parameter $s = EI/(\kappa_s GAL^2)$ and more so when the columns ends are highly restrained such as fixed-pinned columns and fixed-fixed columns. Furthermore, for columns under an intermediate compressive load acting near the base of the column, the effect of transverse shear deformation becomes significant even when the entire column is slender. This is because the column buckling problem behaves like a stocky column problem when the load is acting over a very short portion of the column. © 2002 Published by Elsevier Science Ltd.

Keywords: Column; Buckling; Transverse shear deformation; Timoshenko; Stability criteria; Intermediate axial load

1. Introduction

Elastic buckling solutions for columns of various loading, restraint and boundary conditions are well documented in the open literature. For example, the Handbook of Structural Stability (1970) [1] contains comprehensive buckling solutions of not

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only columns, but also of beams, arches, rings, plates and shells. However, most of the buckling solutions are derived on the basis of the Euler-Bernoulli assumption that neglects the effect of transverse shear deformation. This effect becomes significant when dealing with stocky columns or built-up columns. Its neglect can lead to a non-conservative design because the shear deformation effect reduces the buckling capacities of columns and sometimes this neglect has resulted in major disasters, e.g. the first Quebec Bridge. Thus, it is important for designers to be wary of using the Euler solutions for design checks against buckling when dealing with stocky and built-up columns.

Recently, Wang et al. (2002) [2] revisited the Euler buckling problem of columns subjected to intermediate and end concentrated axial loads. Practical situations where one finds intermediate axial loads on columns include the lifting of columns with the attachment placed at some point along the column length or columns supporting a crane rail or an intermediate floor. This study extends the work to include the effect of transverse shear deformation on the buckling solutions. To do this, the Timoshenko column theory has been adopted. The theory frees the Euler assumption that constrained the normals of the cross-section before buckling to remain normal to the deformed centroidal axis by allowing the normal to rotate with respect to the centroidal axis; thereby admitting a non-zero constant shear strain.

In this paper, the exact stability criteria for the elastic buckling of Timoshenko columns with intermediate and end concentrated axial loads are derived. Various combinations of intermediate loads to end axial loads are treated. The combinations include situations where one segment of the column could be in tension while the other segment in compression and also where one segment experiences no axial load while the other segment a compressive load. The new exact stability criteria (a) elucidate the effect of transverse shear deformation on the buckling loads, and (b) provide comprehensive buckling solutions for columns with various boundary conditions, intermediate load positions and ratios of intermediate to end axial load magnitudes.

2. Problem formulation and governing equations

Consider a column of flexural rigidity EI, shear rigidity GA and length L. It is subjected to an end compressive force P_2 and a concentrated axial compressive axial force P_1 at a distance $x = aL$ from the bottom end as shown in Fig. 1.

According to the Timoshenko column theory, the governing equations are given by [3]

$$\kappa_s GA\frac{d\phi}{dx} + (\kappa_s GA - N)\frac{d^2w}{dx^2} = 0, \tag{1a}$$

$$EI\frac{d^2\phi}{dx^2} - \kappa_s GA\left(\phi + \frac{dw}{dx}\right) = 0, \tag{1b}$$

where N is the axial compressive force in the segment considered, w the transverse

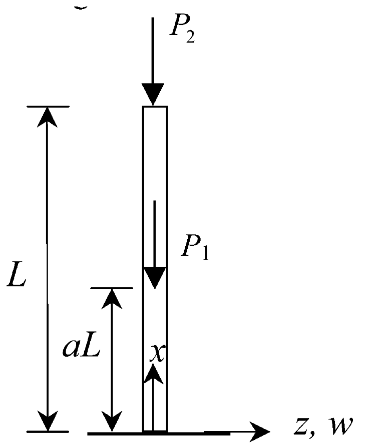

Fig. 1. Column with intermediate and end concentrated axial loads.

displacement, ϕ the rotation and κ_s the shear correction factor that compensates for the error in assuming constant transverse shear strains (and hence constant transverse shear stresses) through the column thickness. For example, $\kappa_s = 1/2$ for a thin tube and $\kappa_s = 9/10$ for a column with solid circular cross-section [4]. In the case of built-up columns, the expressions for $\kappa_s GA$ cast in terms of the geometrical dimensions and patterns of the component members may be obtained from Timoshenko and Gere (1960) [3] and Gjelsvik (1991) [5].

By introducing the following nondimensional parameters,

$$\bar{x} = \frac{x}{L}, \bar{w} = \frac{w}{L}, s = \frac{EI}{\kappa_s GAL^2}, t^2 = \frac{NL^2}{EI}, \tag{2a–d}$$

the governing equations in Eqs. (1a, b) may be expressed as

$$\frac{d\phi}{d\bar{x}} + (1-st^2)\frac{d^2\bar{w}}{d\bar{x}^2} = 0, \tag{3a}$$

$$s\frac{d^2\phi}{d\bar{x}^2} - \phi - \frac{d\bar{w}}{d\bar{x}} = 0. \tag{3b}$$

In solving the buckling problems at hand, we divide the column into two segments, *i.e.* segment 1 (*i.e.* $0 \leq \bar{x} \leq a$) and segment 2 (*i.e.* $a \leq \bar{x} \leq 1$). Also we identify the following cases of problems.

- Case 1: $P_2 > 0$ and $(P_1 + P_2) > 0$, implying that both segments 1 and 2 are in compression.
- Case 2(a): $P_2 > 0$ and $(P_1 + P_2) < 0$, implying that segment 2 is in compression while segment 1 is in tension.
- Case 2(b): $P_2 < 0$ and $(P_1 + P_2) > 0$, implying that segment 2 is in tension while segment 1 is in compression.

- Case 3(a): $P_2 = 0$ and $P_1 > 0$, implying that segment 2 has no axial load while segment 1 is in compression.
- Case 3(b): $P_2 > 0$ and $(P_1 + P_2) = 0$, implying that segment 2 is in compression while segment 1 has no axial load.

It should be noted that Cases 2a and 2b are related by suitable parametric changes. This also applies for Cases 3a and 3b. In view of this, the sequel will only focus on Cases 1, 2a and 3a. Note that for buckling at least one of the segments must be in compression. While columns supporting two floor levels may be subjected to compressive forces in two segments, we may have a tension force in a segment of the columns in lifting operations. This explains the consideration of the above cases.

Through some algebraic manipulations, one can easily uncouple the two governing equations (3a) and (3b) into two fourth-order differential equations that involve $\bar{w}$ and ϕ. One may refer to Appendix A for a detailed derivation of the general solutions for $\bar{w}$ and ϕ for cases where the considered segment is in compression (*i.e.* $N > 0$), in tension (*i.e.* $N < 0$) or where the segment is not subjected to any axial force (*i.e.* $N = 0$). The governing equations and the general solutions of w and ϕ for the two segments for Cases 1, 2a and 3a are summarized below.

Case 1: $P_2 > 0$ and $(P_1 + P_2) > 0$The governing equations are given by: For $0 \le \bar{x} \le a$,

$$\frac{d^4\bar{w}_1}{d\bar{x}^4} + k_1^2\frac{d^2\bar{w}_1}{d\bar{x}^2} = 0,\ \frac{d^4\phi_1}{d\bar{x}^4} + k_1^2\frac{d^2\phi_1}{d\bar{x}^2} = 0, \tag{4a, b}$$

where

$$k_1^2 = \frac{{t_1}^2}{1-s{t_1}^2},\ t_1^2 = \frac{(P_1 + P_2)L^2}{EI}. \tag{4c, d}$$

For $a \le \bar{x} \le 1$,

$$\frac{d^4\bar{w}_2}{d\bar{x}^4} + k_2^2\frac{d^2\bar{w}_2}{d\bar{x}^2} = 0,\ \frac{d^4\phi_2}{d\bar{x}^4} + k_2^2\frac{d^2\phi_2}{d\bar{x}^2} = 0, \tag{5a, b}$$

where

$$k_2^2 = \frac{{t_2}^2}{1-s{t_2}^2},\ t_2^2 = \frac{P_2L^2}{EI}. \tag{5c, d}$$

The general solutions for Eqs. (4a,b) and (5a,b) are given by

$$\bar{w}_1(\bar{x}) = A_1\sin k_1\bar{x} + A_2\cos k_1\bar{x} + A_3\bar{x} + A_4, \tag{6a}$$

$$\phi_1(\bar{x}) = \frac{{t_1}^2}{k_1}A_2\sin k_1\bar{x} - \frac{{t_1}^2}{k_1}A_1\cos k_1\bar{x} - A_3, \tag{6b}$$

$$\bar{w}_2(\bar{x}) = B_1\sin k_2\bar{x} + B_2\cos k_2\bar{x} + B_3\bar{x} + B_4, \tag{6c}$$

$$\phi_2(\bar{x}) = \frac{{t_2}^2}{k_2}B_2\sin k_2\bar{x} - \frac{{t_2}^2}{k_2}B_1\cos k_2\bar{x} - B_3, \tag{6d}$$

where the subscripts 1 and 2 denote quantities in segments 1 and 2, respectively.

Case 2: $P_2 > 0$ and $(P_1 + P_2) < 0$ The governing equations are given by: For $0 \leq \bar{x} \leq a$,

$$\frac{d^4\bar{w}_1}{d\bar{x}^4} - k_1^2\frac{d^2\bar{w}_1}{d\bar{x}^2} = 0, \quad \frac{d^4\phi_1}{d\bar{x}^4} - k_1^2\frac{d^2\phi_1}{d\bar{x}^2} = 0, \tag{7a, b}$$

where

$$k_1^2 = \frac{{t_1}^2}{1 + s{t_1}^2}, \quad t_1^2 = \frac{|P_1 + P_2|L^2}{EI}. \tag{7c, d}$$

For $a \leq \bar{x} \leq 1$,

$$\frac{d^4\bar{w}_2}{d\bar{x}^4} + k_2^2\frac{d^2\bar{w}_2}{d\bar{x}^2} = 0, \quad \frac{d^4\phi_2}{d\bar{x}^4} + k_2^2\frac{d^2\phi_2}{d\bar{x}^2} = 0, \tag{8a, b}$$

where

$$k_2^2 = \frac{{t_2}^2}{1 - s{t_2}^2}, \quad t_2^2 = \frac{P_2 L^2}{EI}. \tag{8c, d}$$

The general solutions to Eqs. (7a,b) and (8a,b) are given by

$$\bar{w}_1(\bar{x}) = A_1 \sinh k_1\bar{x} + A_2 \cosh k_1\bar{x} + A_3\bar{x} + A_4, \tag{9a}$$

$$\phi_1(\bar{x}) = -\frac{{t_1}^2}{k_1}A_2 \sinh k_1\bar{x} - \frac{{t_1}^2}{k_1}A_1 \cosh k_1\bar{x} - A_3, \tag{9b}$$

$$\bar{w}_2(\bar{x}) = B_1 \sin k_2\bar{x} + B_2 \cos k_2\bar{x} + B_3\bar{x} + B_4, \tag{9c}$$

$$\phi_2(\bar{x}) = \frac{{t_2}^2}{k_2}B_2 \sin k_2\bar{x} - \frac{{t_2}^2}{k_2}B_1 \cos k_2\bar{x} - B_3. \tag{9d}$$

Case 3: $P_2 = 0$ and $P_1 > 0$

The governing equations are given by:For $0 \leq \bar{x} \leq a$,

$$\frac{d^4\bar{w}_1}{d\bar{x}^4} + k_1^2\frac{d^2\bar{w}_1}{d\bar{x}^2} = 0, \quad \frac{d^4\phi_1}{d\bar{x}^4} + k_1^2\frac{d^2\phi_1}{d\bar{x}^2} = 0, \tag{10a, b}$$

where

$$k_1^2 = \frac{{t_1}^2}{1 - s{t_1}^2}, \quad t_1^2 = \frac{P_1 L^2}{EI}. \tag{10c, d}$$

For

$$a \leq \bar{x} \leq 1$$

,

$$\frac{d^4\bar{w}_2}{d\bar{x}^4} = 0, \quad \frac{d^4\phi_2}{d\bar{x}^4} = 0. \tag{11a, b}$$

The general solutions for Eqs. (10a,b) and (11a,b) are given by:

$$\bar{w}_1(\bar{x}) = A_1 \sin k_1 \bar{x} + A_2 \cos k_1 \bar{x} + A_3 \bar{x} + A_4, \tag{12a}$$

$$\phi_1(\bar{x}) = \frac{t_1^{\,2}}{k_1} A_2 \sin k_1 \bar{x} - \frac{t_1^{\,2}}{k_1} A_1 \cos k_1 \bar{x} - A_3, \tag{12b}$$

$$\bar{w}_2(\bar{x}) = B_1 \bar{x}^3 + B_2 \bar{x}^2 + B_3 \bar{x} + B_4, \tag{12c}$$

$$\phi_2(\bar{x}) = -3(\bar{x}^2 + 2s) B_1 - 2B_2 \bar{x} - B_3, \tag{12d}$$

where $s = EI/(\kappa_s GAL^2)$ [as defined in Eq. (2c)].

From Eqs. (6a–d), (9a–d) and (12a–d), it is clear that the buckling problem involves a total of eight constants (*i.e.* A_i and B_i where i =1, 2, 3, 4.). Four of the constants can be solved from each of the two boundary conditions at each end of the column (*i.e.* at $\bar{x} = 0$ and $\bar{x} = 1$). Now, the classical boundary conditions are

$$\text{For free end: } \bar{V}_j = \phi_j + (1 - s t_j^{\,2}) \frac{d\bar{w}_j}{d\bar{x}} = 0,\ \bar{M}_j = \frac{d\phi_j}{d\bar{x}} = 0, \tag{13a, b}$$

$$\text{For pinned end: } \bar{w} = 0,\ \bar{M}_j = \frac{d\phi_j}{d\bar{x}} = 0, \tag{14a, b}$$

$$\text{For fixed end: } \bar{w} = 0,\ \phi = 0, \tag{15a, b}$$

where the subscript j denotes the segment number (*i.e.* $j = 1$ or 2), $\bar{V}_j$ and $\bar{M}_j$ are the nondimensionalized shear force and the nondimensionalized bending moment corresponding to the segment j.

After the substitution of the boundary conditions into the general solutions for $\bar{w}$ and ϕ, there remains four constants which can be evaluated by the four continuity conditions at the segment boundary $\bar{x} = a$. These continuity conditions for deflection, slope, bending moment and shear force, are respectively,

$$\bar{w}_1 = \bar{w}_2, \tag{16}$$

$$\phi_1 = \phi_2, \tag{17}$$

$$\frac{d\phi_1}{d\bar{x}} = \frac{d\phi_2}{d\bar{x}}, \tag{18}$$

$$\phi_1 + (1 - s t_1^{\,2}) \frac{d\bar{w}_1}{d\bar{x}} = \phi_2 + (1 - s t_2^{\,2}) \frac{d\bar{w}_2}{d\bar{x}}, \text{ for Case 1,} \tag{19a}$$

$$\phi_1 + (1 + s t_1^{\,2}) \frac{d\bar{w}_1}{d\bar{x}} = \phi_2 + (1 - s t_2^{\,2}) \frac{d\bar{w}_2}{d\bar{x}}, \text{ for Case 2a,} \tag{19b}$$

$$\phi_1 + (1 - s t_1^{\,2}) \frac{d\bar{w}_1}{d\bar{x}} = \phi_2 + \frac{d\bar{w}_2}{d\bar{x}}, \text{ for Case 3a.} \tag{19c}$$

In view of the boundary conditions in Eqs. (13a) to (15b) and the continuity relations in Eqs. (16) to (19c), one can develop an eigenvalue problem of the following form

$$[A]\{C\} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{20}$$

By setting the determinant of the above 8×8 matrix $[A]$ to zero, one obtains the stability criterion of the buckling problem concerned.

3. Stability criteria

As an illustrative example in determining the stability criterion, consider a cantilevered column that is clamped at $\bar{x} = 0$ and is subjected to only an axial compressive force P_1 at $\bar{x} = a$ (*i.e.* Case 3a). In view of the boundary conditions in Eqs. (13a, b), (15a, b), the continuity conditions in Eqs. (16–18), (19c) and the general solutions given by Eqs. (12a–d), the matrix $[A]$ is given by

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{t_1^2}{k_1} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6s & 0 & 0 & 0 \\ \sin(ak_1) & \cos(ak_1) & a & 1 & -a^3 & -a^2 & -a & -1 \\ -\frac{t_1^2}{k_1}\cos(ak_1) & \frac{t_1^2}{k_1}\sin(ak_1) & -1 & 0 & 3a^2+6s & 2a & 1 & 0 \\ t_1^2\sin(ak_1) & t_1^2\cos(ak_1) & 0 & 0 & 6a & 2 & 0 & 0 \\ 0 & 0 & -st_1^2 & 0 & 6s & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

By setting the determinant of the above matrix to zero, one obtains the stability criterion as

$$\cos(k_1 a) = 0. \tag{22}$$

It is interesting that the above stability criterion for fixed-free Timoshenko column under only an intermediate load has exactly the same form as its Euler column counterpart [3,6]. In view of this, we have the following buckling load relationship

$$P^T = \frac{P^E}{1 + \dfrac{P^E}{\kappa_s GA}} \tag{23}$$

This same buckling load relationship was shown to be valid for end loaded fixed-free columns, pinned-pinned columns, fixed-fixed columns and fixed-ended columns with sliding restraint at the top, and columns with equal rotational restraints at the ends [6,7,8]. Note that the superscripts T and E refer to quantities in Timoshenko and Euler columns, respectively.

The stability criteria for the three cases, (*i.e.* Cases 1, 2a and 3a) and for all combinations of boundary conditions are derived in the same manner as described above. Tables 1 to 3 present the stability criteria for Cases 1, 2a and 3a, respectively.

4. Results and discussion

Based on the stability criteria presented in Tables 1 to 3, one can easily generate numerical results to examine how the effect of transverse shear deformation, boundary conditions, and the influence of different intermediate load magnitudes and positions have on the critical buckling loads. Figures 2 to 5 show the interaction between

Table 1
Stability criteria for columns with compressive intermediate and end loads

Boundary Conditions	Stability Criteria for Case 1 Problem*
Fixed-Free	$1-\sqrt{\beta}\tan(k_1a)\tan[(1-a)k_1\sqrt{\beta}] = 0$
Pinned-Pinned	$\eta[(1-a)\eta + a]\tan(k_1a) + \sqrt{\beta}\left[(1-a)\eta^2 + a\eta-\frac{k_1}{t_1^2}\eta(\eta-1)^2\tan(k_1a)\right]\tan[(1-a)k_1\sqrt{\beta}] = 0$
Fixed-Pinned	$(1-a)t_1^2\eta+at_1^2-k_1\tan(k_1a) + \sqrt{\beta}\{k_1(\eta^2-2\eta+2)+\{2k_1(\eta-1)\sec(k_1a)+t_1^2[(1-a)\eta+a]\tan(k_1a)\}\}\tan[(1-a)k_1\sqrt{\beta}]=0$
Pinned-Fixed	$\{-k_1\beta[(1-a) + a\beta] + \{2-2\beta + \beta^2-2(1-\beta)\sec[(1-a)k_1\sqrt{\beta}]\}\tan(k_1a)\} -\{\beta + k_1\beta[(1-a) + a\beta]\tan(k_1a)\}\tan[(1-a)k_1\sqrt{\beta}] = 0$
Fixed-Fixed	$2k_1\sqrt{\beta}\{\eta^2-\eta + 1 + (\eta-1)\sec(k_1a)-\eta[\eta-1 + \sec(k_1a)]\sec[(1-a)k_1\sqrt{\beta}]\} + \sqrt{\beta}\eta t_1^2\left[(1-a) + \frac{a}{\eta}\right]\tan(k_1a) + [(1-a)t_1^2\eta + at_1^2-k_1(\beta\eta^2 + 1)\tan(k_1a)]\tan[(1-a)k_1\sqrt{\beta}] = 0$

* Note that $k_1^2 = t_1^2/(1-st_1^2)$, $k_2^2 = t_2^2/(1-st_2^2)$, $\beta = k_2^2/k_1^2$, $\eta = (t_1/t_2)^2$.

Table 2
Stability criteria for columns subjected to tensile intermediate loadand compressive end load

Boundary Conditions	Stability Criteria for Case 2a Problem*
Fixed-Free	$1-\sqrt{\beta}\tanh(k_1a)\tan[(1-a)k_1\sqrt{\beta}] = 0$
Pinned-Pinned	$\sqrt{\beta}t_1{}^2[a-(1-a)\eta]\tan[(1-a)k_1\sqrt{\beta}]$ $+ \{t_1{}^2[(1-a)\eta-a]-k_1\sqrt{\beta}(1+\eta)^2\tan[(1-a)k_1\sqrt{\beta}]\}\tanh(k_1a) = 0$
Fixed-Pinned	$\left\{a-(1-a)\eta + \dfrac{k_1\sqrt{\beta}}{t_1{}^2}[\eta^2+2\eta+2-2(1+\eta)\operatorname{sech}(k_1a)]\tan[(1-a)k_1\sqrt{\beta}]\right\}$ $+\left\{-\dfrac{k_1}{t_1{}^2}+\sqrt{\beta}[(1-a)\eta-a]\tan[(1-a)k_1\sqrt{\beta}]\right\}\tanh(k_1a) = 0$
Pinned-Fixed	$\sqrt{\beta}\{at_1{}^2-(1-a)t_1{}^2\eta+k_1\sqrt{\beta}\eta^2\tan[k_1\sqrt{\beta}(1-a)]\}$ $-\{k_1\sqrt{\beta}\{2\eta^2+2\eta+1-2\eta(1+\eta)\sec[k_1\sqrt{\beta}(1-a)]\}$ $+t_1{}^2[(1-a)\eta-a]\tan[k_1\sqrt{\beta}(1-a)]\}\tanh(k_1a) = 0$
Fixed-Fixed	$-2(1+\eta+\eta^2)+2\{1+\eta-\eta\sec[k_1\sqrt{\beta}(1-a)]\}\operatorname{sech}(k_1a)$ $+\eta\left\{2(1+\eta)\sec[k_1\sqrt{\beta}(1-a)]+\dfrac{t_1{}^2}{k_1\sqrt{\beta}}\left[\dfrac{a}{\eta}-(1-a)\right]\tan[k_1\sqrt{\beta}(1-a)]\right\}$ $+\left\{-\dfrac{(1-a)}{k_1}t_1{}^2\eta+\dfrac{at_1{}^2}{k_1}+\dfrac{(\beta\eta^2-1)}{\sqrt{\beta}}\tan[k_1\sqrt{\beta}(1-a)]\right\}\tanh(k_1a) = 0$

* Note that $k_1^2 = t_1{}^2/(1+st_1{}^2)$, $k_2^2 = t_2{}^2/(1+st_2{}^2)$, $\beta = k_2{}^2/k_1^2$, $\eta = (t_1/t_2)^2$.

the end buckling load $t_2{}^2 = P_2L^2/(EI)$ and the intermediate axial load $P_1L^2/(EI)$ for pinned-pinned, fixed-free, fixed-pinned and fixed-fixed columns, respectively. It is worth mentioning a few important points before interpreting the aforementioned graphs. Firstly, the buckling loads associated with $s = 0$ denote the Euler solutions while the buckling loads with $s = 0.01$ apply for stocky and built-up columns where the effect of transverse shear deformation is significant. Next, the pertinent points A to D on the interaction curve are associated with the following physical buckling problems. The x-intercepts (*i.e.* points A) denote solutions to Case 3a problem where $P_2 = 0$ while the y-intercepts denote the critical buckling loads $t_2{}^2 = P_2L^2/(EI)$ of columns under end axial load only (i.e. no applied intermediate axial load or $P_1 = 0$). In addition, region A to B and region B to C in the interaction curves apply

Table 3
Stability criteria for columns with compressive intermediate load only

Boundary Conditions	Stability Criteria for Case 3a Problem*
Fixed-Free	$\cos(k_1 a) = 0$
Pinned-Pinned	$3(1-a)^2 t_1^2 \cos(k_1 a) + k_1\{3(2-a)-(1-a)t_1^2[3s + (1-a)^2]\}\sin(k_1 a) = 0$
Fixed-Pinned	$k_1 t_1^2\{-6(1-a) + \{3(2-a)-(1-a)t_1^2[3s + (1-a)^2]\}\cos(k_1 a)\}-3[k_1^2 + (1-a)^2 t_1^4]\sin(k_1 a) = 0$
Pinned-Fixed	$4t_1^2\{3a-(1-a)t_1^2[3s + (1-a)^2]\}\cos(k_1 a) + k_1\{-12[1 + t_1^2(1-a)] + (1-a)^2 t_1^4[12s + (1-a)^2]\}\sin(k_1 a) = 0$
Fixed-Fixed	$12k_1[2 + t_1^2(1-a)^2] + k_1\{-24-12t_1^2(1-a) + t_1^4(1-a)^2[12s + (1-a)^2]\}\cos(k_1 a) + 4\{3k_1^2(1-a)-3at_1^2 + k_1^2(1-a)[3s + (1-a)^2]\}\sin(k_1 a) = 0$

* Note that $k_1^2 = t_1^2/(1-st_1^2)$.

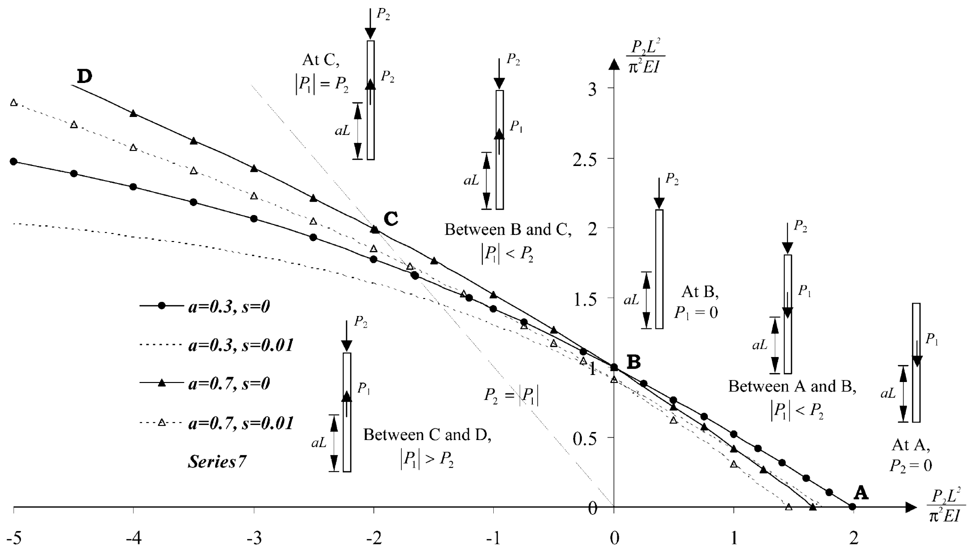

Fig. 2. Interaction curves for pinned-pinned columns with intermediate load at $a = 0.3$ and at $a = 0.7$.

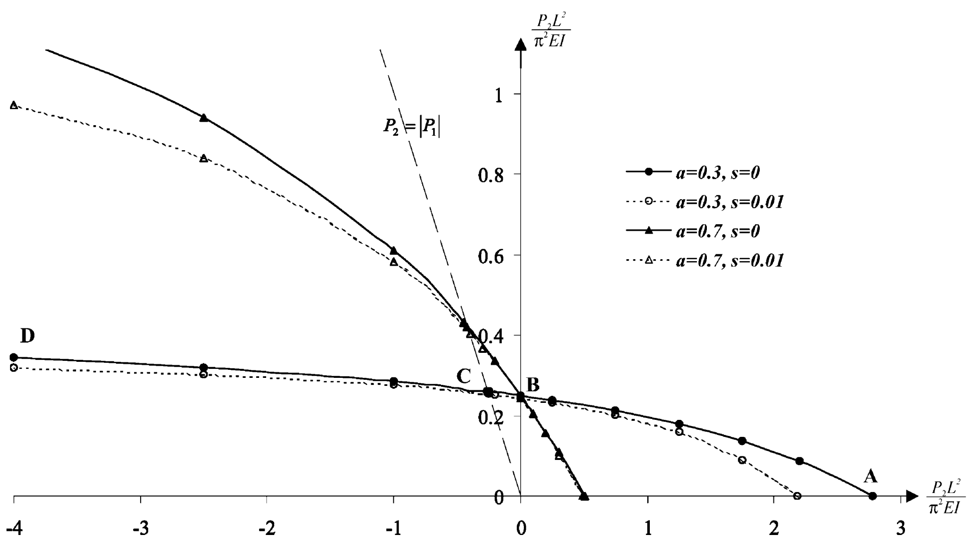

Fig. 3. Interaction curves for fixed-free column with intermediate load at $a = 0.3$ and at $a = 0.7$.

to Case 1 problem whereby both segments are in compression. However, for the region A to B, both P_1 and P_2 are compressive but for region B to C, P_1 though in tension, is smaller than P_2, and thus segment 1 is still in the state of compression. The special point C on the interaction curve denotes the point for which the magnitude of the tensile load P_1 is equal to the compressive end load P_2. Hence, segment 1 has

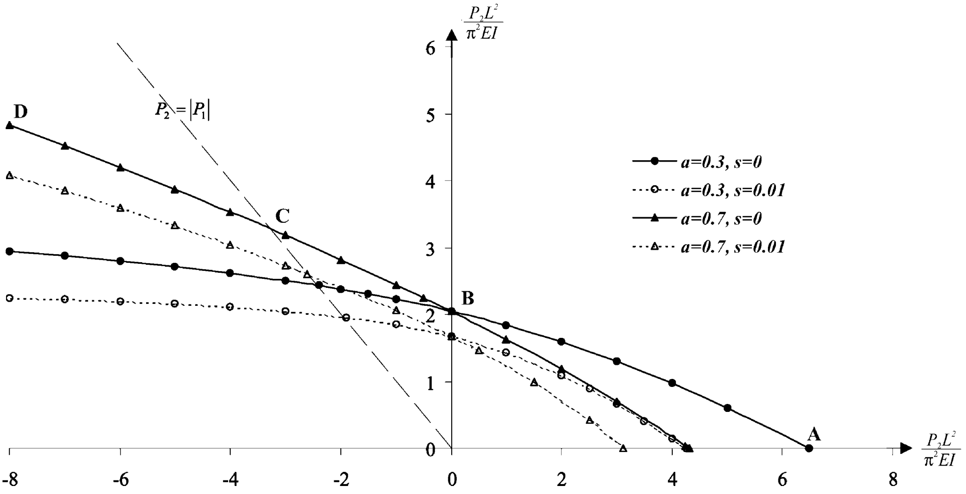

Fig. 4. Interaction curves for fixed-pinned column with intermediate load at $a = 0.3$ and at $a = 0.7$.

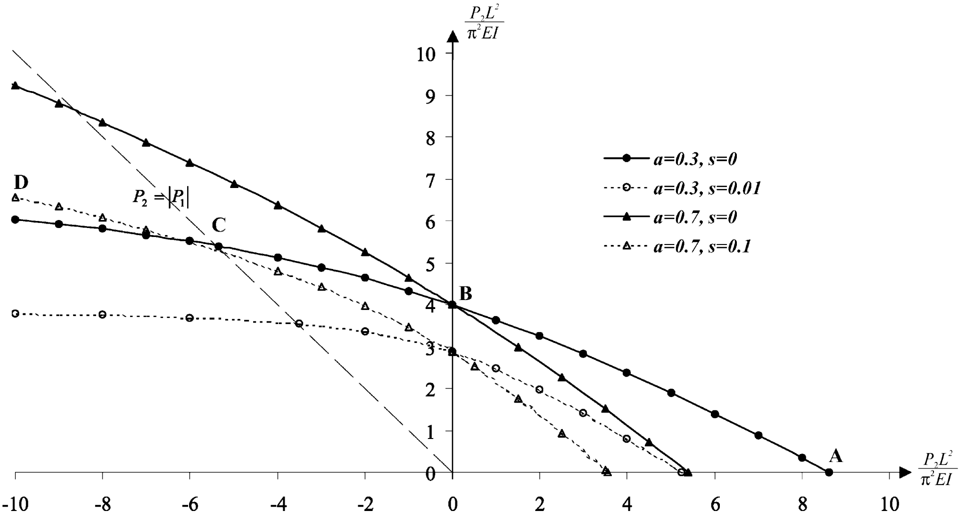

Fig. 5. Interaction curves for fixed-fixed columns with intermediate load at $a = 0.3$ and at $a = 0.7$.

no axial load but segment 2 is subjected to a compressive load of P_2 (*i.e.* the problem corresponds to Case 3b problem). Finally, region C to D on the interaction curve corresponds to the solutions for Case 2a problem where the magnitude of the tensile intermediate load exceeds that of the compressive end load, thus causing segment 1 to be in tension while segment 2 to be in compression.

It can be seen from the interaction curves in Figs. 2 to 5 that in the region A to B (*i.e.* when both P_1 and P_2 are compressive), the critical buckling loads P_2 associated with an intermediate load at $a = 0.3$ are higher than their counterparts with an intermediate load at $a = 0.7$. However, beyond the intersection point B (*i.e.* when P_1 is tensile and P_2 is compressive), the reverse is true. This observation may be explained as follows: When the intermediate load P_1 is compressive, a longer portion of the column is subjected to a greater compressive load when the intermediate load is located at $a = 0.7$ as compared to when it is at $a = 0.3$. Thus, a smaller end compressive load is needed to buckle the former column as compared to the latter one. In contrast, when the intermediate axial load P_1 is tensile in nature, it tends to stabilize the column. Hence for a longer portion of the column being subjected to a tensile intermediate load (say, at $a = 0.7$), the critical buckling load P_2 is expected to be higher than its counterpart with intermediate load at $a = 0.3$. For an intermediate load at a specified location and a given magnitude, we observe that the magnitude of the end buckling load decreases by more or less the same amount when the intermediate load is in tension or in compression. The decrement, however, increases with increasing magnitude of the intermediate load.

The effect of transverse shear deformation can also be observed in this set of interaction curves. It reduces the area enclosed by the interaction curve as the capability of the column to resist either the end or intermediate compressive axial loads against buckling decrease. Note that the end buckling load values at intersection points B for $s = 0$ and for $s = 0.01$ are related to each other by the buckling load relationship given in Eq. (23).

Figures 6 compares the effect of boundary conditions on the interaction curves for fixed-free, pinned-pinned, fixed-pinned and fixed-fixed columns with an intermediate load at the column's mid-height, while Fig. 7 shows the buckling loads for Case 3a problem. It should be noted that the superscript E in Fig. 7 refers to the

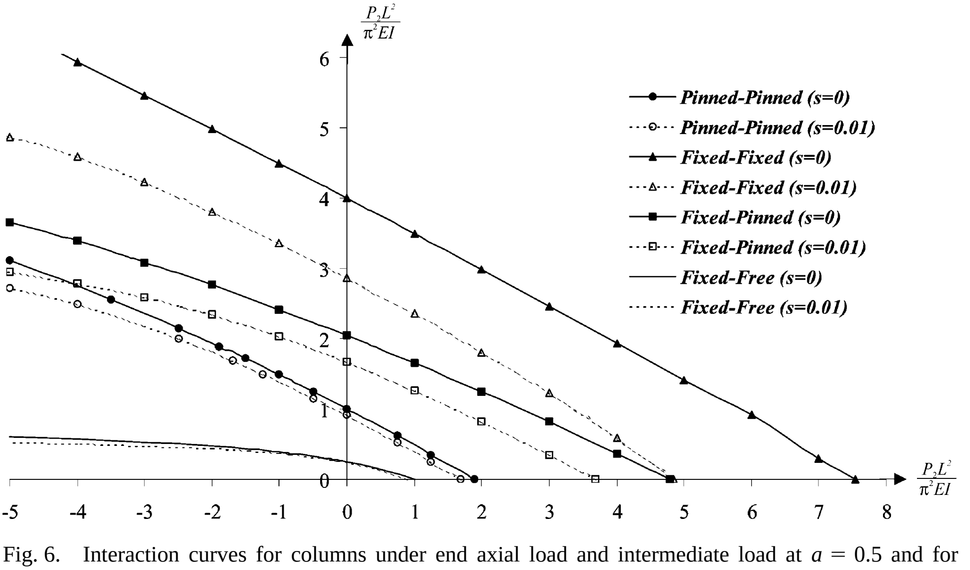

Fig. 6. Interaction curves for columns under end axial load and intermediate load at $a = 0.5$ and for various boundary conditions.

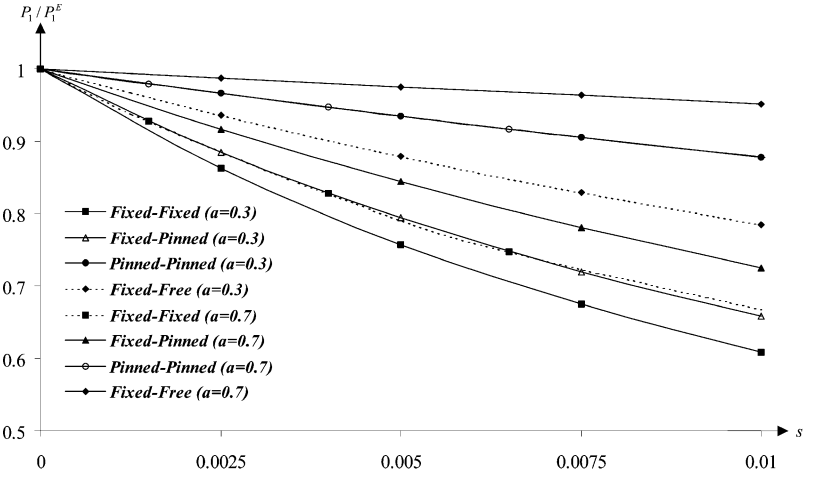

Fig. 7. Variation of buckling load ratios, P_1/P_1^E for Timoshenko columns under a compressive intermediate axial load at $a = 0.3$and at $a = 0.7$.

Euler column where the effect of transverse shear deformation is neglected (*i.e.* $s = 0$). From these two figures, one can observe that the effect of transverse shear deformation in lowering the buckling load is greater for columns with greater restraints at their ends. For example, from Fig. 7, the P_1/P_1^E values are highest for fixed-fixed columns, followed by the values for the fixed-pinned and pinned-pinned columns and are the least for fixed-free columns. Further in Fig. 7, we note that the effect of transverse shear deformation in reducing the Euler buckling load increases with decreasing value of a, i.e. when the intermediate load is located nearer to the bottom end of the column. This is due to the fact that when the intermediate load is located near the column base, the column buckling problem is now behaving like a very stocky column problem even though the entire column is slender.

5. Concluding remarks

Derived herein are the exact stability criteria for the elastic buckling of Timoshenko columns that are subjected to intermediate and end concentrated axial loads. Using these stability criteria, one can readily obtain the buckling loads of stocky and built-up columns. The results presented in graphical forms also provide an insight into the combined effects of transverse shear deformation, the boundary conditions and the intermediate load location on the elastic buckling load. The interaction curves of the end and intermediate axial loads show that the end buckling load is greatly reduced for columns under compressive loads, and with high values of shear parameter s, and when the ends of the columns are highly restrained. For columns under an intermediate compressive load near the base of the column, the effect of transverse shear deformation becomes significant even when the entire column is slender. This is because the column buckling problem behaves like a stocky column problem when the load is acting over a very short portion of the column. The exclusion of the transverse shear deformation in all the aforementioned column problems will lead to an unsafe design, and therefore it is essential that engineers pay attention to the effect of transverse shear deformation in such situations.

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Appendix A. Derivation of general solutions for *w* and *ø*

Consider Case 1 where $N > 0$ (*i.e.* total axial force in segment considered is compressive). By differentiating Eq. (3b) with respect to x and then substituting it into Eq. (3a), one obtains a fourth-order differential equation in terms of rotation ϕ as

$$\frac{d^4\phi}{d\bar{x}^4} + \frac{t^2}{1-st^2}\frac{d^2\phi}{d\bar{x}^2} = 0, \tag{A1a}$$

where

$$s = \frac{EI}{\kappa_s GAL^2},\ t^2 = \frac{NL^2}{EI}. \tag{A1b, c}$$

The general solution for the differential equation given by Eq. (A1a) is

$$\phi(x) = B_1\sin(k_c\bar{x}) + B_2\cos(k_c\bar{x}) + B_3\bar{x} + B_4, \tag{A1d}$$

where

$$k_c^2 = \frac{t^2}{1-st^2}, \tag{A1e}$$

and B_i, $i = 1,2,3,4$ are arbitrary constants.

Similarly, by differentiating Eq. (3a) with respect to x, and followed by substituting it into Eq. (3b) and after some algebraic manipulations, the uncoupled differential equation in terms of transverse deflection w is

$$\frac{d^4\bar{w}}{d\bar{x}^4} + \frac{t^2}{1-st^2}\frac{d^2\bar{w}}{d\bar{x}^2} = 0, \tag{A2a}$$

The general solution to the differential equation (A2a) is

$$\bar{w}(\bar{x}) = A_1\sin(k_c\bar{x}) + A_2\cos(k_c\bar{x}) + A_3\bar{x} + A_4, \tag{A2b}$$

where A_i, $i = 1,2,3,4$ are arbitrary constants.

The substitution of Eqs. (A1d) and (A2b) into Eqs. (3a) and (3b) furnishes

$$B_1 = A_2k_c(1-st^2),\quad B_2 = -A_1k_c(1-st^2),\qquad B_3 = 0,\quad B_4 = -A_3 \tag{A3a–d}$$

In view of Eqs. (A3a–d), Eq. (A1d) can be re-written as

$$\phi = \frac{t^2}{k_c}A_2\sin(k_c\bar{x})-\frac{t^2}{k_c}A_1\cos(k_c\bar{x})-A_3 \tag{A4}$$

Next we consider Case 2 where $N < 0$ (*i.e.* total axial force in segment considered is tensile). In view of Eqs. (3a) and (3b), the governing equations for this case in which the column is subjected to a tensile axial load can be expressed as

$$\frac{d\phi}{d\bar{x}} + (1 + st^2)\frac{d^2\bar{w}}{d\bar{x}^2} = 0, \tag{A5a}$$

$$s\frac{d^2\phi}{d\bar{x}^2}-\phi-\frac{d\bar{w}}{d\bar{x}} = 0. \tag{A5b}$$

where

$$t^2 = \frac{|N|L^2}{EI}. \tag{A5c}$$

By employing similar algebraic manipulations as for Case 1, one obtains the following uncoupled differential equations for w and ϕ

$$\frac{d^4\phi}{d\bar{x}^4}-k_t\frac{d^2\phi}{d\bar{x}^2} = 0, \tag{A6a}$$

$$\frac{d^4\bar{w}}{d\bar{x}^4}-k_t\frac{d^2\bar{w}}{d\bar{x}^2} = 0, \tag{A6b}$$

where

$$k_t^2 = \frac{t^2}{1 + st^2}. \tag{A6c}$$

In view of the above differential equations, the rotations and transverse displacement are given, respectively, by

$$\phi(\bar{x}) = B_1\sinh(k_t\bar{x}) + B_2\cosh(k_t\bar{x}) + B_3\bar{x} + B_4, \tag{A7a}$$

$$\bar{w}(\bar{x}) = A_1\sinh(k_t\bar{x}) + A_2\cosh(k_t\bar{x}) + A_3\bar{x} + A_4, \tag{A7b}$$

Noting Eqs. (A5a), (A5b), (A7a) and (A7b), the constants B_i can be expressed in terms of A_i as

$$B_1 = -A_2k_t(1 + st^2),\quad B_2 = -A_1k_t(1 + st^2),\qquad B_3 = 0,\quad B_4 = -A_3. \tag{A8a–d}$$

Thus the general solution for the rotation is given by

$$\phi(\bar{x}) = -\frac{t^2}{k_t}A_2\sinh(k_t\bar{x})-\frac{t^2}{k_t}A_1\cosh(k_t\bar{x})-A_3. \tag{A9}$$

Finally for Case 3 where $N = 0$ (*i.e.* no axial force in segment considered). In view of Eqs. (3a) and (3b), the governing equations for this case, where the column segment is not subjected to any axial load, are given by

$$\frac{d\phi}{d\bar{x}} + \frac{d^2\bar{w}}{d\bar{x}^2} = 0, \tag{A10a}$$

$$s\frac{d^2\phi}{d\bar{x}^2}-\phi-\frac{d\bar{w}}{d\bar{x}} = 0. \tag{A10b}$$

Similarly, with some algebraic manipulations, one obtains

$$\frac{d^4\phi}{d\bar{x}^4} = 0, \tag{A11a}$$

$$\frac{d^4w}{d\bar{x}^4} = 0. \tag{A11b}$$

The general solutions to Eqs. (A11a) and (A11b) are, respectively, given by

$$\phi(\bar{x}) = B_1\bar{x}^3 + B_2\bar{x}^2 + B_3\bar{x} + B_4, \tag{A12a}$$

$$\bar{w}(\bar{x}) = A_1\bar{x}^3 + A_2\bar{x}^2 + A_3\bar{x} + A_4. \tag{A12b}$$

The substitution of Eqs. (A12a,b) into Eqs. (A10a,b) leads to

$$\begin{aligned} &B_1 = 0 \qquad B_2 = -3A_1 \\ &B_3 = -2A_2 \quad B_4 = -6sA_1-A_3. \end{aligned} \tag{A13a–d}$$

Thus, the general solution for the rotation is furnished as

$$\phi(\bar{x}) = -3(\bar{x}^2 + 2s)A_1-2A_2\bar{x}-A_3. \tag{A14}$$

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